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The eigenvalues of the principal minor of a Hermitian random matrix

By

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Abstract

This is a summary of the paper [8]. We study the difference between the empirical eigenvalue distribution of a unitarily invariant Hermitian random matrix and that of its principal minor. Under some technical assumptions, we prove that the difference can be described by the Markov–Krein correspondence as the size of the matrix goes to infinity.

§ 1. Introduction

Let X_N be a hermitian random matrix of size N whose distribution on the matrix space $M_N(\mathbb{C})$ is invariant under conjugacy by any unitary matrix. Let $\Lambda_N = (\lambda_1^{(N)} \le \cdots \le \lambda_N^{(N)})$ be its eigenvalues. Then a diagonalization $X_N = U_N D_N {U_N}^*$ exists, where $D_N = \mathrm{diag}(\lambda_1^{(N)}, \lambda_2^{(N)}, \dots, \lambda_N^{(N)})$ and U_N is a Haar unitary random matrix of size N and independent of D_N (see [6, Proposition 6.1]). We denote by $\mathfrak{m}_N$ the *empirical eigenvalue distribution* of X_N:

$$\mathfrak{m}_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^{(N)}}.$$

Given a Schwartz distribution ζ on $\mathbb{R}$, let $\mathrm{M}_k(\zeta)$ denote the evaluation of ζ by the test function x^k and call it the k-th moment of ζ, as long as it is well defined. If ζ is a probability measure, then $\mathrm{M}_k(\zeta)$ is the usual k-th moment.

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Assumption 1.1. The following are assumed throughout this paper.

(A1) $\sup_{N\geq 1} \mathbb{E}[\mathrm{M}_k(\mathfrak{m}_N)] < \infty, \quad k \in 2\mathbb{N}.$

(A2) There exists a (non-random) probability distribution $\mathfrak{m}$ on $\mathbb{R}$ having finite moments of all orders such that $\mathfrak{m}_N$ converges in moments in probability to $\mathfrak{m}$:

$$\lim_{N\to\infty} \mathbb{P}[|\mathrm{M}_k(\mathfrak{m}_N) - \mathrm{M}_k(\mathfrak{m})| \geq \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0.$$

Let $\tilde{X}_N$ be the principal minor of X_N made by removing the last row and column of X_N. Cauchy's interlacing law says that the eigenvalues $\tilde{\Lambda}_N = (\tilde{\lambda}_1^{(N)} \leq \cdots \leq \tilde{\lambda}_{N-1}^{(N)})$ of $\tilde{X}_N$ interlace with Λ_N (see [16, Exercise 1.3.14]):

$$\lambda_1^{(N)} \leq \tilde{\lambda}_1^{(N)} \leq \lambda_2^{(N)} \leq \tilde{\lambda}_2^{(N)} \leq \cdots \leq \lambda_{N-1}^{(N)} \leq \tilde{\lambda}_{N-1}^{(N)} \leq \lambda_N^{(N)}.$$

It is not hard to see under Assumption 1.1 that the empirical eigenvalue distribution $\tilde{\mathfrak{m}}_N$ of $\tilde{X}_N$ also converges in moments in probability to the same limit $\mathfrak{m}$. Our main purpose is to investigate the difference of $\mathfrak{m}_N$ and $\tilde{\mathfrak{m}}_N$. Let us investigate the rescaled difference $N(\mathfrak{m}_N - \tilde{\mathfrak{m}}_N)$, which can be calculated as

$$N(\mathfrak{m}_N - \tilde{\mathfrak{m}}_N) = \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}} - \frac{N}{N-1}\sum_{j=1}^{N-1} \delta_{\tilde{\lambda}_j^{(N)}} = \underbrace{\sum_{i=1}^{N} \delta_{\lambda_i^{(N)}} - \sum_{j=1}^{N-1} \delta_{\tilde{\lambda}_j^{(N)}}}_{=:\widehat{\tau}_N} - \tilde{\mathfrak{m}}_N.$$

Because $\tilde{\mathfrak{m}}_N$ converges to $\mathfrak{m}$, the problem is the convergence of the signed measure $\widehat{\tau}_N$.

To describe the convergence of $\widehat{\tau}_N$, recall the notions of *free cumulants* and *Markov–Krein correspondence.* The free cumulants $\mathrm{R}_n(\mathfrak{m}), n \in \mathbb{N}$, of $\mathfrak{m}$ are the real numbers defined by the recursive formula

$$\mathrm{M}_k(\mathfrak{m}) = \sum_{\rho\in \mathrm{NC}(k)} \mathrm{R}_\rho(\mathfrak{m}), \tag{1.1}$$

where $\mathrm{NC}(k)$ is the set of non-crossing partitions on the set $\{1,2,\ldots,k\}$ and $\mathrm{R}_\rho(\mathfrak{m}) := \prod_{B\in\rho} \mathrm{R}_{|B|}(\mathfrak{m})$. Free cumulants are helpful for computing free convolutions and are well studied in free probability. For further details, the reader is referred to [15]. On the other hand, let τ be the Schwartz distribution determined by formula

$$\int_{\mathbb{R}} \frac{1}{1-zx}\, d\mathfrak{m}(x) = \exp\left[\int_{\mathbb{R}} \log\frac{1}{1-zx}\, d\tau(x)\right], \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.2}$$

The correspondence between $\mathfrak{m}$ and τ gives a bijection between the set of probability measures on $\mathbb{R}$ and the set of certain Schwartz distributions, and is called the *Markov–Krein correspondence*; see [12] for further details.

Here are three remarks on the Markov–Krein correspondence. Firstly, in many examples, τ is a signed measure, and in such a case it is called the Rayleigh measure of $\mathfrak{m}$. In general, τ is the Schwartz-distributional derivative of a so-called Rayleigh function.

Secondly, because we assumed that $\mathfrak{m}$ has finite moments of all orders, so is τ (see [1, Theorem A (d)] and [12, Section 3.4]). The relationship between the two kinds of moments $\mathrm{M}_k(\mathfrak{m})$ and $\mathrm{M}_k(\tau), k \in \mathbb{N}$, is exactly the one between the complete symmetric functions and Newton's power sums, i.e.,

$$\mathrm{M}_k(\tau) = k\,\mathrm{M}_k(\mathfrak{m}) - \sum_{r=1}^{k-1} \mathrm{M}_r(\tau)\,\mathrm{M}_{k-r}(\mathfrak{m}), \qquad k \in \mathbb{N}. \tag{1.3}$$

Thirdly, the Markov–Krein correspondence appears in different contexts to describe interlacing sequences: limit shapes of large random Young diagrams [13, 17, 3, 4]; roots of two consecutive orthogonal polynomials of large degrees [11]; eigenvalues of large random matrices and of their principal minor [11, 5] (the present paper deals with this category). There are also situations where the distribution τ above appears as a probability measure: Poisson–Dirichlet processes (see [12, Section 4.1] and references therein); self-decomposable distributions for monotone convolution [7]; Harish-Chandra–Izykson–Zuber integral of rank one at a high temperature regime [14]. The reason why the same correspondence appears in different contexts seems still unclear.

§ 2. The main result

The main result of [8] is as follows.

Theorem 2.1. *Let $\mathfrak{m}_N, \mathfrak{m}, \widehat{\tau}_N, \tau$ be as in Section 1. Then $\mathrm{M}_k(\widehat{\tau}_N)$ converges to $\mathrm{M}_k(\tau)$ in L^2 as $N \to \infty$ for every $k \in \mathbb{N}$. In particular, $\widehat{\tau}_N$ converges to τ in moments in probability.*

This result is known as a folklore theorem and is announced in [10] as a conjecture. Similar results are previously obtained in [11] for randomly rotated real Wigner matrices and then in [5] for Wigner and Wishart matrices (without random rotation).

The proof is based on a free probability technique and is sketched below; for further details see [8]. We first compute the expectation of $\mathrm{M}_k(\widehat{\tau}_N)$, which is given by

$$\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)] = \mathbb{E} \circ \mathrm{Tr}[D_N^k] - \mathbb{E} \circ \mathrm{Tr}[(D_N U_N P_N U_N{}^*)^k], \tag{2.1}$$

where $P_N = \mathrm{diag}(1, 1, \ldots, 1, 0)$. *Weingarten calculus* fits this problem and yields the asymptotic expansion of $\mathbb{E} \circ \mathrm{Tr}[(D_N U_N P_N U_N{}^*)^k]$ in variable N^{-1}. The first order term of the expansion is $\mathbb{E} \circ \mathrm{Tr}[D_N^k]$ that is of order $O(N)$ and exactly cancels the first term of (2.1). The second order term, which is the crucial part, turns out to be

$$-\sum_{\rho \in \mathrm{NC}(k)} (k+1-|\rho|)\, \mathrm{R}_\rho(\mathfrak{m}), \tag{2.2}$$

where $|\rho|$ is the number of blocks of ρ. To prove the main theorem, expression (2.2) should equal $-\mathrm{M}_k(\tau)$, and indeed it is. This equality can be easily proved by combining combinatorial formulas for complete symmetric functions. The higher order terms will vanish in the limit and are unnecessary for the proof. Considering the above, we obtain $\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)] = \mathrm{M}_k(\tau) + o(1)$ as $N \to \infty$. To complete the proof of L^2 convergence $\mathrm{M}_k(\widehat{\tau}_N) \to \mathrm{M}_k(\tau)$, we also need to compute and estimate $\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2]$. Although the computation is more involved, Weingarten calculus also works for this case.

As a final remark, the recent paper [9] investigated equation (2.1) from the viewpoint of noncommutative independence. As a result, the equality of (2.2) and $-\mathrm{M}_k(\tau)$ was generalized to a multivariate situation, see [9, Theorem 6.1].

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