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Similarity and isospectral transformation of max-plus matrices

By

Yuki Nishida*, Sennosuke Watanabe**and Yoshihide Watanabe***

Abstract

Max-plus algebra is a semiring with addition "max" and multiplication "+". It has been used to describe ultradiscrete integrable systems. In this paper, we discuss transformations of max-plus matrices. We first define the similarity transformation and show that such transformation preserves the maximum eigenvalue of the matrix. Unlike the case of conventional linear algebra, the similarity transformation does not induce an equivalence relation on max-plus matrices. We develop the concept of unitary-pair semigroups so that the transformation becomes symmetric and transitive.

§ 1. Introduction

Max-plus algebra $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is a semiring with addition $\oplus$ and multiplication $\otimes$ defined by $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}_{\max}$, respectively. Max-plus algebra appears in many real-world problems, for example, steelworks [7], train timetable [14, 17], operation in emergency call center [2]. Hence, it is applied to many problems, such as combinatorial optimization [4] and discrete event systems [12, 21].

The algebraic geometry on max-plus algebra is called tropical geometry [19, 22]. The adjective "tropical" is in honor of the works of Imre Simon [32]. Tropical geometry can be derived from algebraic geometry over the fields with the valuation through

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Supported by JSPS KAKENHI No. 22K13964.
*Department of Science, Technology and Informatics, Kyoto Prefectural University, Kyoto 606-8522, Japan.
e-mail: y-nishida@kpu.ac.jp
**Department of Informatics, The University of Fukuchiyama, Fukuchiyama 620-0886, Japan.
e-mail: sewatana@fukuchiyama.ac.jp
***Applied Ultrasonic Research Center, Doshisha University, Kyotanabe 610-0394, Japan.
e-mail: yowatana@mail.doshisha.ac.jp

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"tropicalization", which is the limiting process in some sense. Such a limiting process is called ultradiscretization in the research field of integrable systems [34]. Applying this limiting process to the dependent variables of discrete integrable systems, we have obtained "ultradiscrete integrable systems" described by max-plus operations. For example, the box-ball system [33], which is a kind of cellular automata, can be obtained by such process from the Lotka-Volterra equation [34] or the Toda lattice equation [23]. An application of the max-plus eigenvalue problem to these equations is found in [30].

In this paper, we discuss the similarity transformation from A to B of the form $A \otimes P = P \otimes B$ and the relation to the eigenvalue problem. Here, arithmetics of max-plus matrices are defined as in the conventional algebra by replacing the addition and the multiplication with $\oplus$ and $\otimes$, respectively. We first prove that for such transformation from A to B, the maximum eigenvalue of A coincides with that of B under some regularity condition for P. However, other eigenvalues that are not maximum may be different. This fact is explained in terms of graph theory. Each max-plus square matrix is associated with a weighted digraph so that the weight of an edge (i, j) corresponds to the (i, j) entry of the matrix. Then, the maximum eigenvalue of a matrix is identical to the maximum average weight of circuits in the associated graph. Further, each eigenvalue comes from the average weight of some circuit, but it depends on the strong connectivity of the digraph. In particular, an irreducible matrix, whose associated graph is strongly connected, has exactly one eigenvalue. If we transpose the matrix, then the directions of edges in the associated graph are reversed and hence the strong connectivity of the graph will be changed. This causes that the left and right eigenvalues of a max-plus matrix are not the same in general.

Max-plus spectral theory originated in [8] and developed with studies on the periodicity of matrix powers [11, 29]. In the computational aspect, a method to find all eigenvalues and eigenvectors of reducible matrices was given in [5]. To cope with the problem that a max-plus matrix has a very few eigenvalues and eigenvectors, the authors recently introduced algebraic eigenvectors with respect to the roots of the characteristic polynomial [25]. These vectors are shown to have analogous properties to the conventional ones [26]. It is known that some discrete integrable systems are related to numerical algorithms for computing eigenvalues of usual matrices by interpreting the recurrence equations as similarity transformations [31]. This is extended to the relation between ultradiscrete integrable systems and eigenvalue problems of max-plus matrices. For example, an isospectral transformation of a max-plus tridiagonal matrix is given by the ultradiscrete Toda equation [35], that of a symmetric tridiagonal matrix is by the ultradiscrete Lotka-Volterra equation [15], and that of a lower Hessenberg banded matrix is by the ultradiscrete hungry Toda equation [**?**].

In the definition of the similarity transformation $A \otimes P = P \otimes B$, an important issue is which kind of regularity we should impose on the matrix P. If we follow the similarities in conventional linear algebra, the invertible matrix P may be a possible choice. However, in max-plus algebra, it is not appropriate since most square matrices do not have their inverses. Some kinds of the regularity of max-plus matrices are proposed in the literature such as [1, 13, 18]. We demonstrate the difference and the intensities of these regularities by presenting examples. Also, we discuss the similarity transformations defined by the matrix P with such kinds of regularities. In any case, we must note that the relation derived from the similarity transformation $A \otimes P = P \otimes B$ does not define the equivalence relation since it is not symmetric. To obtain the equivalence relation we consider two equations $A \otimes P = P \otimes B$ and $B \otimes Q = Q \otimes A$ at the same time. The matrix Q is expected to be close to the inverse of P so that we impose the condition that all diagonal of $P \otimes Q$ and $Q \otimes P$ are 0, and the determinants of both of them are 0 as well. Moreover, in order to have the transitive law, we introduce the notion of a unitary-pair semigroup consisting of pairs of matrices imitating the pair of P and its inverse. Equivalence relation induced by unitary-pair semigroup includes the similarity transformation by invertible matrices and the transformation into a block diagonal form in spectral theory.

§ 2. Preliminaries on max-plus algebra

Max-plus algebra is the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ with two operations $\oplus$ and $\otimes$, defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b$$

for $a, b \in \mathbb{R}_{\max}$. By regarding $\oplus$ and $\otimes$ as addition and multiplication, respectively, max-plus algebra is a semiring. Here, $\varepsilon := -\infty$ is the identity element for addition, and $e := 0$ is the identity element for multiplication.

Let $\mathbb{R}^n_{\max}$ and $\mathbb{R}^{m\times n}_{\max}$ be the set of n-dimensional max-plus column vectors and the set of $m \times n$ max-plus matrices, respectively. The operations $\oplus$ and $\otimes$ are extended to max-plus vectors and matrices as in conventional linear algebra. For $A, B \in \mathbb{R}^{m\times n}_{\max}$, the matrix sum $A \oplus B \in \mathbb{R}^{m\times n}_{\max}$ is defined by

$$[A \oplus B]_{ij} = [A]_{ij} \oplus [B]_{ij},$$

where $[A]_{ij}$ indicates the (i, j) entry of A. For $A \in \mathbb{R}^{\ell\times m}_{\max}$ and $B \in \mathbb{R}^{m\times n}_{\max}$, the matrix product $A \otimes B \in \mathbb{R}^{\ell\times n}_{\max}$ is defined by

$$[A \otimes B]_{ij} = \bigoplus_{k=1}^{m} [A]_{ik} \otimes [B]_{kj}.$$

For $A \in \mathbb{R}_{\max}^{m\times n}$ and $c \in \mathbb{R}_{\max}$, the scalar multiplication of A by c is defined by

$$[c \otimes A]_{ij} = c \otimes [A]_{ij}.$$

The n-dimensional max-plus zero vector and the $m \times n$ zero matrix are denoted by $\mathcal{E}_n$ and $\mathcal{E}_{m,n}$, respectively, and the max-plus unit matrix of order n is denoted by I_n.

§ 2.1. Regularity of max-plus square matrices

Here, we summarize some kinds of "regularity" of max-plus square matrices. Many characteristics of regular matrices in conventional linear algebra are not equivalent in max-plus algebra. Several kinds of non-equivalent rank function on max-plus matrices are presented and compared in [1]. Some regularity of max-plus matrices can be defined by these rank functions.

A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called *regular* if it contains finite (i.e., non-ε) entry in each row and each column [18]. This property is also referred to as *doubly* $\mathbb{R}$-*astic* in [9]. This simple definition is sometimes used in the eigenvalue problem. The class of regular matrices seems to be too large. For example, all finite matrices become regular.

A concept that is analogous to regularity in conventional linear algebra is stated in terms of the independence of column (or row) vectors [10]. Let $S \subset \mathbb{R}_{\max}^{n}$ be a finite set of vectors. It is called *dependent* if there exists a vector $\boldsymbol{x} \in S$ that can be expressed as a linear combination of others, that is,

$$\boldsymbol{x} = \bigoplus_{\boldsymbol{u} \in S\setminus\{\boldsymbol{x}\}} c_{\boldsymbol{u}} \otimes \boldsymbol{u}, \qquad c_{\boldsymbol{u}} \in \mathbb{R}_{\max}.$$

If S is not dependent, then it is called *independent*. A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called *doubly full-rank* if both row vectors and column vectors are independent.

Another definition of regularity comes from the determinant of a matrix [13]. For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we define the determinant of A by

$$\det A := \bigoplus_{\pi \in S_n} \bigotimes_{i=1}^{n} a_{i\pi(i)},$$

where S_n denotes the symmetric group of order n. The right-hand side of the above equation is the maximum over $n!$ permutations. The matrix A is called *singular* if the maximum is attained by at least two permutations; otherwise A is called *non-singular*.

Lastly, a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called *invertible* if there exists $B \in \mathbb{R}_{\max}^{n\times n}$ such that $A \otimes B = B \otimes A = I_n$. The matrix B is called the *inverse* of A and denoted by $A^{\otimes(-1)}$. In max-plus algebra, $A \in \mathbb{R}_{\max}^{n\times n}$ is invertible if and only if A is a *generalized permutation matrix* [9], that is, there exists a permutation $\sigma \in S_n$ such that $[A]_{ij} \neq \varepsilon$ if and only if $j = \sigma(i)$. In this case, $A^{\otimes(-1)}$ is also a generalized inverse matrix with $[A^{\otimes(-1)}]_{ji} = -[A]_{ij}$ for $j = \sigma(i)$.

For these four kinds of regularity, we have the following relationship:

$$\text{regular} \quad \overset{(1)}{\Longleftarrow} \quad \text{doubly full-rank} \quad \overset{(2)}{\Longleftarrow} \quad \text{non-singular} \quad \overset{(3)}{\Longleftarrow} \quad \text{invertible.}$$

The proofs of (1) and (3) are very simple. Indeed, if the ith column vector $\boldsymbol{a}_i$ of A is the zero vector $\mathcal{E}_n$, then it can be expressed as a trivial linear combination of other columns:

$$\boldsymbol{a}_i = \bigoplus_{j \neq i} \varepsilon \otimes \boldsymbol{a}_j.$$

A similar result holds for rows. Hence, if A is not regular, then it is not doubly full-rank, which proves (1). For an invertible matrix $A \in \mathbb{R}_{\max}^{n \times n}$, take a permutation $\sigma \in S_n$ such that $[A]_{ij} \neq \varepsilon$ if and only if $j = \sigma(i)$. Then, we see that $\bigotimes_{i=1}^{n} a_{i\pi(i)} \neq \varepsilon$ if and only if $\pi = \sigma$. Thus, the maximum in the definition of the determinant is attained only by σ. This implies A is non-singular, proving (3). The proof of (2) is rather nontrivial, see [28].

Example 2.1. We consider four matrices

$$A_1 = \begin{pmatrix} 1\,3\,3 \\ 2\,1\,2 \\ 0\,\varepsilon\,0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0\,0\,\varepsilon \\ \varepsilon\,0\,0 \\ 0\,\varepsilon\,0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \varepsilon & \varepsilon\,2 \\ -3\,\varepsilon\,\varepsilon \\ \varepsilon & 1\,\varepsilon \end{pmatrix}.$$

All of them are regular because no row or column is the zero vector. The matrix A_1 is not doubly full-rank because the third column is the sum of the other two:

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 1 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

The matrix A_2 is doubly full-rank but singular because the maximum in the determinant is attained by two terms:

$$\begin{aligned} \det A_2 &= 0 \otimes 0 \otimes 0 \oplus 0 \otimes 0 \otimes 0 \oplus \varepsilon \otimes \varepsilon \otimes \varepsilon \oplus 0 \otimes 0 \otimes \varepsilon \oplus 0 \otimes \varepsilon \otimes 0 \oplus \varepsilon \otimes 0 \otimes 0 \\ &= 0 \oplus 0 \oplus \varepsilon \oplus \varepsilon \oplus \varepsilon \oplus \varepsilon. \end{aligned}$$

The matrix A_3 is non-singular but not invertible. Indeed, suppose $B = (b_{ij}) \in \mathbb{R}_{\max}^{3 \times 3}$ is an inverse of A. Then, from the $(1,2)$ entry of $A \otimes B = I_3$, we have

$$1 \otimes b_{12} \oplus 0 \otimes b_{22} \oplus 0 \otimes b_{32} = \varepsilon,$$

which implies $b_{12} = b_{22} = b_{32} = \varepsilon$. Then, the $(2,2)$ entry of $A \otimes B$ is computed as

$$0 \otimes b_{12} \oplus 1 \otimes b_{22} \oplus 0 \otimes b_{32} = \varepsilon,$$

which contradicts the fact that $[A \otimes B]_{22} = [I_3]_{22} = 0$. Hence, A_3 must not have an inverse matrix. The matrix A_4 is an example of a generalized permutation matrix, which is invertible. The inverse of A_4 is

$$A_4^{\otimes(-1)} = \begin{pmatrix} \varepsilon & 3 & \varepsilon \\ \varepsilon & \varepsilon & -1 \\ -2 & \varepsilon & \varepsilon \end{pmatrix}.$$

§ 2.2. Max-plus matrices and graphs

For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$, we define a weighted digraph $\mathcal{G}(A) := (V, E, w)$ associated with A as follows. The sets of the vertices and edges are $V = \{1, 2, \ldots, n\}$ and $E = \{(i, j) \mid a_{ij} \neq \varepsilon\}$, respectively, and the weight function $w : E \to \mathbb{R}$ is defined by $w((i, j)) = a_{ij}$ for $(i, j) \in E$. A sequence of vertices $\mathcal{P} = (i_0, i_1, \ldots, i_\ell)$ is called a *path* if $(i_k, i_{k+1}) \in E$ for $k = 0, 1, \ldots, \ell - 1$. It is called an i_0-i_ℓ *path* if its start and end should be specified. The sets of the vertices and edges in $\mathcal{P}$ are denoted by $V(\mathcal{P})$ and $E(\mathcal{P})$, respectively. The number $\ell(\mathcal{P}) := \ell$ is called the *length* of $\mathcal{P}$. The sum $w(\mathcal{P}) := \sum_{k=0}^{\ell-1} w((i_k, i_{k+1}))$ is called the *weight* of $\mathcal{P}$. A path $\mathcal{C} = (i_0, i_1, \ldots, i_\ell)$ with $i_\ell = i_0$ is called a *circuit*. In particular, if $i_k \neq i_{k'}$ for $1 \leq k < k' \leq \ell$, then $\mathcal{C}$ is called an *elementary circuit*. The length and weight of a circuit are defined similarly to a path. The *average weight* of a circuit $\mathcal{C}$ is defined by $w(\mathcal{C})/\ell(\mathcal{C})$.

For $A \in \mathbb{R}_{\max}^{n \times n}$ and a positive integer k, let $A^{\otimes k}$ denote the product $\overbrace{A \otimes A \otimes \cdots \otimes A}^{k \text{ times}}$. The (i, j) entry of $A^{\otimes k}$ is identical to the maximum weight of all i-j paths with length k in $\mathcal{G}(A)$. We consider the formal matrix power series of the form

$$A^* := I_n \oplus A \oplus A^{\otimes 2} \oplus \cdots .$$

The matrix A^* is called the *Kleene star* of A. If there is no circuit with positive weight in $\mathcal{G}(A)$, then A^* is computed as the finite sum

$$A^* = I_n \oplus A \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes n-1}.$$

In this case, the (i, j) entry of A^* is the maximum weight of all i-j paths [16].

§ 2.3. Eigenvalues and eigenvectors

For a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, a scalar λ is called a *(right) eigenvalue* of A if there exists a vector $\boldsymbol{x} \neq \mathcal{E}_n$ satisfying

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}.$$

This vector $\boldsymbol{x}$ is called a *(right) eigenvector* of A with respect to λ. Here, we summarize the results in the literature on the max-plus eigenvalue problem, e.g., [3, 6, 18].

Proposition 2.2. *For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, the maximum average weight of all elementary circuits in $\mathcal{G}(A)$ is the maximum eigenvalue of A.*

Let $\lambda(A)$ be the maximum eigenvalue of A. A circuit in $\mathcal{G}(A)$ with average weight $\lambda(A)$ is called *critical*. Vertices and edges of a critical circuit are called *critical vertices* and *critical edges*, respectively. The set of all critical vertices and edges are denoted by $V^c(A)$ and $E^c(A)$. The subgraph $\mathcal{G}^c(A) = (V^c(A), E^c(A))$ of $\mathcal{G}(A)$ is called the *critical graph*.

Proposition 2.3. *The kth column of $((-\lambda(A)) \otimes A)^*$ is an eigenvector of A with respect to $\lambda(A)$ if and only if $k \in V^c(A)$.*

Example 2.4. We consider a matrix

$$A = \begin{pmatrix} \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & 3 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & 1 & 4 \\ \varepsilon & \varepsilon & 5 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 \end{pmatrix}.$$

The associated graph $\mathcal{G}(A)$ is shown in Figure 1. The elementary circuits in $\mathcal{G}(A)$ are

$$(1,2,1),\ (1,2,3,1),\ (3,4,3),\ (4,4) \text{ and } (5,5),$$

whose average weights are

$$\frac{2+0}{2} = 1,\ \frac{2+3+1}{3} = 2,\ \frac{1+5}{2} = 3,\ \frac{1}{1} = 1 \text{ and } \frac{4}{1} = 4,$$

respectively. Thus, the maximum eigenvalue of A is 4. To find an eigenvector, we compute the Kleene star of

$$B := (-4) \otimes A = \begin{pmatrix} \varepsilon & -2 & \varepsilon & \varepsilon & \varepsilon \\ -4 & \varepsilon & -1 & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & -3 & 0 \\ \varepsilon & \varepsilon & 1 & -3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

We note that the maximum (average) weight of circuits in $\mathcal{G}(B)$ becomes 0 since the weights of all edges are decreased by 4. We see that

$$B^* = I_5 \oplus B \oplus B^{\otimes 2} \oplus B^{\otimes 3} \oplus B^{\otimes 4} = \begin{pmatrix} 0 & -2 & -3 & -6 & -3 \\ -4 & 0 & -1 & -4 & -1 \\ -3 & -5 & 0 & -3 & 0 \\ -2 & -4 & 1 & 0 & 1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}$$

Since the critical circuit of $\mathcal{G}(B)$ is $(5,5)$, the last column of B^* is an eigenvector of A with respect to the eigenvalue 4.

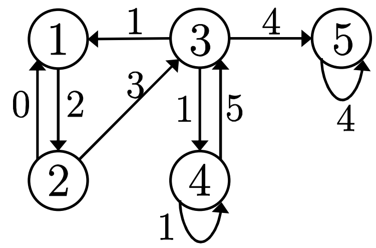

Figure 1. Associated graph $\mathcal{G}(A)$ for Example 2.4.

To find all eigenvalues of a matrix, we focus on the strong connectivity of the associated graph. A directed graph $\mathcal{G}$ is called *strongly connected* if there exists an i-j path for any vertex i, j of $\mathcal{G}$. A matrix A is called *irreducible* if the associated graph $\mathcal{G}(A)$ is strongly connected; otherwise, it is called *reducible*. If A is reducible, then $\mathcal{G}(A)$ is decomposed into strongly connected components $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r$. By definition, any elementary circuit of $\mathcal{G}(A)$ is contained in exactly one of these components. For each component $\mathcal{G}_i$, let $\lambda(\mathcal{G}_i)$ be the maximum average weight of circuits in $\mathcal{G}_i$. If $\mathcal{G}_i$ has no circuit, we set $\lambda(\mathcal{G}_i) = \varepsilon$. A component $\mathcal{G}_i$ is called *spectral* if it is not reachable from any other components $\mathcal{G}_j$ such that $\lambda(\mathcal{G}_i) < \lambda(\mathcal{G}_j)$, that is, there is no path from a vertex of $\mathcal{G}_j$ to that of $\mathcal{G}_i$.

Proposition 2.5 ([5]). *For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r$ be strongly connected components of $\mathcal{G}(A)$. Then, the set of all eigenvalues of a matrix A is*

$$\{\lambda(\mathcal{G}_i) \mid \mathcal{G}_i \text{ is spectral}\}.$$

In particular, if A is irreducible, then A has exactly one eigenvalue $\lambda(A)$.

Example 2.6. We again consider the matrix A in Example 2.4. The associated graph $\mathcal{G}(A)$ is reducible and it is decomposed into strongly connected components $\mathcal{G}_1$ and $\mathcal{G}_2$ with vertices $\{1,2,3,4\}$ and $\{5\}$, respectively. The maximum average weight for these components are $\lambda(\mathcal{G}_1) = 3$ and $\lambda(\mathcal{G}_2) = 4$. Since there is no path from vertex 5 to the others, both $\mathcal{G}_1$ and $\mathcal{G}_2$ are spectral. Hence, the eigenvalues of A are 3 and 4. For the eigenvalue 3, we compute an eigenvector. As in the previous example, we consider a matrix $C = (-3) \otimes A$. Then, we have

$$C^* = \begin{pmatrix} 0 & -1 & -1 & -3 & \infty \\ -2 & 0 & 0 & -2 & \infty \\ -2 & -3 & 0 & -2 & \infty \\ 0 & -1 & 2 & 0 & \infty \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \infty \end{pmatrix}.$$

We remark that some entries of C^* do not converge and tend to infinity because $\mathcal{G}(C)$ has the circuit $(5, 5)$ with positive weight. Nevertheless, the columns corresponding to the critical circuit $(3, 4, 3)$ of $\mathcal{G}_1$ are still in $\mathbb{R}^5_{\max}$. Hence, the third and fourth columns of C^* are eigenvectors of A with respect to the eigenvalue 3. Note that these vectors are essentially the same, that is,

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = 2 \otimes \begin{pmatrix} -3 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

So far, we have dealt with right eigenvalues and right eigenvectors. Left eigenvalues and left eigenvectors are similarly defined by the equation for a row vector $\boldsymbol{x}$:

$$\boldsymbol{x} \otimes A = \lambda \otimes \boldsymbol{x}.$$

We can see that left eigenvalues and eigenvectors of A are right eigenvalues and eigenvectors of $A^\top$. The associated graph $\mathcal{G}(A^\top)$ is comprised of all edges of $\mathcal{G}(A)$, but the directions of edges are reversed. Hence, each circuit of $\mathcal{G}(A^\top)$ corresponds one-to-one to that of $\mathcal{G}(A^\top)$. In particular, the maximum average weights of circuits of $\mathcal{G}(A)$ and $\mathcal{G}(A^\top)$ are the same. This means that the maximum left eigenvalue is equal to the maximum right one. Thus, the maximum eigenvalue need not be distinguished by the adjective “left” or “right”. On the other hand, surprisingly, not all left eigenvalues of A become the right ones. For example, let us consider the matrix A in Example 2.4. The decomposition of $\mathcal{G}(A^\top)$ into strongly connected components induces the same partition of vertices as $\mathcal{G}(A)$. However, we have an edge $(5, 3)$ in $\mathcal{G}(A^\top)$. This means that the component $\mathcal{G}_1$ with vertex set $\{1, 2, 3, 4\}$ is not spectral any longer. Thus, $\lambda(\mathcal{G}_1) = 3$ is not an left eigenvalue of A.

§ 3. Similarity transformation of max-plus matrices

In this section, we discuss the similarity transformation of max-plus matrices. As described in the previous section, only generalized permutation matrices have their inverses. Hence, to define the similarity transformation from A to B by $P^{\otimes(-1)} \otimes A \otimes P = B$ seems to be inappropriate. Instead, we consider the transformation induced by the following equation:

$$A \otimes P = P \otimes B.$$

We first show that the maximum eigenvalue is preserved by this transformation.

Theorem 3.1. *For $A, B \in \mathbb{R}_{\max}^{n \times n}$, suppose there exists a regular matrix $P \in \mathbb{R}_{\max}^{n \times n}$ such that*

$$A \otimes P = P \otimes B.$$

Then, any right eigenvalue of B is that of A, and any left eigenvalue of A is that of B. In particular, the maximum eigenvalue of A and B coincide.

Proof. Let λ be a right eigenvalue of B and $\boldsymbol{x}$ be a right eigenvector with respect to λ. Then, we have

$$A \otimes (P \otimes \boldsymbol{x}) = P \otimes B \otimes \boldsymbol{x} = \lambda \otimes (P \otimes \boldsymbol{x}).$$

Since $\boldsymbol{x}$ is a right eigenvector, $[\boldsymbol{x}]_j \neq \varepsilon$ for some index j. Further, if P is regular, $[P]_{ij} \neq \varepsilon$ for some row i because the jth column is not the zero vector. Hence, the ith entry of $P \otimes \boldsymbol{x}$ is not ε, which implies that $P \otimes \boldsymbol{x} \neq \mathcal{E}_n$. Thus, $P \otimes \boldsymbol{x}$ is a right eigenvector of A with respect to the right eigenvalue λ.

The assertion for a left eigenvalue is similarly proved. Since the maximum eigenvalue is both a left and right eigenvalue, the maximum eigenvalue of B is that of A. $\square$

Example 3.2. Let us consider two matrices

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ 0 & 3 & 3 \\ \varepsilon & \varepsilon & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ 3 & 4 & 3 \\ 2 & 2 & 3 \end{pmatrix}.$$

Taking a regular matrix

$$P = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

we have

$$A \otimes P = P \otimes B = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ 2 & 3 & 3 \\ 3 & 4 & 3 \end{pmatrix}.$$

We see that the maximum eigenvalues of both A and B are 4. We can easily check that $\boldsymbol{x} = (0, \varepsilon, \varepsilon)$ is a left eigenvector of A with respect to the left eigenvalue 1. Moreover, we can also check that $\boldsymbol{x} \otimes P = (1, \varepsilon, \varepsilon)$ is a left eigenvector of B with respect to the left eigenvalue 1. On the other hand, although A has a right eigenvalue 3 and a right eigenvector $(\varepsilon, 0, \varepsilon)^\top$ with respect to it, it is not a right eigenvalue of B because $\mathcal{G}(B)$ has only two strongly connected components $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\lambda(\mathcal{G}_1) = 1$ and $\lambda(\mathcal{G}_2) = 4$.

We may consider other kinds of "regularity" for the transformation matrix P in Theorem 3.1. If P is invertible and $A \otimes P = P \otimes B$, then we have $B \otimes P^{\otimes(-1)} = P^{\otimes(-1)} \otimes A$. Hence, we have the following result.

Corollary 3.3. *For $A, B \in \mathbb{R}_{\max}^{n \times n}$, suppose there exists an invertible matrix $P \in \mathbb{R}_{\max}^{n \times n}$ such that*

$$A \otimes P = P \otimes B.$$

Then, the sets of right (left) eigenvalues of A and B coincide.

If P is doubly full-rank or non-singular, the assertion of Theorem 3.1 also holds because P is regular in either case. However, the transformation defined by doubly full-rank or non-singular matrices is not transitive. As demonstrated in the following example, the product of doubly full-rank (non-singular) matrices is not always doubly full-rank (non-singular).

Example 3.4. Let us consider two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easily checked that both A and B are doubly full-rank and non-singular. On the other hand, we obtain

$$A \otimes B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

which is neither doubly full-rank nor non-singular.

As we have seen, the relation $A \otimes P = P \otimes B$ seems appropriate to define the similarity transformation from A to B. However, this is not symmetric, that is, the existence of a matrix Q such that $B \otimes Q = Q \otimes A$ is not guaranteed. To make the term "transformation" more appropriate, we introduce a framework that gives an equivalence relation for max-plus matrices. The basic idea is to define that A and B are equivalent if $A \otimes P = P \otimes B$ and $B \otimes Q = Q \otimes A$ for some matrices P and Q. In the conventional algebra, Q can be taken as P^{-1}. Hence, in max-plus algebra, it is expected that $P \otimes Q$, as well as $Q \otimes P$, is close to the identity matrix. From this observation, we propose the following definition.

Definition 3.5. A subset $\mathfrak{U} \subset \mathbb{R}_{\max}^{n\times n} \times \mathbb{R}_{\max}^{n\times n}$ is called a *unitary-pair semigroup* if it has the properties 1–4.

1 $(I_n, I_n) \in \mathfrak{U}$.

2 If $(P, Q) \in \mathfrak{U}$, then $(Q, P) \in \mathfrak{U}$.

3 If $(P_1, Q_1) \in \mathfrak{U}$ and $(P_2, Q_2) \in \mathfrak{U}$, then $(P_1 \otimes P_2, Q_2 \otimes Q_1) \in \mathfrak{U}$.

4 If $(P, Q) \in \mathfrak{U}$, then all diagonal entries of $P \otimes Q$ and $Q \otimes P$ are 0. In addition, both $\det(P \otimes Q)$ and $\det(Q \otimes P)$ are attained only by the identity permutation.

Property 2 corresponds to the equality $(P^{-1})^{-1} = P$ in the conventional linear algebra. Property 3 is needed to ensure the transitivity of the transformation. Property 4 means that the diagonal entries of $P \otimes Q$ and I_n are identical, and the other entries are so small that they can be ignored. This implies that Q is like an inverse of P. We remark that property 4 ensures that both P and Q are regular. Indeed, if the ith column of P is $\mathcal{E}_n$, then the ith column of $Q \otimes P$ is also $\mathcal{E}_n$. Other cases can be shown similarly. Moreover, we can see that both P and Q are non-singular using the fact that $\det(P \otimes Q) = \det P \otimes \det Q$ if $P \otimes Q$ is non-singular [27].

Definition 3.6. Let $\mathfrak{U} \subset \mathbb{R}_{\max}^{n\times n} \times \mathbb{R}_{\max}^{n\times n}$ be a unitary-pair semigroup. Two matrices $A, B \in \mathbb{R}_{\max}^{n\times n}$ are called $\mathfrak{U}$-*equivalent*, denoted by $A \sim_{\mathfrak{U}} B$, if there exists a pair $(P, Q) \in \mathfrak{U}$ such that

$$A \otimes P = P \otimes B \quad \text{and} \quad B \otimes Q = Q \otimes A.$$

The reflexive law follows from $(I_n, I_n) \in \mathfrak{U}$. The symmetric law is trivial. To show the transitive law, suppose that there exists $(P_1, Q_1) \in \mathfrak{U}$ such that

$$A \otimes P_1 = P_1 \otimes B \quad \text{and} \quad B \otimes Q_1 = Q_1 \otimes A,$$

and $(P_2, Q_2) \in \mathfrak{U}$ such that

$$B \otimes P_2 = P_2 \otimes C \quad \text{and} \quad C \otimes Q_2 = Q_2 \otimes B.$$

Then, we have

$$A \otimes (P_1 \otimes P_2) = P_1 \otimes B \otimes P_2 = (P_1 \otimes P_2) \otimes C,$$
$$C \otimes (Q_2 \otimes Q_1) = Q_2 \otimes B \otimes Q_1 = (Q_2 \otimes Q_1) \otimes A.$$

Since $(P_1 \otimes P_2, Q_2 \otimes Q_1) \in \mathfrak{U}$ by Definition 3.5, we have shown the transitive law. Hence, the relation $\sim_{\mathfrak{U}}$ is an equivalence relation.

Remark. Property 4 in Definition 3.5 is not used to ensure that $\sim_{\mathfrak{U}}$ is an equivalence relation. This property is introduced to restrict the class of unitary-pair semigroups. Indeed, without this property, $\mathbb{R}_{\max}^{n\times n} \times \mathbb{R}_{\max}^{n\times n}$ itself becomes a unitary-pair semigroup.

We present some examples of unitary-pair semigroups.

(1) The set

$$\mathfrak{U}_n^{\mathrm{GP}} = \{(P, P^{\otimes(-1)}) \mid P \in \mathbb{R}_{\max}^{n\times n} \text{ is a generalized permutation matrix}\}$$

is obviously a unitary-pair semigroup. Indeed, $P \otimes P^{\otimes(-1)} = P^{\otimes(-1)} \otimes P = I_n$ and any product of generalized permutation matrices is also a generalized permutation matrix. Hence, the equivalence relation defined by unitary-pair semigroups is an extension of the similarity transformation by invertible matrices. Two matrices $A, B \in \mathbb{R}_{\max}^{n\times n}$ are $\mathfrak{U}_n^{\mathrm{GP}}$-equivalent if and only if there exists an invertible matrix P such that $A \otimes P = P \otimes B$.

(2) Let $P \in \mathbb{R}_{\max}^{n\times n}$ be a Kleene star of some matrix. This is equivalent to $P^{\otimes 2} = P$ and all diagonal entries of P are 0. If P is non-singular, the set

$$\mathfrak{U}_P^{\mathrm{KS}} = \{(I_n, I_n), (P, P)\}$$

is a unitary-pair semigroup. This kind of unitary-pair semigroup appears when we consider the transformation derived from eigenvectors.

(3) To obtain a broader class of unitary-pair semigroup, we focus on non-positive matrices whose determinants are 0. For a non-singular matrix $P \in \mathbb{R}_{\max}^{n\times n}$, let $\sigma_P \in S_n$ denote the permutation that attains $\det P$. Further, we say that P is *strictly normalized* if $[P]_{i\sigma_P(i)} = 0$ for $i = 1, 2, \ldots, n$ and $[P]_{ij} < 0$ for all other entries. We note that $\det P = 0$ if P is strictly normalized. The set

$$\mathfrak{U}_n^{\mathrm{SN}} = \left\{(P, Q) \in \mathbb{R}_{\max}^{n\times n} \times \mathbb{R}_{\max}^{n\times n} \mid P \text{ and } Q \text{ are strictly normalized, } \sigma_P = \sigma_Q^{-1}\right\}$$

is a unitary-pair semigroup. This follows from the next lemma.

Lemma 3.7. *Let $P, Q \in \mathbb{R}_{\max}^{n\times n}$ be strictly normalized matrices. Then $P \otimes Q$ is strictly normalized and $\sigma_{P\otimes Q} = \sigma_Q \sigma_P$.*

Proof. For any indices i, j, we have

$$[P \otimes Q]_{ij} = \bigoplus_{k=1}^{n} [P]_{ik} \otimes [Q]_{kj} \leq 0.$$

The equality holds if and only if $[P]_{ik} = [Q]_{kj} = 0$ for some k, which implies $k = \sigma_P(i)$ and $j = \sigma_Q(k) = \sigma_Q\sigma_P(i)$. Hence, we have

$$\det(P \otimes Q) = \bigotimes_{i=1}^{n} [P \otimes Q]_{i\,\sigma_Q\sigma_P(i)} = 0.$$

This means $\sigma_{P\otimes Q} = \sigma_Q\sigma_P$ and $P \otimes Q$ is strictly normalized. $\square$

If $(P, Q) \in \mathfrak{U}_n^{\mathrm{SN}}$, then $\det(P \otimes Q) = 0$ and $\sigma_{P\otimes Q} = \sigma_Q\sigma_P = \mathrm{id}_n$, where id_n denotes the identity permutation. Since all entries of $P \otimes Q$ are non-positive, all diagonal entries of $P \otimes Q$ must be 0. A similar argument holds for $Q \otimes P$. Hence, $\mathfrak{U}_n^{\mathrm{SN}}$ satisfies property 4 in Definition 3.5. Further, for $(P_1, P_2), (Q_1, Q_2) \in \mathfrak{U}_n^{\mathrm{SN}}$, we see that $\sigma_{P_1\otimes P_2} = \sigma_{P_2}\sigma_{P_1} = (\sigma_{Q_1}\sigma_{Q_2})^{-1} = \sigma^{-1}_{Q_2\otimes Q_1}$. Hence, property 3 in Definition 3.5 is satisfied.

(4) Finally, we propose a method to generate a new unitary-pair semigroup from a given unitary-pair semigroup using a generalized permutation matrix. Let $D \in \mathbb{R}_{\max}^{n\times n}$ be a fixed generalized permutation matrix and $\mathfrak{U}$ be a unitary-pair semigroup. The set

$$\mathfrak{U}_D = \left\{ (D^{\otimes(-1)} \otimes P \otimes D, D^{\otimes(-1)} \otimes Q \otimes D) \mid (P, Q) \in \mathfrak{U} \right\}$$

is also a unitary-pair semigroup. It is shown by the following observation. For any non-singular matrix A and a generalized permutation matrix D, we can easily verify that $[D^{\otimes(-1)} \otimes A \otimes D]_{ii} = [A]_{\sigma_D^{-1}(i)\sigma_D^{-1}(i)}$. In addition, we have

$$\begin{aligned}
\det(D^{\otimes(-1)} \otimes A \otimes D) &= \bigoplus_{\pi\in S_n} \bigotimes_{i=1}^{n} [D^{\otimes(-1)} \otimes A \otimes D]_{i\pi(i)} \\
&= \bigoplus_{\pi\in S_n} \bigotimes_{i=1}^{n} (-[D]_{\sigma_D^{-1}(i)\,i}) \otimes [A]_{\sigma_D^{-1}(i)\,\sigma_D^{-1}\pi(i)} \otimes [D]_{\sigma_D^{-1}\pi(i)\,\pi(i)} \\
&= \bigotimes_{i=1}^{n} (-[D]_{i\sigma_D(i)}) \otimes \left(\bigoplus_{\tau\in S_n} \bigotimes_{i=1}^{n} [A]_{i\tau(i)} \right) \otimes \bigotimes_{i=1}^{n} [D]_{i\sigma_D(i)} \\
&= \det A.
\end{aligned}$$

Here, we replace $\sigma_D^{-1}\pi\sigma_D$ with τ for each $\pi \in S_n$. Hence, we see that $\sigma_{D^{\otimes(-1)}\otimes A\otimes D} = \sigma_D\sigma_A\sigma_D^{-1}$. Since

$$\det((D^{\otimes(-1)} \otimes P \otimes D) \otimes (D^{\otimes(-1)} \otimes Q \otimes D)) = \det(D^{\otimes(-1)} \otimes (P \otimes Q) \otimes D),$$

we take $A = P \otimes Q$ and $A = Q \otimes P$ to prove property 4 of Definition 3.5. Similarly, we take $A = (P_1 \otimes P_2) \otimes (Q_2 \otimes Q_1)$ and $A = (Q_2 \otimes Q_1) \otimes (P_1 \otimes P_2)$ to prove property 3 of Definition 3.5.

Example 3.8. We consider the equivalence relation derived from the unitary-pair semigroup $\mathfrak{U}_n^{\mathrm{SN}}$. Let $A \in \mathbb{R}_{\max}^{n\times n}$ be a non-positive matrix with 0 only on the diagonal. We can easily verify that A^* is strictly normalized. In addition, we have $A \otimes A^* = (A \oplus I_n) \otimes A^* = A^*$. For another non-positive matrix $B \in \mathbb{R}_{\max}^{n\times n}$ with 0 only on the diagonal, we have

$$A \otimes (A^* \otimes B^*) = A^* \otimes B^* = (A^* \otimes B^*) \otimes B,$$
$$B \otimes (B^* \otimes A^*) = B^* \otimes A^* = (B^* \otimes A^*) \otimes A.$$

Here, both $A^* \otimes B^*$ and $B^* \otimes A^*$ are strictly normalized. Hence, A is $\mathfrak{U}_n^{\mathrm{SN}}$-equivalent to B. In particular all $n \times n$ non-positive matrices with 0 only on the diagonal are $\mathfrak{U}_n^{\mathrm{SN}}$-equivalent. This class contains I_n

Example 3.9. We present an example that is derived from the theory of Jordan canonical forms in max-plus algebra [24]. We consider a matrix

$$A = \begin{pmatrix} \varepsilon & 4 & 3 & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & 3 & \varepsilon & 0 \end{pmatrix}.$$

The associated graph $\mathcal{G}(A)$ is illustrated in Figure 2. Since the maximum average weight of circuits in $\mathcal{G}(A)$ is 4 and the $\mathcal{G}(A)$ is strongly connected, A has exactly one eigenvalue $\lambda(A) = 4$. The critical circuits in $\mathcal{G}(A)$ are $(1, 2, 1)$ and $(3, 4, 5, 3)$. By computing

$$((-4) \otimes A)^* = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -2 & -2 & -1 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 \end{pmatrix},$$

we have two independent right eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}.$$

The number of right independent eigenvectors is less than the dimension of the matrix.

To cope with this problem, we compute $(((-4)\otimes A)^{\otimes 6})^*$ instead:

$$P := (((-4)\otimes A)^{\otimes 6})^* = \begin{pmatrix} 0 & -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ -2 & -2 & -3 & 0 & -2 \\ -2 & -2 & -3 & -2 & 0 \end{pmatrix}.$$

Here, the exponent 6 comes from the least common multiple of the lengths of critical circuits $(1, 2, 1)$ and $(3, 4, 5, 3)$. Then, columns of P are independent and become right eigenvectors of $A^{\otimes 6}$ with respect to $\lambda(A)^{\otimes 6}$. The same holds for rows of P. Further, $\det P^* = 0$ is attained only by the identity permutation. By computation, we can verify that

$$A\otimes P = P \otimes \begin{pmatrix} \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \end{pmatrix} \quad \text{and} \quad P\otimes A = \begin{pmatrix} \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \end{pmatrix} \otimes P.$$

This shows that A is equivalent to a block diagonal matrix:

$$A \sim_{\mathfrak{U}_P^{\mathrm{KS}}} \left(\begin{array}{cc|ccc} \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \end{array}\right).$$

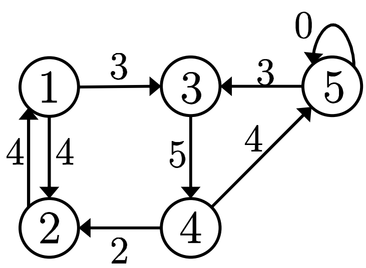

Figure 2. Associated graph $\mathcal{G}(A)$ for Example 3.9.

§ 4. Concluding remarks

In this paper, we discuss the similarity transformation of max-plus matrices. Although the similarity transformation $A\otimes P = P\otimes B$ preserves the maximum eigenvalue, the other eigenvalues of A and B are different in general. One of the main reasons is that the similarity transformation is not symmetric in general. To resolve this problem, we propose a pair of matrices that imitates a pair of a matrix and its inverse. By defining a set of these pairs as a unitary-pair semigroup, we obtain a transformation that is both symmetric and transitive, and hence induces an equivalence relation. It is expected in the future to enlarge the class of matrices that can be used in the similarity transformation. Moreover, it is also a challenging problem to construct a transformation of matrices that preserves the roots of the characteristic polynomial as well as eigenvalues.

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