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Space of initial conditions for the four-dimensional Garnier system revisited

By

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Abstract

A geometric study is given for the 4-dimensional Garnier system. By the resolution of indeterminacy, the group of its Bäklund transformations is lifted to a group of pseudo-isomorphisms between rational varieties obtained from $\mathbb{P}^2 \times \mathbb{P}^2$ by 10 or 21 blow-ups. The root basis is discussed in the Néron-Severi bilattices for the space with 10 blow-ups.

§ 1. Introduction

§ 1.1. Background and overview

The method of Okamoto-Sakai's spaces of initial conditions (SIC in short) is a powerful tool for studying the classification and symmetry of the Painlevé equations in two dimensions [12, 13]. However, studies of higher dimensional initial value spaces are scarce due to the theoretical and computational complexity.

The Garnier system is a natural generalization of the Painlevé equations to higher order and obtained as the monodromy preserving deformation for a Fuchsian ODE of the Schlesinger type:

$$\frac{d}{dx}\mathbf{y}(x) = \sum_{i=1}^{n+2} \frac{A_i}{x - z_i}\mathbf{y}(x), \tag{1.1}$$

where n is a positive integer, $\mathbf{y}$ is two-dimensional and the eigenvalues of A_i's and $A_\infty = -\sum_{i=1}^{n+2} A_i$ are different with each other. More concretely, the monodromy

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preserving deformation is given by so called the Schlesinger equation:

$$\frac{\partial A_i}{\partial z_j}=\frac{[A_j,A_i]}{z_j-z_i}\quad (j\neq i),\quad \frac{\partial A_i}{\partial z_i}=-\sum_{j\neq i}\frac{[A_j,A_i]}{z_j-z_i} \tag{1.2}$$

[Jimbo-Miwa-Môri-Sato 1980]. Since the singular points of the Fuchsian equation (z_1, $\dots, z_{n+2},\infty$) can be normalized to $(0,1,\infty,z_1',\dots,z_n')$ by a Möbius transformation for x, the Garnier system has n independent variables, or more precisely, they are n-compatible systems of ODEs. On the other hand the Garnier system has $2n$ dependent variables, since the dimension of the moduli space of ODEs of the form of (1.1) is $2n$. It is well known that the Garnier system has Painlevé property, i.e. movable singularities are at most poles [6].

In particular, when $n = 2$, the Garnier system is a commuting pair of four-dimensional systems of ordinary differential equations. In [11] H. Kimura and K. Okamoto showed that it can be written in a polynomial Hamiltonian form of two directions as

$$\frac{dq_i}{ds_j}=\frac{\partial H_j}{\partial p_i},\ \frac{dp_i}{ds_j}=-\frac{\partial H_j}{\partial q_i}\quad (i,j=1,2)$$

with the Hamiltonians

$$\begin{aligned}
s_1(s_1-1)H_1 =&\Big(q_1(q_1-1)(q_1-s_1)-\frac{s_1(s_1-1)}{s_1-s_2}q_1q_2\Big)p_1^2\\
&+2q_1q_2\Big(q_1+\frac{s_1(s_2-1)}{s_1-s_2}\Big)p_1p_2+q_1q_2\Big(q_2-\frac{s_2(s_1-1)}{s_1-s_2}\Big)p_2^2\\
&-\Big\{(\kappa_0-d)q_1(q_1-1)+\kappa_1q_1(q_1-s_1)+\theta_1(q_1-1)(q_1-s_1)\\
&\quad+\theta_2q_1\Big(q_1+\frac{s_1(s_2-1)}{s_1-s_2}\Big)-\theta_1\frac{s_1(s_1-1)}{s_1-s_2}q_2\Big\}p_1\\
&+\Big((2\alpha_0+\kappa_\infty)q_1q_2+\theta_2q_1\frac{s_2(s_1-1)}{s_1-s_2}-\theta_1q_2\frac{s_1(s_2-1)}{s_1-s_2}\Big)p_2\\
&+\alpha_0(\alpha_0+\kappa_\infty)q_1\\
H_2 =&\{\text{replacing as } q_1\leftrightarrow q_2,\ p_1\leftrightarrow p_2,\ s_1\leftrightarrow s_2,\ \theta_1\leftrightarrow\theta_2 \text{ in } H_1\},
\end{aligned}$$

where κ_0, κ_1, κ_∞, θ_1, θ_2 and α_0 are parameters independent from the Hamiltonian flows, and d is given by $d=2\alpha_0+\kappa_0+\kappa_1+\kappa_\infty+\theta_1+\theta_2$.

The Hamiltonians can be written using the Hamiltonian for the sixth Painlevé equation

$$\begin{aligned}
&s(s-1)H_{\mathrm{VI}}(q,p,s,\kappa_0,\kappa_1,\kappa_\infty,\theta,\alpha_0)\\
=&q(q-1)(q-s)p^2-\Big\{-(2\alpha_0+\kappa_1+\kappa_\infty+\theta)q(q-1)+\kappa_1q(q-s)\\
&+\theta_1(q-1)(q-s)\Big\}p+\alpha_0(\alpha_0+\kappa_\infty)q
\end{aligned}$$

as

$$
\begin{aligned}
s_1(s_1-1)H_1 =& s_1(s_1-1)H_{\mathrm{VI}}(q_1,p_1,s_1,\kappa_0,\kappa_1,\kappa_\infty,\theta_1,\alpha_0)\\
&+(2q_1p_1+q_2p_2+2\alpha_0+\kappa_\infty)q_1q_2p_2+\frac{s_1(s_1-1)}{s_1-s_2}(-q_1p_1+\theta_1)q_2p_1\\
&+\frac{s_1(s_2-1)}{s_1-s_2}(2q_1p_1-\theta_1)q_2p_2+\frac{s_2(s_1-1)}{s_1-s_2}\big(-q_2p_2^2+\theta_2(p_2-p_1)\big)q_1.
\end{aligned}
$$

In this paper we construct the space of initial conditions for this four-dimensional Garnier system, where a fiber space $\pi : X \to B$ is called the space of initial conditions for the ODE system φ if

(i) φ is regularly defined at any point in X;

(ii) For any point x in X, any path γ in B passing through $\pi(x)$ can be lifted to $\gamma' \subset X$ by φ, where $\pi : \gamma' \to \gamma$ is an isomorphism.

The most typical examples of the SIC are those for the Painlevé equations[12], while H. Kimura constructed the SIC for the Garnier system in n variables in [10].

In particular, for the sixth Painlevé equation (the Garnier system with $n = 1$), the SIC was constructed by blowing up a Hirzebrugh surface 8 times and excluding 5 irreducible curves (called vertical leaves), where B is $\{t \in \mathbb{C} \mid t \neq 0, 1\}$. For the Garnier system with $n = 2$, it was constructed by blowing up a 4-dimensional minimal projective variety 10 times, where B is $\{(t_1,t_2) \in \mathbb{C}^2 \mid t_i \neq 0,1\ (i=1,2),\ t_1 \neq t_2\}$. In the $n = 2$ case, M. Suzuki also constructed by gluing 13 affine spaces $\mathbb{C}^4$ using the Bäcklund transformations in [15] and Y. Sasano (reported as he) constructed the SIC by 13 times blowing up from $\mathbb{P}^4$ in [14]. Note that these systems have the Painlevé property, which guarantees that any path in B can be globally lifted.

Our approach to constructing the SIC is similar to Kimura's, but follows the way of H. Sakai constructing the SIC for discrete Painlevé systems [13][1]. Our approach differs from Kimura's at the following points:

(i) we blow up from $\mathbb{P}^2 \times \mathbb{P}^2$ instead of twisted four-dimensional variety;

(ii) we resolve the indeterminacy of Bäcklund transformations instead of the system of ODEs.

In the discrete setting, the SIC is a compact variety without excluding vertical leaves. Thus, not only does this approach provide a good computation prospect, but also allows us to easily see the homology and cohomology structure and to find the root lattice in this structure.

[1] A concrete expression of the SIC for the sixth Painlevé equation from $\mathbb{P}^1 \times \mathbb{P}^1$ can be found in [5] and [8], for example.

For a discrete dynamical system $\varphi_n : X_n \to X_{n+1}$, $n \in \mathbb{Z}$, the sequence of manifolds $\{X_n\}_{n\in\mathbb{Z}}$ is called a SIC for $\{\varphi_n\}_{n\in\mathbb{Z}}$ if $\varphi_n : X_n \to X_{n+1}$ is a pseudo-isomorphism for all $n \in \mathbb{Z}$ (see Section 1.3 below).[2]

§ 1.2. Bäcklund transformations

We use birational symmetries (Bäcklund transformations) of the Garnier sytem found by H. Kimura [9] and T. Tsuda [16]. Let us denote $(\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \theta_2, \alpha_0, s_1, s_2)$ by α and $w(f)$ by $\bar{f}$ for a birational action w. Bäcklund transformations change parameters as Table 1.

Table 1. Action on parameters

	$\bar{\kappa}_0$	$\bar{\kappa}_1$	$\bar{\kappa}_\infty$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\alpha}_0$	$\bar{s}_1$	$\bar{s}_2$
w_{κ_0}	$-\kappa_0$	κ_1	κ_∞	θ_1	θ_2	$\alpha_0 + \kappa_0$	s_1	s_2
w_{κ_1}	κ_0	$-\kappa_1$	κ_∞	θ_1	θ_2	$\alpha_0 + \kappa_1$	s_1	s_2
w_{κ_∞}	κ_0	κ_1	$-\kappa_\infty$	θ_1	θ_2	$\alpha_0 + \kappa_\infty$	s_1	s_2
w_{θ_1}	κ_0	κ_1	κ_∞	$-\theta_1$	θ_2	$\alpha_0 + \theta_1$	s_1	s_2
w_{θ_2}	κ_0	κ_1	κ_∞	θ_1	$-\theta_2$	$\alpha_0 + \theta_2$	s_1	s_2
w_{α_0}	$d - \kappa_0$	$d - \kappa_1$	$-\kappa_\infty$	$-\theta_1$	$-\theta_2$	$-\alpha_0$	s_1	s_2
σ_1	κ_1	κ_0	κ_∞	θ_1	θ_2	α_0	s_1^{-1}	s_2^{-1}
σ_2	κ_0	κ_∞	κ_1	θ_1	θ_2	α_0	$\frac{s_1}{s_1-1}$	$\frac{s_2}{s_2-1}$
σ_3	κ_0	κ_1	θ_1	κ_∞	θ_2	α_0	s_1^{-1}	$s_1^{-1}s_2$
σ_4	κ_0	κ_1	κ_∞	θ_2	θ_1	α_0	s_2	s_1

The actions on dependent variables are as Table 2, where

$$d = 2\alpha_0 + \kappa_0 + \kappa_1 + \kappa_\infty + \theta_1 + \theta_2$$

$$Q_{12} = q_1 + q_2 - 1 \tag{1.3}$$

$$Q^s_{12} = q_1/s_1 + q_2/s_2 - 1 \tag{1.4}$$

$$P_{12} = q_1p_1 + q_2p_2 + \alpha_0.$$

[2]Some of the contents of this paper have already been published in the proceedings for a special lecture of the infinite integrable systems session of the Mathematical Society of Japan in 2021, but this paper is the first publication of the proofs, details and accompanying results.

Table 2. Action on (q, p) variables

	$\bar{q}_1$	$\bar{q}_2$	$\bar{p}_1$	$\bar{p}_2$
w_{κ_0}	q_1	q_2	$p_1 - \frac{\kappa_0}{s_1 Q_{12}^s}$	$p_2 - \frac{\kappa_0}{s_2 Q_{12}^s}$
w_{κ_1}	q_1	q_2	$p_1 - \frac{\kappa_1}{Q_{12}}$	$p_2 - \frac{\kappa_1}{Q_{12}}$
w_{κ_∞}	q_1	q_2	p_1	p_2
w_{θ_1}	q_1	q_2	$p_1 - \theta_1/q_1$	p_2
w_{θ_2}	q_1	q_2	p_1	$p_2 - \theta_2/q_2$
w_{α_0}	$\frac{s_1 p_1(q_1 p_1 - \theta_1)}{P_{12}(P_{12}+\kappa_\infty)}$	$\frac{s_2 p_2(q_2 p_2 - \theta_2)}{P_{12}(P_{12}+\kappa_\infty)}$	$-q_1 p_1/\bar{q}_1$	$-q_2 p_2/\bar{q}_2$
σ_1	$s_1^{-1} q_1$	$s_2^{-1} q_2$	$s_1 p_1$	$s_2 p_2$
σ_2	$\frac{q_1}{Q_{12}}$	$\frac{q_2}{Q_{12}}$	$Q_{12}(p_1 - P_{12})$	$Q_{12}(p_2 - P_{12})$
σ_3	q_1^{-1}	$-q_1^{-1} q_2$	$-q_1 P_{12}$	$-q_1 p_2$
σ_4	q_2	q_1	p_2	p_1

Similar to the case of the sixth Painlevé equation, let us introduce new coordinates $(q_i, r_i) = (q_i, q_i p_i)$, then these actions are written more simply as Table 3, where

$$R_{12} = r_1 + r_2 + \alpha_0. \tag{1.5}$$

Table 3. Action on (q, r) variables

	$\bar{q}_1$	$\bar{q}_2$	$\bar{r}_1$	$\bar{r}_2$
w_{κ_0}	q_1	q_2	$r_1 - \frac{\kappa_0 q_1}{s_1 Q_{12}^s}$	$r_2 - \frac{\kappa_0 q_2}{s_2 Q_{12}^s}$
w_{κ_1}	q_1	q_2	$r_1 - \frac{\kappa_1 q_1}{Q_{12}}$	$r_2 - \frac{\kappa_1 q_2}{Q_{12}}$
w_{κ_∞}	q_1	q_2	r_1	r_2
w_{θ_1}	q_1	q_2	$r_1 - \theta_1$	r_2
w_{θ_2}	q_1	q_2	r_1	$r_2 - \theta_2$
w_{α_0}	$\frac{s_1 r_1(r_1 - \theta_1)}{q_1 R_{12}(R_{12}+\kappa_\infty)}$	$\frac{s_2 r_2(r_2 - \theta_2)}{q_2 R_{12}(R_{12}+\kappa_\infty)}$	$-r_1$	$-r_2$
σ_1	$s_1^{-1} q_1$	$s_2^{-1} q_2$	r_1	r_2
σ_2	$\frac{q_1}{Q_{12}}$	$\frac{q_2}{Q_{12}}$	$r_1 - q_1 R_{12}$	$r_2 - q_2 R_{12}$
σ_3	q_1^{-1}	$-q_1^{-1} q_2$	$-R_{12}$	r_2
σ_4	q_2	q_1	r_2	r_1

In the next section we will consider $\mathbb{P}^2 \times \mathbb{P}^2$ using this coordinate system.

§ 1.3. Basic facts

In this paper, we use the following basic facts about birational maps between higher dimensional varieties; see § 2 of [3] for details.

Pseudo-isomorphisms

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties. For a birational map $f : \mathcal{X} \to \mathcal{Y}$, let $I(f)$ denote the indeterminate set (i.e. the set of points where f is not defined) of f in $\mathcal{X}$.

We say a sequence of birational maps $\varphi_n : \mathcal{X}_n \to \mathcal{X}_{n+1}$ for smooth projective varieties $\mathcal{X}_n$ $(n \in \mathbb{Z})$ to be algebraically stable if

$$(\varphi_{n+k-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n)^* = \varphi_n^* \circ \varphi_{n+1}^* \circ \cdots \circ \varphi_{n+k-1}^*$$

holds as a mapping from the Picard group of $\mathcal{X}_{n+k}$ to that of $\mathcal{X}_n$ for any integers n and $k \geq 1$.

Proposition 1.1 ([2, 1]). *A sequence of birational maps $\varphi_n : \mathcal{X}_n \to \mathcal{X}_{n+1}$ for smooth projective varieties $\mathcal{X}_n$ $(n \in \mathbb{Z})$ is algebraically stable if and only if there do not exist integers n and $k \geq 1$ and a divisor D on $\mathcal{X}_{n-1}$ such that $\varphi(D \setminus I(\varphi_{n-1})) \subset I(\varphi_{n+k-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n)$.*[3]

We call a birational mapping $f : \mathcal{X} \to \mathcal{Y}$ a *pseudo-isomorphism* if f is isomorphic except on finite number of subvarieties of codimension two at least. This condition is equivalent to that there is no prime divisor pulled back to the zero divisor by f or f^{-1}. Hence, if φ_n is a pseudo-isomorphism for each n, then $\{\varphi_n\}_{n\in\mathbb{Z}}$ and $\{\varphi_n^{-1}\}_{n\in\mathbb{Z}}$ are algebraically stable.

Proposition 1.2 ([4]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties and φ a pseudo-isomorphism from $\mathcal{X}$ to $\mathcal{Y}$. Then φ acts on the Néron-Severi bi-lattice as an automorphism preserving the intersections.*

The Néron-Severi bi-lattice of a smooth rational variety $\mathcal{X}$ is isomorphic to $H^2(\mathcal{X},\mathbb{Z}) \times H_2(\mathcal{X},\mathbb{Z})$ which is explicitly given in the following.

Blow-ups

Recall that in local coordinates $U \subset \mathbb{C}^N$, the blow-up along a subvariety V of dimension $N-d$, $d \geq 2$, written as

$$x_1 - f_1(x_{d+1}, \dots, x_N) = \cdots = x_d - f_d(x_{d+1}, \dots, x_N) = 0,$$

where f_i's are holomorphic functions, is a birational morphism $\pi : X \to U$ such that $X = \{U_i\}$ is an open variety given by

$$U_i = \{(u_1^{(i)}, \dots, u_d^{(i)}, x_{d+1}, \dots, x_N) \in \mathbb{C}^N\} \quad (i = 1, \dots, d)$$

[3]This statement is a non-autonomous analog of a proposition shown in [2, 1]. The proof does not change except in notations.

with $\pi : U_i \to U$:

$$\begin{aligned}(x_1, \dots, x_N) =&(u_1^{(i)} u_i^{(i)} + f_1, \dots, u_{i-1}^{(i)} u_i^{(i)} + f_{i-1}, u_i^{(i)} + f_i,\\ &u_{i+1}^{(i)} u_i^{(i)} + f_{i+1}, \dots, u_d^{(i)} u_i^{(i)} + f_d, x_{d+1}, \dots, x_N).\end{aligned}$$

It is convenient to write the coordinates of U_i as

$$\left(\frac{x_1 - f_1}{x_i - f_i}, \dots, \frac{x_{i-1} - f_{i-1}}{x_i - f_i}, x_i - f_i, \frac{x_{i+1} - f_{i+1}}{x_i - f_i}, \dots, \frac{x_d - f_d}{x_i - f_i}, x_{d+1}, \dots, x_N\right).$$

The exceptional divisor E is written as $u_i = 0$ in U_i and each point in the center of the blow-up corresponds to a subvariety isomorphic to $\mathbb{P}^{d-1}$: $(x_1 - f_1 : \cdots : x_d - f_d)$. Thus E is locally a direct product $V \times \mathbb{P}^{d-1}$. We called such $\mathbb{P}^{d-1}$ a fiber of the exceptional divisor. (In algebraic setting the affine charts often need to be embedded into higher dimensional space.)

Case of $\mathbb{P}^2 \times \mathbb{P}^2$

Let $\mathcal{X}$ be a rational variety obtained by K successive blowups from $\mathbb{P}^2 \times \mathbb{P}^2$ and

$$(\mathbf{x}_1, \mathbf{x}_2) = (x_{10} : x_{11} : x_{12},\ x_{20} : x_{21} : x_{22})$$

the direct product of homogeneous coordinate chart. Let $\mathcal{H}_i$ denote the total transform of the class of a hyper-plane $\mathbf{c}_i \mathbf{x}_i = c_{i0}x_{i0} + c_{i1}x_{i1} + c_{i2}x_{i2} = 0$, where $\mathbf{c}_i = (c_{i0} : c_{i1} : c_{i2})$ is a constant vector in $\mathbb{P}^2$, and $\mathcal{E}_k$ the total transform of the k-th exceptional divisor class.

Let h_i denote the total transforms of the class of a line

$$\{\mathbf{x} \mid \mathbf{x}_j = \mathbf{c}_j (\forall j \neq i),\ \mathbf{x}_i = s\mathbf{a}_i + t\mathbf{b}_i (\exists (s : t) \in \mathbb{P}^1)\},$$

where $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_j$'s are constant vectors in $\mathbb{P}^2$, and e_k the class of a line in a fiber of the k-th blow-up. Then, the Picard group $\simeq H^2(\mathcal{X}, \mathbb{Z})$ and its Poincaré dual $\simeq H_2(\mathcal{X}, \mathbb{Z})$ are lattices

$$H^2(\mathcal{X}, \mathbb{Z}) = \bigoplus_{i=1}^{2} \mathbb{Z}\, \mathcal{H}_i \oplus \bigoplus_{k=1}^{K} \mathbb{Z}\, \mathcal{E}_k, \quad H_2(\mathcal{X}, \mathbb{Z}) = \bigoplus_{i=1}^{2} \mathbb{Z}\, h_i \oplus \bigoplus_{k=1}^{K} \mathbb{Z}\, e_k \tag{1.6}$$

and the intersection form is given by

$$\langle \mathcal{H}_i, h_j \rangle = \delta_{ij}, \quad \langle \mathcal{E}_k, e_l \rangle = -\delta_{kl}, \quad \langle \mathcal{H}_i, e_k \rangle = \langle \mathcal{E}_k, h_i \rangle = 0 \tag{1.7}$$

for $i, j = 1, 2$ and $1 \leq k, l \leq K$.

Moreover, the anti-canonical divisor class of $\mathcal{X}$ is given by

$$-\mathcal{K}_{\mathcal{X}} = 3\,\mathcal{H}_1 + 3\,\mathcal{H}_2 - \sum_{k=1}^{K} (3 - d_k)\, \mathcal{E}_k, \tag{1.8}$$

where d_k is the dimension of the center manifold for the k-th blow-up.

§ 2. Construction of the space of initial conditions

First of all, let us compactify the phase space $(q_1, q_2, p_1, p_2) \in \mathbb{C}^4$ to $(Q_0 : Q_1 : Q_2) \times (R_0 : R_1 : R_2) \in \mathbb{P}^2 \times \mathbb{P}^2$, where the original coordinates correspond to $Q_0 \neq 0$ and $R_0 \neq 0$ through $(q_1, q_2, r_1, r_2) = (Q_1/Q_0, Q_2/Q_0, R_1/R_0, R_2/R_0)$ as usual. Let $\mathcal{X}_\alpha$ be a rational projective variety obtained by successive 10 blow-ups from $\mathbb{P}^2 \times \mathbb{P}^2$ with the center of each blow-up C_i $(i = 1, \ldots, 10)$ given as follows.

$$
\begin{array}{ll}
C_1 : q_1 = r_1 = 0 & U_1 : (u_1, q_2, v_1, r_2) = (q_1, q_2, r_1/q_1, r_2) \\
C_2 : q_1 = r_1 - \theta_1 = 0 & U_2 : (u_2, q_2, v_2, r_2) = (q_1, q_2, (r_1 - \theta_1)/q_1, r_2) \\
C_3 : q_2 = r_2 = 0 & U_3 : (q_1, u_3, r_1, v_3) = (q_1, q_2, r_1, r_2/q_2) \\
C_4 : q_2 = r_2 - \theta_2 = 0 & U_4 : (q_1, u_4, r_1, v_4) = (q_1, q_2, r_1, (r_2 - \theta_2)/q_2) \\
C_5 : Q_0 = R_{12} = 0 & U_5 : (u_5, q_{2,1}, r_1, v_5) = (1/q_1, q_2/q_1, r_1, q_1 R_{12}) \\
C_6 : Q_0 = R_{12} + \kappa_\infty = 0 & U_6 : (u_6, q_{2,1}, r_1, v_6) = (1/q_1, q_2/q_1, r_1, q_1(R_{12} + \kappa_\infty)) \\
C_7 : R_0 = Q_{12} = A_{12} = 0 & U_7 : (q_1, u_7, v_7, w_7) = (q_1, 1/r_1, Q_{12} r_1, A_{12} r_1) \\
C_8 : u_7 = v_7 - \kappa_1 q_1 = 0 & U_8 : (q_1, u_8, v_8, w_7) = (q_1, 1/r_1, (v_7 - \kappa_1 q_1) r_1, w_7) \\
C_9 : R_0 = Q^s_{12} = A^s_{12} = 0 & U_9 : (q_1, u_9, v_9, w_9) = (q_1, 1/r_1, Q^s_{12} r_1, A^s_{12} r_1) \\
C_{10} : u_9 = v_9 - \kappa_0 q_1/s_1 = 0 & U_{10} : (q_1, u_{10}, v_{10}, w_9) = (q_1, 1/r_1, (v_9 - \kappa_0 q_1/s_1) r_1, w_9)
\end{array}
$$

where Q_{12}, Q^s_{12} are R_{12} are (1.3), (1.4) and (1.5) respectively, and

$$A_{12} = q_2/q_1 - r_2/r_1 \tag{2.1}$$

$$A^s_{12} = (s_1 q_2)/(s_2 q_1) - r_2/r_1. \tag{2.2}$$

Here, C_i is two-dimensional subvariety for $i \neq 7, 9$, while C_7 and C_9 are 1-dimensional.

The following proposition follows immediately from the basic facts in Section 1.3.

Proposition 2.1. *The Picard group ($\simeq H^2(\mathcal{X}_\alpha, \mathbb{Z})$) and its dual $\simeq H_2(\mathcal{X}_\alpha, \mathbb{Z})$ are 12-dimensional lattices*

$$H^2(\mathcal{X}_\alpha, \mathbb{Z}) = \mathbb{Z}\,\mathcal{H}_q \oplus \mathbb{Z}\,\mathcal{H}_r \oplus \bigoplus_{k=1}^{10} \mathbb{Z}\,\mathcal{E}_k, \quad H_2(\mathcal{X}_\alpha, \mathbb{Z}) = \mathbb{Z}\,h_q \oplus \mathbb{Z}\,h_r \oplus \bigoplus_{k=1}^{10} \mathbb{Z}\,e_k,$$

where $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{E}_k$ denote the total transforms of the classes for a hyper-plane $c_0 Q_0 + c_1 Q_1 + c_2 Q_2 = 0$, $c_0 R_0 + c_1 R_1 + c_2 R_2 = 0$ with $(c_0 : c_1 : c_2) \in \mathbb{P}^2$ and the k-th exceptional divisor, and h_q, h_p and e_k denote the total transforms of the classes for a generic line

$$\{(Q, R) = (\mathbf{c}, s\mathbf{a} + t\mathbf{b}) \in \mathbb{P}^2 \times \mathbb{P}^2 \ \mid (s : t) \in \mathbb{P}^1\},$$

$$\{(Q, R) = (s\mathbf{a} + t\mathbf{b}, \mathbf{c}) \in \mathbb{P}^2 \times \mathbb{P}^2 \ \mid (s : t) \in \mathbb{P}^1\},$$

and a generic fiber of the k-th blow-up. The intersection form is given by

$$\langle \mathcal{H}_i, h_j \rangle = \delta_{ij}, \quad \langle \mathcal{E}_k, e_l \rangle = -\delta_{kl}, \quad \langle \mathcal{H}_i, e_k \rangle = \langle \mathcal{E}_k, h_i \rangle = 0. \tag{2.3}$$

The anti-canonical divisor class is

$$-\mathcal{K}_{\mathcal{X}} = 3\,\mathcal{H}_q + 3\,\mathcal{H}_r - \sum_{k=1,2,3,4,5,6,8,10} \mathcal{E}_k - 2\,\mathcal{E}_7 - 2\,\mathcal{E}_9\,.$$

Let A denote the set of generic values of the parameter α. The following theorem holds.

Theorem 2.2. *The Bäcklund transformations $w = w_i$ for $i = \kappa_0, \kappa, \kappa_\infty, \theta_1, \theta_2$ and $w = \sigma_j$ for $j = 1, 2, 3, 4$ can be lifted to pseudo-isomorphisms from $\mathcal{X}_\alpha$ to $\mathcal{X}_{w(\alpha)}$, where w acts a bijection on A.*

Table 4 is the actions of the above w, where we omit the preserved elements and $\mathcal{E}_{i_1,i_2,\ldots,i_k}$ is the abbreviation for $\mathcal{E}_{i_1} + \mathcal{E}_{i_2} + \cdots + \mathcal{E}_{i_k}$.

Table 4. Action on the Picard group

w_{κ_0}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{9,10}$ $\mathcal{E}_9 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{10}, \quad \mathcal{E}_{10} \leftrightarrow \mathcal{H}_q - \mathcal{E}_9$
w_{κ_1}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{7,8}$ $\mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_8, \quad \mathcal{E}_8 \leftrightarrow \mathcal{H}_q - \mathcal{E}_7$
w_{κ_∞}	$\mathcal{E}_5 \leftrightarrow \mathcal{E}_6$
w_{θ_1}	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_2$
w_{θ_2}	$\mathcal{E}_3 \leftrightarrow \mathcal{E}_4$
σ_1	$\mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10}$
σ_2	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{5,7}, \quad \mathcal{E}_5 \leftrightarrow \mathcal{H}_q - \mathcal{E}_7$ $\mathcal{E}_6 \leftrightarrow \mathcal{E}_8, \quad \mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_5$
σ_3	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_5, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_6$
σ_4	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_3, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_4$

Furthermore, direct calculations also allow us to verify the following theorem.

Theorem 2.3. *Let $\mathcal{X}^o_\alpha$ be the open variety obtained by excluding the following* *proper transforms from $\mathcal{X}_\alpha$:*

$$\begin{array}{ll} Q_1 = 0: & \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_2 \\ Q_2 = 0: & \mathcal{H}_q - \mathcal{E}_3 - \mathcal{E}_4 \\ Q_0 = 0: & \mathcal{H}_q - \mathcal{E}_5 - \mathcal{E}_6 \\ R_0 = 0: & \mathcal{H}_r - \mathcal{E}_7 - \mathcal{E}_9 \\ R_0 = Q_{12} = A_{12} = 0: & \mathcal{E}_7 - \mathcal{E}_8 \\ R_0 = Q_{12}^s = A_{12}^s = 0: & \mathcal{E}_9 - \mathcal{E}_{10}, \end{array}$$

then the family $\{\mathcal{X}_\alpha^o\}_{\alpha \in A}$ is a SIC for the Garnier system.

However, we need further blow-ups for w_{α_0} which is lifted to a pseudo-isomorphism. Actually, w_{α_0} maps a hypersurface $Q_0 = 0$ to a subvariety $Q_1 = Q_2 = 0$ whose codimension is two, which means that we need blow-up along $Q_1 = Q_2 = 0$. Note that both $Q_0 = 0$ and $Q_1 = Q_2 = 0$ are included in $\mathcal{X}_\alpha \setminus \mathcal{X}_\alpha^o$, vertical leaves.

Let us blow-up $\mathbb{P}^2 \times \mathbb{P}^2$ along $C_{11}, \dots, C_{21}$, where

$$\begin{array}{l} C_{11}: Q_0 = Q_2 = 0 \quad U_{11}: (u_{11}, v_{11}, r_1, r_2) = (1/q_1, q_2, r_1, r_2) \\ C_{12}: Q_0 = Q_1 = 0 \quad U_{12}: (u_{12}, v_{12}, r_1, r_2) = (1/q_2, q_1, r_1, r_2) \\ C_{13}: q_1 = q_2 = 0 \quad U_{13}: (u_{13}, v_{13}, r_1, r_2) = (q_1, q_2/q_1, r_1, r_2) \\ C_{14}: q_1 = q_2 - 1 = R_0 = R_1 = 0 \\ \qquad U_{14}: (x_{14}, w_{14}, v_{14}, u_{14}) = (q_1 r_2, (q_2 - 1) r_2, r_1, 1/r_2) \\ C_{15}: q_1 = q_2/s_2 - 1 = R_0 = R_1 = 0 \\ \qquad U_{15}: (x_{15}, w_{15}, v_{15}, u_{15}) = (q_1 r_2, (q_2/s_2 - 1) r_2, r_1, 1/r_2) \\ C_{16}: q_2 = q_1 - 1 = R_0 = R_2 = 0 \\ \qquad U_{16}: (x_{16}, w_{16}, v_{16}, u_{16}) = (q_2 r_1, (q_1 - 1) r_1, r_2, 1/r_1) \\ C_{17}: q_2 = q_1/s_1 - 1 = R_0 = R_2 = 0 \\ \qquad U_{17}: (x_{17}, w_{17}, v_{17}, u_{17}) = (q_2 r_1, (q_1/s_1 - 1) r_1, r_2, 1/r_1) \\ C_{18}: Q_0 = Q_1 + Q_2 = R_0 = R_1 + R_2 = 0 \\ \qquad U_{18}: (x_{18}, w_{18}, v_{18}, u_{18}) = (r_1/q_1, (q_2/q_1 + 1) r_1, r_2 + r_1, 1/r_1) \\ C_{19}: Q_0 = Q_1/s_1 + Q_2/s_2 = R_0 = R_1 + R_2 = 0 \\ \qquad U_{19}: (x_{19}, w_{19}, v_{19}, u_{19}) = (r_1/q_1, (q_2/q_1 + s_2/s_1) r_1, r_2 + r_1, 1/r_1) \\ C_{20}: q_1 + s_1(s_2 - 1)/(s_1 - s_2) = q_2 + s_2(s_1 - 1)/(s_2 - s_1) = R_0 = 0 \\ \qquad U_{20}: (v_{20}, w_{20}, u_{20}, r_{2,1}) \\ \qquad = ((q_1 + s_1(s_2 - 1)/(s_1 - s_2)) r_1, (q_2 + s_2(s_1 - 1)/(s_2 - s_1)) r_1, 1/r_1, r_2/r_1) \end{array}$$

and $C_{21} = w_{\alpha_0}(C_{20})$.

Then, all the Bäcklund transformations in Table 3 are lifted to pseudo-isomorphisms as Table 5.

Table 5. Action on the Picard group with 21 blow-ups

w_{κ_0}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{9,10,15,17,19,20}$ $\mathcal{E}_9 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{10,15,17,19,20}, \quad \mathcal{E}_{10} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{9,15,17,19,20}$
w_{κ_1}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{7,8,14,16,18,20}$ $\mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{8,14,16,18,20}, \quad \mathcal{E}_8 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{7,14,16,18,20}$
w_{κ_∞}	$\mathcal{E}_5 \leftrightarrow \mathcal{E}_6$
w_{θ_1}	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_2$
w_{θ_2}	$\mathcal{E}_3 \leftrightarrow \mathcal{E}_4$
w_{α_0}	$\mathcal{H}_q \leftrightarrow 2\,\mathcal{H}_q + 2\,\mathcal{H}_r - \mathcal{E}_{1,2,3,4,5,6,11,12,13,14,15,16,17,18,19}$ $\mathcal{E}_1 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{1,14,15}, \quad \mathcal{E}_2 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{2,14,15}, \quad \mathcal{E}_3 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{3,16,17}$ $\mathcal{E}_4 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{4,16,17}, \quad \mathcal{E}_5 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{5,18,19}, \quad \mathcal{E}_6 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{6,18,19}$ $\mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10}$ $\mathcal{E}_{11} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{1,2,12,13,14,15}, \quad \mathcal{E}_{12} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{3,4,11,13,16,17}$ $\mathcal{E}_{13} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{5,6,11,12,18,19} \quad \mathcal{E}_{20} \leftrightarrow \mathcal{E}_{21}$
σ_1	$\mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10}$ $\mathcal{E}_{14} \leftrightarrow \mathcal{E}_{15}, \quad \mathcal{E}_{16} \leftrightarrow \mathcal{E}_{17}, \quad \mathcal{E}_{18} \leftrightarrow \mathcal{E}_{19}$
σ_2	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{5,7,14,16,18,19}, \quad \mathcal{E}_5 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{7,14,16,18,20}$ $\mathcal{E}_6 \leftrightarrow \mathcal{E}_8, \quad \mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{5,11,12,18,19} \quad \mathcal{E}_{11} \leftrightarrow \mathcal{E}_{16}, \quad \mathcal{E}_{12} \leftrightarrow \mathcal{E}_{14}, \quad \mathcal{E}_{19} \leftrightarrow \mathcal{E}_{20}$
σ_3	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_5, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_6$ $\mathcal{E}_{11} \leftrightarrow \mathcal{E}_{13}, \quad \mathcal{E}_{14} \leftrightarrow \mathcal{E}_{18}, \quad \mathcal{E}_{15} \leftrightarrow \mathcal{E}_{19}$
σ_4	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_3, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_4$ $\mathcal{E}_{11} \leftrightarrow \mathcal{E}_{12}, \quad \mathcal{E}_{14} \leftrightarrow \mathcal{E}_{16}, \quad \mathcal{E}_{15} \leftrightarrow \mathcal{E}_{17}$

§ 3. Root system

Since the space with 21 blow-ups is too complicated to see the structure, in this section we consider the action on the Néron-Severi bi-lattice for the case of 10 blow-ups.

Let $\mathcal{X}_\alpha$ be the space of initial conditions obtained by the first 10 blow-ups. Define the root vectors α_i $(i = 0, 1, \ldots, 5)$ and the co-root vectors as

$$\begin{aligned}
&\alpha_0 = \tfrac{1}{2}(\mathcal{H}_q + 2\,\mathcal{H}_r - 2\,\mathcal{E}_1 - 2\,\mathcal{E}_3 - 2\,\mathcal{E}_5), && \alpha_1 = \mathcal{H}_q - \mathcal{E}_9 - \mathcal{E}_{10}\\
&\alpha_2 = \mathcal{H}_q - \mathcal{E}_7 - \mathcal{E}_8, && \alpha_3 = \mathcal{E}_5 - \mathcal{E}_6\\
&\alpha_4 = \mathcal{E}_1 - \mathcal{E}_2, && \alpha_5 = \mathcal{E}_3 - \mathcal{E}_4
\end{aligned}$$

$$\begin{aligned}
&\check{\alpha}_0 = h_q - e_1 - e_3 - e_5, && \check{\alpha}_1 = h_r - e_9 - e_{10}\\
&\check{\alpha}_2 = h_r - e_7 - e_8, && \check{\alpha}_3 = e_5 - e_6\\
&\check{\alpha}_4 = e_1 - e_2, && \check{\alpha}_5 = e_3 - e_4
\end{aligned}$$

(see Figure 1).

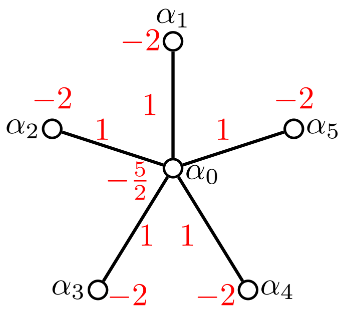

Figure 1. Dynkin diagram

Numbers beside edges denote the intersection number $\langle \alpha_i, \check{\alpha}_j \rangle = \langle \alpha_j, \check{\alpha}_i \rangle$, while numbers beside vertices denote the self-intersection number $\langle \alpha_i, \check{\alpha}_i \rangle$.

Then, w_{α_i}, $i = 1, 2, 3, 4, 5$, acts on the Néron-Severi bi-lattice as

$$w_{\alpha_i}(D) = D - 2\frac{\langle D, \check{\alpha}_i \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\alpha_i, \quad w_{\alpha_i}(d) = d - 2\frac{\langle \alpha_i, d \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\check{\alpha}_i$$

for $D \in H^2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$ and $d \in H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$.

Thus, w_{α_i}, $i = 1, 2, 3, 4, 5$, coincide with $w_{\kappa_0}, w_{\kappa_1}, w_{\kappa_\infty}, w_{\theta_1}, w_{\theta_2}$ respectively, while σ_j, $j = 1, 2, 3, 4$, act on the roots as transposition $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$ respectively.

Remark. The coroots are taken orthogonal to the vertical leaves, while there is not known automatic way to determine the roots, which are determined in a manner consistent with the action of the Bäcklund transformations on the bi-lattice.

According to this remark, there seems to be no reasonable way to determine α_0. Moreover, w_{α_0} was not a pseudo-isomorphism with 10 blow-ups. In fact, if we apply the above formula to w_{α_0} with $D = \mathcal{H}_q$, we obtain

$$\mathcal{H}_q \mapsto \mathcal{H}_q + \frac{2}{5}(\mathcal{H}_q + 2\mathcal{H}_r - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5),$$

which is not in $H^2(\mathcal{X}_\alpha, \mathbb{Z})$, and w_{α_0} can not be realized as a birational map.

Surprisingly, however, the following theorem holds and gives the reason why we take α_0 as above.

Theorem 3.1. *Kac's translation (§6.5 in [7]):*

$$T_{\alpha_i}(D) = D + \langle D, \check{\delta} \rangle \alpha_i + \langle D, \check{\delta} - \check{\alpha}_i \rangle \delta$$

with[4]

$$\delta = 2\alpha_0 + \sum_{i=1}^{5} \alpha_i = 3\,\mathcal{H}_q + 2\,\mathcal{H}_r - \mathcal{E}_{1,2,3,4,5,6,7,8,9,10} \tag{3.1}$$

$$\check{\delta} = 2\check{\alpha}_0 + \sum_{i=1}^{5} \check{\alpha}_i = 2h_q + 2h_r - e_{1,2,3,4,5,6,7,8,9,10} \tag{3.2}$$

can be realized as a pseudo-isomorphism for $i = 1, 2, 3, 4, 5$. *Moreover,* $T_{\alpha_0}^2$ *can be realized as a pseudo-isomorphism.*

Especially, T_{α_1} acts on $(\alpha_0, \alpha_1, \ldots, \alpha_5)$ as

$$T_{\alpha_1} : (\alpha_0, \alpha_1, \ldots, \alpha_5) \mapsto (\alpha_0, \alpha_1, \ldots, \alpha_5) + \delta(-1, 2, 0, 0, 0, 0)$$

and $T_{\alpha_0}^2$ acts on $(\alpha_0, \alpha_1, \ldots, \alpha_5)$ as

$$T_{\alpha_0}^2 : (\alpha_0, \alpha_1, \ldots, \alpha_5) \mapsto (\alpha_0, \alpha_1, \ldots, \alpha_5) + \delta(5, -2, -2, -2, -2, -2).$$

Proof. Using Table 5, it tuns out that the action of

$$T_1 = (w_{\theta_2} w_{\theta_1} w_{\kappa_2} w_{\kappa_0} w_{\alpha_0})^2$$

on the space with 21 blow-ups is trivial on the sub-lattice expanded by $\mathcal{E}_{11}, \ldots, \mathcal{E}_{21}$, and hence T_1 is a pseudo-isomorphism on the space with the first 10 blow-ups. Moreover, the action of T_1 on $\{\mathcal{X}_\alpha\}$ coincides with T_{α_1}. In other words, T_{α_1} is realized as T_1 as a birational map.

Obviously, we have

$$\begin{aligned} &T_{\alpha_1} = T_1, \quad T_{\alpha_2} = \sigma_1 T_1 \sigma_1, \quad T_{\alpha_3} = \sigma_2 \sigma_1 T_1 \sigma_1 \sigma_2, \\ &T_{\alpha_4} = \sigma_3 \sigma_2 \sigma_1 T_1 \sigma_1 \sigma_2 \sigma_3, \quad T_{\alpha_5} = \sigma_4 \sigma_3 \sigma_2 \sigma_1 T_1 \sigma_1 \sigma_2 \sigma_3 \sigma_4, \end{aligned}$$

and using these translations, we can realize $T_{\alpha_0}^2$ as

$$T_{\alpha_0}^2 = T_{-\alpha_1} T_{-\alpha_2} T_{-\alpha_3} T_{-\alpha_4} T_{-\alpha_5},$$

where $T_{-\alpha_i} = T_{\alpha_i}^{-1}$ for $i = 1, 2, 3, 4, 5$. □

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[4]The author does not know geometric interpretation for those null roots.

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