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A recent topic on rogue wave

By

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Abstract

Rogue wave type solutions for the self-dual Yang-Mills equation are constructed by using the Gram type determinant expression of solutions for systems related to toroidal Lie algebra.

§ 1. Introduction

Rogue waves are large and spontaneous waves which are localized in time direction and usually in space direction also[1]. In these years the mathematical structure of such localized wave solutions has been intensively studied for various soliton equations in the KP hierarchies which are associated with the affine Lie algebra symmetry. Other hierarchies of integrable equations can be generated by the representation theory of toroidal Lie algebra[2, 3]. The self-dual Yang-Mills (SDYM) equation is one of the typical examples of equations related with the toroidal Lie algebra symmetry. The rogue waves for this class of equations have not been studied well and we could expect different structure of solutions from the case of KP hierarchies. The explicit expressions for solutions of SDYM equation of SU(2) case are constructed in terms of the Hankel determinant whose matrix elements are given by an arbitrary function satisfying a linear dispersion relation[4]. Recently the theory of Cauchy matrix schemes for KP and AKNS hierarchies is developed to the case of SDYM equation and a broad class of solutions is explicitly constructed[5, 6]. These kinds of explicit expressions are useful to derive concrete solutions such as solitons, breathers and rogue waves.

In order to derive the rogue wave type solutions to SDYM equation, we first construct the Gram type determinant solution whose matrix elements contain arbitrary functions. A direct proof based on the bilinear method is described in an elementary

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way. By taking these arbitrary functions as appropriate polynomials, the Gram type determinant gives the rogue wave type solutions.

§ 2. Gram type determinant solution for SDYM equation

The SDYM equation in the R gauge of Yang is given as

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) = 0, \tag{2.1}$$

with $\det J = 1$, where the gauge field $J = \Big(J_{kl}\Big)_{1\le k,l\le M}$ is an $M \times M$ matrix function of four variables y, $\bar{y}$, z and $\bar{z}$. By scaling we can relax the condition, $\det J = 1$, and replace it by

$$\det J : \text{nonzero constant}. \tag{2.2}$$

Theorem 2.1. *The Gram type determinant solution for SDYM equation (2.1) and (2.2) is given as*

$$J = \frac{G}{f}, \quad G = \Big(g_{kl}\Big)_{1\le k,l\le M}, \tag{2.3}$$

where f is $N \times N$ Gram type determinant and g_{kl} is its bordered determinant,

$$f = \det_{1\le i,j\le N}\Big(m_{ij}\Big) = \begin{vmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \cdots & m_{NN} \end{vmatrix}, \quad g_{kl} = \det\begin{pmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\psi_{kj} & \delta_{kl} \end{pmatrix} = \begin{vmatrix} m_{11} & \cdots & m_{1N} & \varphi_{1l} \\ \vdots & & \vdots & \vdots \\ m_{N1} & \cdots & m_{NN} & \varphi_{Nl} \\ \frac{1}{q_1}\psi_{k1} & \cdots & \frac{1}{q_N}\psi_{kN} & \delta_{kl} \end{vmatrix}, \tag{2.4}$$

$$m_{ij} = \frac{1}{p_i + q_j}\sum_{\mu=1}^{M} \varphi_{i\mu}\psi_{\mu j}, \tag{2.5}$$

$$\varphi_{i\mu} = \varphi_{i\mu}(\bar{y} + p_i z, \bar{z} - p_i y), \quad \psi_{\mu j} = \psi_{\mu j}(\bar{y} - q_j z, \bar{z} + q_j y), \tag{2.6}$$

where $\varphi_{i\mu}$ and $\psi_{\mu j}$ are arbitrary functions of two variables, and p_i and q_j are constants.

Remark. *For the Gram type determinant solution in the above theorem, we have*

$$\det J = \prod_{\nu=1}^{N}\Big(-\frac{p_\nu}{q_\nu}\Big), \tag{2.7}$$

and J^{-1} is given as

$$J^{-1} = \frac{H}{f}, \quad H = \Big(h_{kl}\Big)_{1\le k,l\le M}, \tag{2.8}$$

$$h_{kl} = \det\begin{pmatrix} m_{ij} & \frac{1}{p_i}\varphi_{il} \\ \psi_{kj} & \delta_{kl} \end{pmatrix} = \begin{vmatrix} m_{11} & \cdots & m_{1N} & \frac{1}{p_1}\varphi_{1l} \\ \vdots & & \vdots & \vdots \\ m_{N1} & \cdots & m_{NN} & \frac{1}{p_N}\varphi_{Nl} \\ \psi_{k1} & \cdots & \psi_{kN} & \delta_{kl} \end{vmatrix}, \tag{2.9}$$

since f, g_{kl} and h_{kl} in (2.4) and (2.9) satisfy

$$\sum_{\mu=1}^{M} h_{k\mu} g_{\mu l} = \delta_{kl} f f. \tag{2.10}$$

By using (2.3) and (2.8), the SDYM equation (2.1) is decoupled to two bilinear equations,

$$\sum_{\mu=1}^{M} D_z h_{k\mu} \cdot g_{\mu l} = 2 D_{\bar{y}} f_{kl} \cdot f, \tag{2.11}$$

$$\sum_{\mu=1}^{M} D_y h_{k\mu} \cdot g_{\mu l} = -2 D_{\bar{z}} f_{kl} \cdot f, \tag{2.12}$$

where D is the Hirota bilinear differential operator and f_{kl} is an auxiliary variable which is given as the bordered determinant,

$$f_{kl} = \det \begin{pmatrix} m_{ij} & \varphi_{il} \\ \psi_{kj} & 0 \end{pmatrix} = \begin{vmatrix} m_{11} & \cdots & m_{1N} & \varphi_{1l} \\ \vdots & & \vdots & \vdots \\ m_{N1} & \cdots & m_{NN} & \varphi_{Nl} \\ \psi_{k1} & \cdots & \psi_{kN} & 0 \end{vmatrix}. \tag{2.13}$$

An elementary proof of the facts in above theorem and remark is given in appendix.

§ 3. Rogue wave type solutions

By choosing the arbitrary functions $\varphi_{i\mu}$ and $\psi_{\mu j}$ in the Gram type determinant solution appropriately, we can derive various solutions explicitly. For example the standard N-soliton solution of SDYM equation is obtained by taking

$$\begin{aligned} \varphi_{i\mu}(\bar{y} + p_i z, \bar{z} - p_i y) &= A_{i\mu} \exp(r_i(\bar{y} + p_i z) + u_i(\bar{z} - p_i y)), \\ \psi_{\mu j}(\bar{y} - q_j z, \bar{z} + q_j y) &= B_{\mu j} \exp(s_j(\bar{y} - q_j z) + v_j(\bar{z} + q_j y)), \end{aligned}$$

where p_i, r_i and u_i are wave numbers of i-th soliton, $1/q_i$, $v_i q_i$ and $-s_i q_i$ are their conjugates, $A_{i\mu}$ is the phase parameter of i-th soliton in μ-th component and $B_{\mu i}/q_i$ is its conjugate up to μ-dependent constant multiplication.

Since the rogue wave solutions are typically given by rational functions, we consider the case that $\varphi_{i\mu}$ and $\psi_{\mu j}$ are polynomials. In particular let us take them as first order polynomials,

$$\begin{aligned} \varphi_{i\mu}(\bar{y} + p_i z, \bar{z} - p_i y) &= a_{i\mu}(\bar{y} + p_i z) + b_{i\mu}(\bar{z} - p_i y) + c_{i\mu}, \\ \psi_{\mu j}(\bar{y} - q_j z, \bar{z} + q_j y) &= \alpha_{\mu j}(\bar{y} - q_j z) + \beta_{\mu j}(\bar{z} + q_j y) + \gamma_{\mu j}, \end{aligned}$$

where $a_{i\mu}$, $b_{i\mu}$, $\alpha_{\mu j}$ and $\beta_{\mu j}$ are wave numbers, and $c_{i\mu}$ and $\gamma_{\mu j}$ are phase constants. By choosing these parameters appropriately, some of the components J_{kl} localize in time direction, i.e., one direction in four dimensional space. For simplicity we only study the case of $\bar{y}$ and $\bar{z}$ being complex conjugate of y and z, respectively, and apply the variable transformations, $y = t + \sqrt{-1}u$ and $z = v + \sqrt{-1}w$, where we regard t as time variable. Hereafter $^-$ means complex conjugate. Then we take

$$q_j = \frac{1}{\bar{p}_j}, \quad \alpha_{\mu j} = -\bar{b}_{j\mu}, \quad \beta_{\mu j} = \bar{a}_{j\mu}, \quad \gamma_{\mu j} = \frac{\bar{c}_{j\mu}}{\bar{p}_j},$$

so that the complex conjugate conditions, $\psi_{\mu j} = \frac{1}{\bar{p}_j}\bar{\varphi}_{j\mu}$ and $m_{ij} = \bar{m}_{ji}$, are satisfied. After all, the solution in this case is given as

$$f = \det_{1\le i,j\le N}\Big(m_{ij}\Big), \quad g_{kl} = \det\begin{pmatrix} m_{ij} & \varphi_{il} \\ \bar{\varphi}_{jk} & \delta_{kl}\end{pmatrix}, \quad m_{ij} = \frac{1}{p_i\bar{p}_j + 1}\sum_{\mu=1}^{M}\varphi_{i\mu}\bar{\varphi}_{j\mu},$$
$$\varphi_{i\mu} = (a_{i\mu} - b_{i\mu}p_i)t - (a_{i\mu} + b_{i\mu}p_i)\sqrt{-1}u + (a_{i\mu}p_i + b_{i\mu})v + (a_{i\mu}p_i - b_{i\mu})\sqrt{-1}w + c_{i\mu},$$

where p_i, $a_{i\mu}$, $b_{i\mu}$ and $c_{i\mu}$ are complex constants.

Now let us take

$$b_{i1} = -a_{i1}\bar{p}_i, \qquad a_{i\mu} = b_{i\mu} = 0 \quad (2 \le \mu \le M).$$

Then a_{i1} can be normalized to 1 without loss of generality. Denoting $c_{i1} = \theta_i + \sqrt{-1}\phi_i$, φ_{i1} is given as

$$\varphi_{i1} = (p_i\bar{p}_i + 1)t + \theta_i + \sqrt{-1}[(p_i\bar{p}_i - 1)u + \frac{p_i - \bar{p}_i}{\sqrt{-1}}v + (p_i + \bar{p}_i)w + \phi_i],$$

and $\varphi_{i\mu} = c_{i\mu}$ for $2 \le \mu \le M$. This solution might be regarded as the rogue wave solution. Here p_i is wave number and θ_i and ϕ_i parametrize the position of rogue waves. In the case of $N = 1$, by omitting the indices i and j for notational simplicity, the solution is written as

$$f = \frac{1}{|p|^2 + 1}\sum_{\mu=1}^{M}|\varphi_\mu|^2, \quad g_{kl} = \delta_{kl}f - \bar{\varphi}_k\varphi_l,$$
$$\varphi_\mu = \begin{cases} (p\bar{p} + 1)t + \theta + \sqrt{-1}[(p\bar{p} - 1)u + \frac{p-\bar{p}}{\sqrt{-1}}v + (p + \bar{p})w + \phi], & \mu = 1, \\ c_\mu, & 2 \le \mu \le M. \end{cases}$$

This solution might be regarded as the fundamental rogue wave, since the components of J are localized in time except $(1,1)$-component,

$$J_{kl} - \delta_{kl} = \frac{g_{kl} - \delta_{kl}f}{f} \to 0 \quad (t \to \pm\infty),$$

for $(k,l) \neq (1,1)$. We should note that J_{11} is not localized in time. The solution for $N \geq 2$ describes the superposition of N fundamental rogue waves.

§ 4. Concluding remark

A wide class of solutions for the SDYM equation is explicitly given by using the Gram type determinant which is expressed in terms of arbitrarily many arbitrary functions of two variables. The rogue wave type solutions are obtained by taking these functions as appropriate polynomials. Any number of fundamental rogue waves with any displacements can be superposed in the case of SDYM equation.

Appendix

We give a direct proof of the results in section 2 by using the bilinear method. Let us assume f, g_{kl}, h_{kl} and f_{kl} are defined by (2.4), (2.9) and (2.13) with (2.5). Since m_{ij} satisfies

$$m_{ij} - \sum_{\mu=1}^{M} \varphi_{i\mu} \frac{1}{q_j} \psi_{\mu j} = -\frac{p_i}{q_j} m_{ij},$$

the Gram type determinant f is expressed in M-bordered determinant form,

$$f \prod_{\nu=1}^{N} (-\frac{p_\nu}{q_\nu}) = \det_{1\leq i,j\leq N} \left(-\frac{p_i}{q_j} m_{ij}\right) = \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & 0 \\ \vdots & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & 0 & & 1 \end{vmatrix}. \tag{A.1}$$

Similarly h_{kl} is rewritten as $(N+l, N+k)$-cofactor of $(N+M)\times(N+M)$ determinant,

$$h_{kl} \prod_{\nu=1}^{N} (-\frac{p_\nu}{q_\nu}) = \det \begin{pmatrix} -\frac{p_i}{q_j} m_{ij} & \varphi_{il} \\ -\frac{1}{q_j}\psi_{kj} & \delta_{kl} \end{pmatrix} \tag{A.2}$$

$$= \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} & \varphi_{il} \\ \frac{1}{q_j}\psi_{1j} & 1 & & 0 & \\ \vdots & & \ddots & & \emptyset \\ \frac{1}{q_j}\psi_{Mj} & 0 & & 1 & \\ -\frac{1}{q_j}\psi_{kj} & & {}^t\emptyset & & \delta_{kl} \end{vmatrix} = \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} & \emptyset \\ \frac{1}{q_j}\psi_{1j} & 1 & & 0 & \\ \vdots & & \ddots & & -e_l \\ \frac{1}{q_j}\psi_{Mj} & 0 & & 1 & \\ {}^t\emptyset & & {}^te_k & & 0 \end{vmatrix} = \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & 0 \\ \vdots & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & 0 & & 1 \end{vmatrix}_{N+l,N+k},$$

where $\emptyset$ is zero column vector, e_l is unit column vector with 1 in l-th component and 0 in others, and $|(\cdots)|_{\alpha,\beta}$ means (α,β)-cofactor of the matrix $(\cdots)$. The sizes of vectors $\emptyset$ and e_l depend on the place they appear. Applying the Sylvester's determinant identity to rhs of (A.1), we obtain

$$\frac{f\prod_{\nu=1}^{N}(-\frac{p_\nu}{q_\nu})}{f}=\det_{1\le k,l\le M}\left(\frac{\det\begin{pmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\psi_{kj} & \delta_{kl}\end{pmatrix}}{f}\right).$$

Thus we have proved that J in (2.3) satisfies (2.7). Applying the Laplace expansion to the vanishing $2(N+M)\times 2(N+M)$ determinant,

$$\det\left(\begin{array}{cccc|c|ccc|c|ccc} m_{ij} & \varphi_{i1} & \cdots & & \varphi_{il} & \cdots & \varphi_{iM} & & & & & \\ \frac{1}{q_j}\psi_{1j} & 1 & & & & & & & & 1 & & \\ \vdots & & \ddots & & e_l & \ddots & & & 0 & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & & & & & & 1 & & & & & 1 \\ \hline & & & & \varphi_{il} & & & & m_{ij} & & & \\ & & & & & & & & \frac{1}{q_j}\psi_{1j} & 1 & & \\ & 0 & & & e_l & & 0 & & \vdots & & \ddots & \\ & & & & & & & & \frac{1}{q_j}\psi_{Mj} & & & 1 \end{array}\right)=0,$$

where $(N+k)$-th column is replaced by the vector $\binom{\varphi_{il}}{e_l}$ both on top and bottom, we obtain

$$\delta_{kl}\begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & \\ \vdots & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & & & 1\end{vmatrix}\left|m_{ij}\right|-\sum_{\mu=1}^{M}\begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & \\ \vdots & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & & & 1\end{vmatrix}_{N+\mu,N+k}\begin{vmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\psi_{\mu j} & \delta_{\mu l}\end{vmatrix}=0.$$

Thus by using (A.1) and (A.2) we have proved that (2.10) holds and J^{-1} is actually given as (2.8).

Assuming (2.6), $\varphi_{i\mu}$, $\psi_{\mu j}$ and m_{ij} satisfy

$$\text{(A.3)}\quad \partial_z\varphi_{i\mu}=p_i\partial_{\bar y}\varphi_{i\mu},\quad \partial_z\psi_{\mu j}=-q_j\partial_{\bar y}\psi_{\mu j},\quad \partial_z m_{ij}-\sum_{\mu=1}^{M}\varphi_{i\mu}\frac{1}{q_j}\partial_z\psi_{\mu j}=p_i\partial_{\bar y}m_{ij},$$

$$\text{(A.4)}\quad \partial_y\varphi_{i\mu}=-p_i\partial_{\bar z}\varphi_{i\mu},\quad \partial_y\psi_{\mu j}=q_j\partial_{\bar z}\psi_{\mu j},\quad \partial_y m_{ij}-\sum_{\mu=1}^{M}\varphi_{i\mu}\frac{1}{q_j}\partial_y\psi_{\mu j}=-p_i\partial_{\bar z}m_{ij}.$$

Then we have

$$
\begin{aligned}
D_{\bar y} f_{kl}\cdot f &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\Big(\begin{vmatrix} m_{ij} & \varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix}_{\alpha,\beta} \begin{vmatrix} m_{ij} \end{vmatrix} - \begin{vmatrix} m_{ij} & \varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} \end{vmatrix}_{\alpha,\beta} \Big) \partial_{\bar y} m_{\alpha\beta} \\
&\quad + \Big(\begin{vmatrix} m_{ij} & \partial_{\bar y}\varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij} & \varphi_{il} \\ \partial_{\bar y}\psi_{kj} & 0 \end{vmatrix} \Big) \begin{vmatrix} m_{ij} \end{vmatrix} \\
&= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix}_{\alpha,N+1} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix}_{N+1,\beta} \frac{1}{p_\alpha}\Big(\partial_z m_{\alpha\beta} - \sum_{\mu=1}^{M} \varphi_{\alpha\mu}\frac{1}{q_\beta}\partial_z \psi_{\mu\beta}\Big) \\
&\quad + \Big(\begin{vmatrix} m_{ij} & \frac{1}{p_i}\partial_z\varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij} & \varphi_{il} \\ -\frac{1}{q_j}\partial_z\psi_{kj} & 0 \end{vmatrix} \Big) \begin{vmatrix} m_{ij} \end{vmatrix},
\end{aligned}
$$

where we used the Jacobi's formula and (A.3). Thus we have

$$
\begin{aligned}
D_{\bar y} f_{kl}\cdot f &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \begin{vmatrix} m_{ij} & e_\alpha \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} & \varphi_{il} \\ {}^t e_\beta & 0 \end{vmatrix} \frac{1}{p_\alpha}\partial_z m_{\alpha\beta} \\
&\quad - \sum_{\mu=1}^{M} \begin{vmatrix} m_{ij} & \frac{1}{p_i}\varphi_{i\mu} \\ \psi_{kj} & \delta_{k\mu} \end{vmatrix} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\partial_z\psi_{\mu j} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij} & \frac{1}{p_i}\partial_z\varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} \end{vmatrix}.
\end{aligned} \tag{A.5}
$$

The last term in rhs of (A.5) multiplied by $\prod_{\nu=1}^{N}(-\frac{p_\nu}{q_\nu})$ is rewritten as

$$
\begin{aligned}
&\begin{vmatrix} m_{ij} & \frac{1}{p_i}\partial_z\varphi_{il} \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} \end{vmatrix} \prod_{\nu=1}^{N}\Big(-\frac{p_\nu}{q_\nu}\Big) = \begin{vmatrix} -\frac{p_i}{q_j} m_{ij} & \partial_z\varphi_{il} \\ -\frac{1}{q_j}\psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} \end{vmatrix} \\
&= \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} & \partial_z\varphi_{il} \\ \frac{1}{q_j}\psi_{1j} & 1 & & & \\ \vdots & & \ddots & & \\ \frac{1}{q_j}\psi_{Mj} & & & 1 & \\ -\frac{1}{q_j}\psi_{kj} & & & & 0 \end{vmatrix} \begin{vmatrix} m_{ij} \end{vmatrix} = -\sum_{\mu=1}^{M} \begin{vmatrix} m_{ij} & \varphi_{i1} & \cdots & \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & \\ \vdots & & \ddots & \\ \frac{1}{q_j}\psi_{Mj} & & & 1 \end{vmatrix}_{N+\mu,N+k} \begin{vmatrix} m_{ij} & \partial_z\varphi_{il} \\ \frac{1}{q_j}\psi_{\mu j} & 0 \end{vmatrix},
\end{aligned}
$$

where in the last line we used the identity obtained by Laplace expansion applied to

$$
\det\left(\begin{array}{cc|c|c|c|c}
m_{ij} & \varphi_{i1}\cdots & \partial_z\varphi_{il} & \cdots\ \varphi_{iM} & & \\
\frac{1}{q_j}\psi_{1j} & 1 & & & & 1 \\
\vdots & \ddots & \emptyset & \ddots & 0 & \ddots \\
\frac{1}{q_j}\psi_{Mj} & & & 1 & & 1 \\
\hline
 & & \partial_z\varphi_{il} & & m_{ij} & \\
 & & & & \frac{1}{q_j}\psi_{1j} & 1 \\
 & 0 & \emptyset & 0 & \vdots & \ddots \\
 & & & & \frac{1}{q_j}\psi_{Mj} & 1
\end{array}\right) = 0,
$$

where $(N+k)$-th column is replaced by the vector $\binom{\partial_z\varphi_{il}}{\emptyset}$ on top and bottom. Similarly for the first term in rhs of (A.5), we have

$$\begin{vmatrix} m_{ij} & e_\alpha \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} & \varphi_{il} \\ {}^t e_\beta & 0 \end{vmatrix} \frac{1}{p_\alpha} \prod_{\nu=1}^{N} \left(-\frac{p_\nu}{q_\nu}\right) = \begin{vmatrix} -\frac{p_i}{q_j} m_{ij} & e_\alpha \\ -\frac{1}{q_j}\psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} & \varphi_{il} \\ {}^t e_\beta & 0 \end{vmatrix}$$
$$= \begin{vmatrix} m_{ij} & \varphi_{i1} \cdots & e_\alpha & \cdots \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 & & \\ \vdots & \ddots & \emptyset & \ddots \\ \frac{1}{q_j}\psi_{Mj} & & & 1 \end{vmatrix} \begin{vmatrix} \widetilde{m_{ij}} & & \\ \widetilde{\frac{1}{q_j}\psi_{1j}} & 1 & \\ \vdots & & \ddots \\ \widetilde{\frac{1}{q_j}\psi_{Mj}} & & 1 \end{vmatrix},$$

where in rhs, $(N+k)$-th column of the first determinant is $\binom{e_\alpha}{\emptyset}$, and $\widetilde{m_{ij}}$ and $\widetilde{\frac{1}{q_j}\psi_{\mu j}}$ in the second determinant are same with m_{ij} and $\frac{1}{q_j}\psi_{\mu j}$ except that their β-th columns are replaced by φ_{il} and $\delta_{\mu l}$, respectively. We can farther rewrite it as

$$\begin{vmatrix} m_{ij} & e_\alpha \\ \psi_{kj} & 0 \end{vmatrix} \begin{vmatrix} m_{ij} & \varphi_{il} \\ {}^t e_\beta & 0 \end{vmatrix} \frac{1}{p_\alpha} \prod_{\nu=1}^{N} \left(-\frac{p_\nu}{q_\nu}\right)$$
$$= \delta_{kl} \begin{vmatrix} m_{ij} & \varphi_{i1} \cdots \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 \\ \vdots & \ddots \\ \frac{1}{q_j}\psi_{Mj} & & 1 \end{vmatrix} \Big| m_{ij} \Big|_{\alpha,\beta} - \sum_{\mu=1}^{M} \begin{vmatrix} m_{ij} & \varphi_{i1} \cdots \varphi_{iM} \\ \frac{1}{q_j}\psi_{1j} & 1 \\ \vdots & \ddots \\ \frac{1}{q_j}\psi_{Mj} & & 1 \end{vmatrix}_{N+\mu,N+k} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\psi_{\mu j} & \delta_{\mu l} \end{vmatrix}_{\alpha,\beta},$$

by using the identity obtained by Laplace expansion of

$$\det \left(\begin{array}{ccc|c|cc|cc} m_{ij} & \varphi_{i1} & \cdots & e_\alpha & \cdots & \varphi_{iM} & \widetilde{m_{ij}} & \\ \frac{1}{q_j}\psi_{1j} & 1 & & & & & \widetilde{\frac{1}{q_j}\psi_{1j}} & 1 \\ \vdots & & \ddots & \emptyset & \ddots & & \vdots & \ddots \\ \frac{1}{q_j}\psi_{Mj} & & & & & 1 & \widetilde{\frac{1}{q_j}\psi_{Mj}} & 1 \\ \hline & & & e_\alpha & & & \widetilde{m_{ij}} & \\ & & & & & & \widetilde{\frac{1}{q_j}\psi_{1j}} & 1 \\ & 0 & & \emptyset & 0 & & \vdots & \ddots \\ & & & & & & \widetilde{\frac{1}{q_j}\psi_{Mj}} & 1 \end{array} \right) = 0.$$

Therefore using (A.1) and (A.2), from (A.5) we get

$$\begin{aligned}
D_{\bar{y}} f_{kl} \cdot f = & \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \left(\delta_{kl} f \left| m_{ij} \right|_{\alpha,\beta} - \sum_{\mu=1}^{M} h_{k\mu} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\psi_{\mu j} & \delta_{\mu l} \end{vmatrix}_{\alpha,\beta} \right) \partial_z m_{\alpha\beta} \\
& - \sum_{\mu=1}^{M} h_{k\mu} \begin{vmatrix} m_{ij} & \varphi_{il} \\ \frac{1}{q_j}\partial_z \psi_{\mu j} & 0 \end{vmatrix} - \sum_{\mu=1}^{M} h_{k\mu} \begin{vmatrix} m_{ij} & \partial_z \varphi_{il} \\ \frac{1}{q_j}\psi_{\mu j} & 0 \end{vmatrix} \\
& = \delta_{kl} f \partial_z f - \sum_{\mu=1}^{M} h_{k\mu} \partial_z g_{\mu l}.
\end{aligned}$$

Finally eliminating $\delta_{kl} f \partial_z f$ from the above equation by using z-derivative of (2.10), we obtain (2.11). Similarly (2.12) is derived by replacing z and $\bar{y}$ by $-y$ and $\bar{z}$, respectively, and using (A.4). We can simply denote (2.11) and (2.12) in matrix bilinear form,

$$D_z H \cdot G = 2 D_{\bar{y}} F \cdot f, \quad D_y H \cdot G = -2 D_{\bar{z}} F \cdot f,$$

where G and H are given in (2.3) and (2.8), and F is defined by $F = \Big(f_{kl} \Big)_{1 \le k,l \le M}$. Dividing these by f^2 we get

$$D_z J^{-1} \cdot J = 2 \partial_{\bar{y}} \frac{F}{f}, \quad D_y J^{-1} \cdot J = -2 \partial_{\bar{z}} \frac{F}{f},$$

thus

$$\partial_{\bar{y}} (D_y J^{-1} \cdot J) + \partial_{\bar{z}} (D_z J^{-1} \cdot J) = 0,$$

which is equivalent with (2.1). This completes the proof.

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