Ill-posedness of the two-dimensional stationary Navier–Stokes equations in critical Besov spaces on the whole plane

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Abstract

This paper is the survey of [4], which proved the ill-posedness of the two-dimensional stationary Navier–Stokes equations in the scaling critical Besov spaces.

1 Introduction

We consider the incompressible stationary Navier–Stokes equations on $\mathbb{R}^n$ with $n \geqslant 2$:

$$-\Delta U + \mathbb{P}\,\mathrm{div}(U \otimes U) = \mathbb{P}F, \quad \mathrm{div}\, U = 0, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\mathbb{P} := I + \nabla\,\mathrm{div}(-\Delta)^{-1} = \left\{\delta_{jk} + \partial_{x_j}\partial_{x_k}(-\Delta)^{-1}\right\}_{1\leqslant j,k\leqslant n}$ is the Helmholtz projection onto the divergence-free vector fields. Here $U = U(x) : \mathbb{R}^n \to \mathbb{R}^n$ represents the unknown velocity fields and $P = P(x) : \mathbb{R}^n \to \mathbb{R}$ denotes the unknown pressure of the fluid, while $F = F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is the external force which is a given function. The equation (1.1) possesses the invariant scaling transform; if (F, U) satisfies (1.1), then the scaled functions

$$F_\lambda(x) := \lambda^3 F(\lambda x), \qquad U_\lambda(x) := \lambda U(\lambda x)$$

also solve (1.1) for all $\lambda > 0$. We call that the data space D and the solution space S are scaling critical if

$$\|F_\lambda\|_D = \|F\|_D, \qquad \|U_\lambda\|_S = \|U\|_S \tag{1.2}$$

for all $\lambda > 0$. As the homogeneous Besov spaces $D = \dot{B}^{\frac{n}{p}-3}_{p,q}(\mathbb{R}^n)$ and $S = \dot{B}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)$ ($1 \leqslant p, q \leqslant \infty$) satisfy (1.2) for all dyadic numbers $\lambda > 0$, we regard them as the scaling critical Besov spaces for (1.1).

The aim of this paper is to survey of the result in [4] that considers (1.1) with the two-dimensional case $n = 2$ and proves that (2.1) is ill-posed in the scaling critical Besov spaces $D = \dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$ and $S = F \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ with $1 \leqslant p \leqslant 2$, which is the completely different phenomenon for the high dimensional case $n \geqslant 3$ stated in [8].

Before stating the main result of [4] precisely, we shall recall known results related to our study. In the higher-dimensional cases $\mathbb{R}^n$ with $n \geqslant 3$, Leray [12], Ladyzhenskaya [11], and Fujita [5] proved the existence of solutions to (1.1). For the scaling critical framework, Chen [2] proved the well-posedness of (1.1) from $F = \mathrm{div}\,\widetilde{F}$ with $\widetilde{F} \in L^{\frac{n}{2}}(\mathbb{R}^n)$ to $U \in L^n(\mathbb{R}^n)$.

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Kaneko–Kozono–Shimizu [8] proved that (1.1) is well-posed from $\dot{B}^{\frac{n}{p}-3}_{p,q}(\mathbb{R}^n)$ to $\dot{B}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)$ for all $(p,q) \in [1,n) \times [1,\infty]$, whereas Tsurumi [14,16] showed the ill-posedness for $(p,q) \in (\{n\} \times (2,\infty]) \cup ((n,\infty] \times [1,\infty])$. Li–Yu–Zhu [13] considered the remaining case $(p,q) \in \{n\} \times [1,2]$. For other related results, see Tsurumi [15] for the well-posedness and ill-posedness in the scaling critical Besov spaces on the periodic box $\mathbb{T}^n$ $(n \geqslant 3)$. Yamazaki [18] made use of some symmetric structures and constructed small solution. In [18], he considered (2.1) in the whole plane case and proved that for given external force $F = \nabla^\perp G = (\partial_{x_2} G, -\partial_{x_1} G)$, where G decays like $|G(x)| \leqslant \delta(1+|x|)^{-2}$ with some $0 < \delta \ll 1$ and possesses the following symmetric conditions:

$$G(-x_1,x_2) = G(x_1,-x_2) = G(x_2,x_1) = G(-x_2,x_1) = -G(x_1,x_2), \tag{1.3}$$

there exists a unique small solution to (1.1) in the $L^{2,\infty}(\mathbb{R}^2)$-framework with the vorticity $\operatorname{rot} U$ satisfying the same condition as for G. Guillod–Korobkov–Ren [7] constructed a unique solution to (1.1) for compact supported external forces that do not necessarily have any spatial symmetric structure.

Despite of numerous studies on the two-dimensional stationary Navier–Stokes equations, it was a long-standing open problem whether the two-dimensional Navier–Stokes equations on the whole plane $\mathbb{R}^2$ possesses a unique small solution for a given small external force F in general settings without any symmetric condition. In particular, unlike the higher-dimensional cases, the well-posedness and ill-posedness of stationary Navier–Stokes equations on the whole plane case in the scaling critical framework were completely unsolved. In the paper [4], the author solved the aforementioned open problem in the challenging case $\mathbb{R}^2$ and proved the ill-posedness. The main result of [4] now reads as follows.

Theorem 1.1 (Ill-posedness of (1.1)). *For any $1 \leqslant p \leqslant 2$, (1.1) is ill-posed from $\dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$ to $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ in the sense that the solution map is discontinuous. More precisely, for any $1 \leqslant p \leqslant 2$, there exist a positive constant $\delta_0 = \delta_0(p)$, a positive integer $N_0 = N_0(p)$, and a sequence $\{F_N\}_{N\in\mathbb{N}} \subset \dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$ satisfying*

$$\lim_{N\to\infty} \|F_N\|_{\dot{B}^{\frac{2}{p}-3}_{p,1}} = 0$$

such that if each F_N with $N \geqslant N_0$ generates a solution $U_N \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ of (1.1), then it holds

$$\inf_{N\geqslant N_0} \|U_N\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \geqslant \delta_0.$$

Remark 1.2. We provide some remarks on Theorem 1.1.

1. Theorem 1.1 can be compared with the result of Yamazaki [18], where he constructed a unique small solution to (1.1) in the scaling critical space $L^{2,\infty}(\mathbb{R}^2)$, which is a wider framework than ours, that is $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2) \hookrightarrow L^{2,\infty}(\mathbb{R}^2)$ $(1 \leqslant p \leqslant 2)$. In [18], it is assumed that the small external force has the form $F = \nabla^\perp G$ with some function G satisfying the symmetric condition (1.3), while our sequence of external forces in Theorem 1.1 is given by an anisotropic form as follows:

   $$F_N(x) := -\frac{\delta}{\sqrt{N}} \nabla^\perp \Delta(\Psi(x)\cos(Mx_1)),$$

   for some constants $0 < \delta \ll 1$, $M \gg 1$, and some real valued radial symmetric function $\Psi \in \mathscr{S}(\mathbb{R}^2)$ with $\operatorname{supp}\widehat{\Psi}$ compact. Therefore, it is revealed that the symmetric condition (1.3) is a crucial assumption for the solvability of (1.1).

2. In the higher-dimensional whole space $\mathbb{R}^n$ and periodic box $\mathbb{T}^n$ cases with $n \geqslant 3$, it was shown in [8,15] that (1.1) is well-posed in the scaling critical Besov spaces based on $L^p(\mathbb{R}^n)$ for $1 \leqslant p < n$. Tsurumi [17] revealed that similar results hold for the two-dimensional stationary Navier–Stokes equations on the periodic box $\mathbb{T}^2$. In [17], he showed the well-posed in the nearly scaling critical Besov spaces based on $L^{p+\varepsilon}(\mathbb{T}^2)$ for $1 \leqslant p < 2$ with small $\varepsilon > 0$. By comparing these results and Theorem 1.1, we see that, unlike the higher-dimensional cases, the solvability is different in the two-dimensional case when the domain is the periodic box $\mathbb{T}^2$ and the whole plane $\mathbb{R}^2$. This implies that in the two-dimensional case, information at the spatial infinity of (1.1) affects the solvability of (1.1), which may be attributed to the fact that the fundamental solution of the two-dimensional Stokes equations increases logarithmically.

We elaborate upon the difficulty that we meet when we prove Theorem 1.1. Following the standard ill-posedness argument as proposed in [1, 14, 19], we may construct a sequence $\{F_N\}_{N\in\mathbb{N}} \subset \dot{B}^{\frac{2}{p}-3}_{p,q}(\mathbb{R}^2)$ of the external force satisfying

$$\lim_{N\to\infty} \|F_N\|_{\dot{B}^{\frac{2}{p}-3}_{p,q}} = 0, \qquad \lim_{N\to\infty} \left\|U_N^{(1)}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} = 0, \qquad \liminf_{N\to\infty} \left\|U_N^{(2)}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} > 0,$$

where $U_N^{(1)}$ and $U_N^{(2)}$ are the first and second iterations, respectively, defined as

$$U_N^{(1)} := (-\Delta)^{-1}\mathbb{P}F_N, \qquad U_N^{(2)} := -(-\Delta)^{-1}\mathbb{P}\,\mathrm{div}(U_N^{(1)} \otimes U_N^{(1)}).$$

We formally decompose the corresponding solution U_N of (1.1) with the external force F_N as

$$U_N = U_N^{(1)} + U_N^{(2)} + W_N,$$

where the perturbation W_N is a solution to

$$\begin{aligned} -\Delta W_N + \mathbb{P}\,\mathrm{div}\Big(& U_N^{(1)} \otimes U_N^{(2)} + U_N^{(2)} \otimes U_N^{(1)} + U_N^{(2)} \otimes U_N^{(2)} \\ &+ U_N^{(1)} \otimes W_N + U_N^{(2)} \otimes W_N + W_N \otimes U_N^{(1)} + W_N \otimes U_N^{(2)} + W_N \otimes W_N\Big) = 0. \end{aligned} \tag{1.4}$$

However, in the whole plane case $\mathbb{R}^2$, it seems hard to find a function space $X \subset \mathscr{S}'(\mathbb{R}^2)$ in which the following nonlinear estimate holds:

$$\left\|(-\Delta)^{-1}\mathbb{P}\,\mathrm{div}(U \otimes V)\right\|_X \leqslant C\|U\|_X\|V\|_X. \tag{1.5}$$

In particular, the author [3] implied that (1.5) fails for all scaling critical Besov spaces $X = \dot{B}^{\frac{2}{p}-1}_{p,q}(\mathbb{R}^2)$ $(1 \leqslant p, q \leqslant \infty)$. Thus, it seems difficult to construct a function W_N obeying (1.4) and establish its suitable estimate. Consequently it is hard to prove the desired ill-posedness by the standard argument.

Let us mention the idea to overcome the aforementioned difficulties and prove Theorem 1.1. Inspired by the general observation that the stationary solutions should be the large time behavior of *nonstationary* solutions, we consider the *nonstationary* Navier–Stokes equations. Then, in contrast to the stationary problem, which possesses difficulties in the singularity of $(-\Delta)^{-1}$ at the origin in the frequency side, we see that, for the nonstationary Navier–Stokes equations, the heat kernel $\{e^{t\Delta}\}_{t>0}$ relaxes the singularity on the low-frequency part, and we may obtain the nonlinear estimate

$$\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,\mathrm{div}(u(\tau) \otimes v(\tau))d\tau\right\|_X \leqslant C\|u\|_X\|v\|_X$$

with $X = \widetilde{L^r}(0,T; \dot{B}^{\frac{2}{p}-1+\frac{2}{r}}_{p,q}(\mathbb{R}^2))$ for some p, q, r and all $0 < T \leqslant \infty$. Motivated by these facts, we suppose to contrary that (1.1) is well-posed and consider the *nonstationary* Navier–Stokes equations with the stationary external forces. Then, we may show that a contradiction appears from the behavior of the *nonstationary* solutions in large times.

2 Nonstationary analysis

Let us consider the *nonstationary* incompressible Navier–Stokes equations with the *stationary* external force:

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}\operatorname{div}(u \otimes u) = \mathbb{P}F, & t > 0, x \in \mathbb{R}^2, \\ \operatorname{div} u = 0, & t \geqslant 0, x \in \mathbb{R}^2, \\ u(0,x) = 0, & x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

Here, $u = u(t,x) : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the unknown *nonstationary* velocity of the fluid, and $F = F(x) : \mathbb{R}^2 \to \mathbb{R}^2$ is the given *stationary* external force. By the Duhamel principle and

$$\int_0^t e^{(t-\tau)\Delta}\mathbb{P}F d\tau = (-\Delta)^{-1}\left(1 - e^{t\Delta}\right)\mathbb{P}F,$$

(2.1) is formally equivalent to

$$u(t) = (-\Delta)^{-1}\left(1 - e^{t\Delta}\right)\mathbb{P}F + \mathcal{D}[u,u](t), \tag{2.2}$$

where the nonlinear Duhamel term $\mathcal{D}[\cdot,\cdot]$ is defined by

$$\mathcal{D}[u,v](t) := -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}(u(\tau) \otimes v(\tau))d\tau.$$

We say that u is a mild solution to (2.1) if u satisfies (2.2).

2.1 Global ill-posedness

Since the external force in (2.1) does not depends on time, it is excepted that the solution to (2.1) does not decay in time. However, it is difficult to close the nonlinear estimates in the scaling critical spaces that include functions non-decaying in time such as $\widetilde{L^\infty}(0,\infty;\dot{B}^{\frac{2}{p}-1}_{p,q}(\mathbb{R}^2))$. Thus, it is hard to construct a bounded-in-time global solution to (2.1). In this subsection, we justify the above consideration in the sense that for every $1 \leqslant p \leqslant 2$, the solution map $\dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2) \ni F \mapsto u \in \widetilde{C}([0,\infty);\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$ is *discontinuous* even if it exists. More precisely we show that there exist two sequences $\{F_N\}_{N\in\mathbb{N}} \subset \dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$ of external forces and $\{T_N\}_{N\in\mathbb{N}} \subset (0,\infty)$ of times satisfying

$$\lim_{N\to\infty} \|F_N\|_{\dot{B}^{\frac{2}{p}-3}_{p,1}} = 0, \qquad \lim_{N\to\infty} T_N = \infty,$$

such that (2.1) with the external force F_N admits a solution $u_N \in \widetilde{C}([0,T_N];\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$ satisfying

$$\liminf_{N\to\infty} \|u_N(T_N)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} > 0.$$

In this paper, we call this phenomenon as the *global ill-posedness.* The aim of this subsection is to prove the following theorem.

Theorem 2.1 ([4, Theorem 3.1]). *Let $1 \leqslant p \leqslant 2$. Then, there exist two positive constants $\delta_1 = \delta_1(p)$ and $K_1 = K_1(p)$ such that for any $0 < \delta \leqslant \delta_1$, there exists a sequence $\{F_{\delta,N}\}_{N\in\mathbb{N}} \subset \dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$ of external forces such that the following two statements are true:*

(i) For any $N \in \mathbb{N}$, it holds

$$\|F_{\delta,N}\|_{\dot{B}^{\frac{2}{p}-3}_{p,1}} \leqslant \frac{K_1\delta}{\sqrt{N}}.$$

(ii) Let $T_N := 2^{2N}$. Then, for each integer $N \geqslant 3$, (2.1) with the external force $F_{\delta,N}$ admits a mild solution $u_{\delta,N} \in \widetilde{C}([0,T_N]; \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$ satisfying

$$\liminf_{N\to\infty} \|u_{\delta,N}(T_N)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} > \frac{\delta^2}{K_1}, \quad \limsup_{N\to\infty} \|u_{\delta,N}\|_{\widetilde{L^\infty}(0,T_N;\dot{B}^{\frac{2}{p}-1}_{p,1})} < K_1\delta^2. \tag{2.3}$$

Remark 2.2. For the nonstationary Navier–Stokes equations in $\mathbb{R}^n$ with $n \geqslant 3$, it is possible to construct a small global-in-time unique solution for small external force that is bounded-in-time but does not decay as $t \to \infty$. We refer to [6, 9] and references therein for the time periodic setting. Thus, the assertion of Theorem 2.1 is one of phenomena inherent to two-dimensional flows.

We shall sketch the outline of the proof of Theorem 2.1. We first follow the standard ill-posedness argument used in studies such as [1, 19] and formally decompose the solution $u_{\delta,N}$ as

$$u_{\delta,N} = u^{(1)}_{\delta,N} + u^{(2)}_{\delta,N} + w_{\delta,N},$$

where $u^{(1)}_{\delta,N}$ and $u^{(2)}_{\delta,N}$ denote the first and second iterations, respectively, which are defined by

$$u^{(1)}_{\delta,N}(t) := (-\Delta)^{-1}\left(1 - e^{t\Delta}\right)\mathbb{P}F_{\delta,N}, \qquad u^{(2)}_{\delta,N}(t) := \mathcal{D}\left[u^{(1)}_{\delta,N}, u^{(1)}_{\delta,N}\right](t)$$

and $w_{\delta,N}$ is the perturbation solving

$$\begin{cases} \partial_t w_{\delta,N} - \Delta w_{\delta,N} + \mathbb{P}\operatorname{div}\Big(w_{\delta,N}\otimes w_{\delta,N} + u^{(1)}_{\delta,N}\otimes u^{(2)}_{\delta,N} + u^{(2)}_{\delta,N}\otimes u^{(1)}_{\delta,N} + u^{(2)}_{\delta,N}\otimes u^{(2)}_{\delta,N} \\ \qquad\qquad + u^{(1)}_{\delta,N}\otimes w_{\delta,N} + u^{(2)}_{\delta,N}\otimes w_{\delta,N} + w_{\delta,N}\otimes u^{(1)}_{\delta,N} + w_{\delta,N}\otimes u^{(2)}_{\delta,N}\Big) = 0, \\ \operatorname{div} w_{\delta,N} = 0, \\ w_{\delta,N}(0,x) = 0. \end{cases}$$

Then, choosing a suitable sequence $\{F_{\delta,N}\}_{N\in\mathbb{N}} \subset \dot{B}^{\frac{2}{p}-3}_{p,1}(\mathbb{R}^2)$, we may see that

$$\|F_{\delta,N}\|_{\dot{B}^{\frac{2}{p}-3}_{p,1}} \leqslant C\frac{\delta}{\sqrt{N}}, \qquad \left\|u^{(1)}_{\delta,N}\right\|_{\widetilde{L^\infty}(0,\infty;\dot{B}^{\frac{2}{p}-1}_{p,1})} \leqslant C\frac{\delta}{\sqrt{N}}, \tag{2.4}$$

whereas the second iteration satisfies

$$\left\|u^{(2)}_{\delta,N}(T_N)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \geqslant c\delta^2, \qquad \left\|u^{(2)}_{\delta,N}\right\|_{\widetilde{L^\infty}(0,T_N;\dot{B}^{\frac{2}{p}-1}_{p,1})} \leqslant C\delta^2 \tag{2.5}$$

for sufficiently large N. It is relatively easy to obtain (2.4) and (2.5), while the most difficult part of the proof is how to construct and control the perturbation $w_{\delta,N}$. To this end, we consider the estimate of $w_{\delta,N}$ in

$$\widetilde{C}([0,T_N];\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)) \cap \widetilde{L^N}(0,T_N;\dot{B}^{\frac{2}{p}-1+\frac{2}{N}}_{p,2}(\mathbb{R}^2)).$$

Here, the choice of the auxiliary space $\widetilde{L^N}(0,T_N;\dot{B}^{\frac{2}{p}-1+\frac{2}{N}}_{p,2}(\mathbb{R}^2))$ is the most crucial idea of the proof. Indeed, choosing the Lebesgue exponent of the time integral as N, we see that the $L^N(0,T_N)$-norm of functions are bounded by the $L^\infty(0,T_N)$-norm with the constant independent of N. More precisely, it holds

$$\|f\|_{L^N(0,T_N)} \leqslant T_N^{\frac{1}{N}}\|f\|_{L^\infty(0,T_N)} = 4\|f\|_{L^\infty(0,T_N)}$$

for all $f \in L^\infty(0,T_N)$. On the other hand, choosing the interpolation index as $q=2$ in the auxiliary Chemin–Lerner space $\widetilde{L^N}(0,T_N;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}}(\mathbb{R}^2))$, we may use a pair of estimates

$$\begin{aligned}\|\mathcal{D}[u,v]\|_{\widetilde{L^\infty}(0,T;\dot{B}_{p,1}^{\frac{2}{p}-1})} \leqslant{} & CN\|u\|_{\widetilde{L^N}(0,T;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})}\|v\|_{\widetilde{L^N}(0,T;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})} \\ & + C\|u\|_{\widetilde{L^\infty}(0,T;\dot{B}_{p,1}^{\frac{2}{p}-1})}\|v\|_{\widetilde{L^\infty}(0,T;\dot{B}_{p,1}^{\frac{2}{p}-1})},\end{aligned} \tag{2.6}$$

$$\|\mathcal{D}[u,v]\|_{\widetilde{L^N}(0,T;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})} \leqslant C\sqrt{N}\|u\|_{\widetilde{L^N}(0,T;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})}\|v\|_{\widetilde{L^N}(0,T;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})} \tag{2.7}$$

proved in [4, Lemma 2.5] above. Then, keeping these facts in mind and making use of the iterative argument via (2.6) and (2.7), we may obtain the existence of the perturbation $w_{\delta,N}$ and the estimate

$$\|w_{\delta,N}\|_{\widetilde{L^\infty}(0,T_N;\dot{B}_{p,1}^{\frac{2}{p}-1})} \leqslant C\delta^3, \qquad \|w_{\delta,N}\|_{\widetilde{L^N}(0,T_N;\dot{B}_{p,2}^{\frac{2}{p}-1+\frac{2}{N}})} \leqslant C\frac{\delta^3}{\sqrt{N}} \tag{2.8}$$

for sufficiently small δ. Collecting (2.4), (2.5), and (2.8), we obtain the solution $u_{\delta,N}$ satisfying the desired estimate (2.3).

2.2 Global solutions around the stationary flow

In contrast to the previous subsection, if we assume that the stationary problem (1.1) possesses a solution U for some external force F and then consider the nonstationary Navier–Stokes equations (2.1) with the same external force F as for U. Under this assumption, we may prove that (2.1) admits a bounded-in-time global solution.

Theorem 2.3 ([4, Theorem 3.3]). *Let $1 \leqslant p < 4$ and $1 \leqslant q < \infty$. Then, there exist a positive constant $\delta_2 = \delta_2(p,q)$ and an absolute positive constant K_2 such that if a given external force $F \in \dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ generates a solution $U \in \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ to (1.1) satisfying*

$$\|U\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant \delta_2,$$

then (2.1) with the same external force F admits a global mild solution u in the class

$$u \in \widetilde{C}([0,\infty);\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)), \qquad \|u\|_{\widetilde{L^\infty}(0,\infty;\dot{B}_{p,q}^{\frac{2}{p}-1})} \leqslant K_2\|U\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}.$$

Assuming the existence of the stationary solution, we consider the perturbation $v = u - U$, which should solve

$$\begin{cases} \partial_t v - \Delta v + \mathbb{P}\operatorname{div}(U\otimes v + v\otimes U + v\otimes v) = 0, & t>0, x\in\mathbb{R}^2, \\ \operatorname{div} v = 0, & t \geqslant 0, x\in\mathbb{R}^2, \\ v(0,x) = -U(x), & x\in\mathbb{R}^2, \end{cases} \tag{2.9}$$

then (2.9) possesses no external force that does not decay as $t\to\infty$, which implies that the solution v of (2.9) is expected to decay as $t\to\infty$ and belong to some time integrable function spaces. Since the nonlinear estimate is closed in

$$\widetilde{C}([0,\infty);\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)) \cap \widetilde{L^r}(0,\infty;\dot{B}_{p,q}^{\frac{2}{p}-1+\frac{2}{r}}(\mathbb{R}^2)) \tag{2.10}$$

for some $2<r<\infty$ (see [4, Lemma 2.4]), we may establish the global solution v to (2.9) in the class (2.10). We then obtain the desired solution by $u := v + U$.

3 Proof of Theorem 1.1

Now, we are in a position to present the proof of our main result.

Proof of Theorem 1.1. Let δ_1 and δ_2 be the positive constants appearing in Theorems 2.1 and 2.3, respectively. Let K_0 be the positive constant satisfying

$$\sup_{0\leqslant t\leqslant T}\|\mathcal{D}[u,v](t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}} \leqslant K_0\|u\|_{\widetilde{L^\infty}(0,T;\dot{B}^{\frac{2}{p}-1}_{p,1})}\sup_{0\leqslant t\leqslant T}\|v(t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}} \tag{3.1}$$

for all $T>0$, $u\in\widetilde{C}([0,T];\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$, and $v\in C([0,T];\dot{B}^{\frac{2}{p}-1}_{p,\infty}(\mathbb{R}^2))$. See [4, Lemma 2.4] for the proof of (3.1). Let K_1 and K_2 be the positive constants appearing in Theorem 2.1 and Theorem 2.3, respectively. We define

$$\delta_0 := \min\left\{\delta_2,\delta_3,\frac{\delta_3^2}{2K_1K_2}\right\};\quad \delta_3 := \min\left\{\delta_1,\frac{1}{2K_0(K_1+K_2)}\right\}.$$

We consider the sequence $F_N := F_{\delta_3,N}$, which is defined in Theorem 2.1 with δ replaced by δ_3. Note that Theorem 2.1 yields

$$\|F_N\|_{\dot{B}^{\frac{2}{p}-3}_{p,1}} \leqslant \frac{K_1\delta_3}{\sqrt{N}} \to 0 \qquad \text{as } N\to\infty.$$

Let us consider the nonstationary Navier–Stokes equations

$$\begin{cases}\partial_t u-\Delta u+\mathbb{P}\operatorname{div}(u\otimes u)=\mathbb{P}F_N, & t>0, x\in\mathbb{R}^2,\\ \operatorname{div}u=0, & t\geqslant 0, x\in\mathbb{R}^2,\\ u(0,x)=0, & x\in\mathbb{R}^2.\end{cases} \tag{3.2}$$

By Theorem 2.1, there exists a $N_0=N_0(p)\in\mathbb{N}$ such that for each $N\in\mathbb{N}$ with $N\geqslant N_0$, (3.2) possesses a solution $u_N=u_{\delta_3,N}\in\widetilde{C}([0,T_N];\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$ satisfying

$$\|u_N(T_N)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \geqslant \frac{\delta_3^2}{K_1},\qquad \|u_N\|_{\widetilde{L^\infty}(0,T_N;\dot{B}^{\frac{2}{p}-1}_{p,1})} \leqslant K_1\delta_3. \tag{3.3}$$

Here, we have set $T_N := 2^{2N}$.

Assume to contrary that there exist an integer $N'\geqslant N_0$ and a solution $U_{N'}\in\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ of (1.1) with the external force $F_{N'}$ satisfying

$$\|U_{N'}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} < \delta_0. \tag{3.4}$$

Then, by (3.4) and Theorem 2.3, $F_{N'}$ generates a global-in-time solution $\widetilde{u}_{N'}\in\widetilde{C}([0,\infty);\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))$ to the nonstationary Navier–Stokes equations (3.2) satisfying

$$\|\widetilde{u}_{N'}\|_{\widetilde{L^\infty}(0,\infty;\dot{B}^{\frac{2}{p}-1}_{p,1})} \leqslant K_2\|U_{N'}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \leqslant K_2\delta_3. \tag{3.5}$$

Next, we show that these two solutions $\widetilde{u}_{N'}$ and $u_{N'}$ coincides on $[0,T_{N'}]$. Since $\widetilde{u}_{N'}-u_{N'}$ enjoys $\widetilde{u}_{N'}-u_{N'}=\mathcal{D}[\widetilde{u}_{N'},\widetilde{u}_{N'}-u_{N'}]+\mathcal{D}[\widetilde{u}_{N'}-u_{N'},u_{N'}]$ we see by (3.1) that

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T_{N'}}\|\widetilde{u}_{N'}(t)-u_{N'}(t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}}\\ &\leqslant K_0\|(u_{N'},\widetilde{u}_{N'})\|_{\widetilde{L^\infty}(0,T_{N'};\dot{B}^{\frac{2}{p}-1}_{p,1})}\sup_{0\leqslant t\leqslant T_{N'}}\|\widetilde{u}_{N'}(t)-u_{N'}(t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}}\\ &\leqslant K_0\left(K_1+K_2\right)\delta_3\sup_{0\leqslant t\leqslant T_{N'}}\|\widetilde{u}_{N'}(t)-u_{N'}(t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}}\\ &\leqslant \frac{1}{2}\sup_{0\leqslant t\leqslant T_{N'}}\|\widetilde{u}_{N'}(t)-u_{N'}(t)\|_{\dot{B}^{\frac{2}{p}-1}_{p,\infty}},\end{aligned}$$

which implies $\widetilde{u}_{N'}(t) = u_{N'}(t)$ for all $0 \leqslant t \leqslant T_{N'}$. Hence, it follows from (3.3) and (3.5) that

$$\begin{aligned} \|U_{N'}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} &\geqslant \frac{1}{K_2}\|\widetilde{u}_{N'}\|_{\widetilde{L^\infty}(0,T_{N'};\dot{B}^{\frac{2}{p}-1}_{p,1})} \\ &\geqslant \frac{1}{K_2}\|\widetilde{u}_{N'}(T_{N'})\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} = \frac{1}{K_2}\|u_{N'}(T_{N'})\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \geqslant \frac{\delta_3^2}{K_1K_2} \geqslant 2\delta_0, \end{aligned}$$

which contradicts (3.4). Thus, we complete the proof. □

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