# Singular and fractional integral operators on bi-predual spaces of generalized Campanato spaces with variable growth condition*

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## 1 Introduction

This report is an announcement of [32].

Let $\mathrm{BMO}(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ and $C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ be the set of all functions of bounded mean oscillation, the Hardy space and the set of all infinitely differentiable functions with compact support on the Euclidean space $\mathbb{R}^d$, respectively. Then the following duality assertions are known:

$$\left(\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{BMO}(\mathbb{R}^d)}\right)^* = H^1(\mathbb{R}^d), \quad \left(H^1(\mathbb{R}^d)\right)^* = \mathrm{BMO}(\mathbb{R}^d), \tag{1.1}$$

where $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{BMO}(\mathbb{R}^d)}$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ with respect to $\mathrm{BMO}(\mathbb{R}^d)$. For the duality assertions in (1.1), see [7, 8] and [5, 9], respectively. See also [6, 10].

The second duality in (1.1) was extended to generalized Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ with variable growth condition by [15], where $p \in [1,\infty)$ and $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathrm{BMO}(\mathbb{R}^n)$. The function space $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ was introduced in [19] to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^d)$ and studied in [12, 14, 16], etc. Moreover, it has been proved that $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ is the dual space of the Hardy space $H^{p(\cdot)}(\mathbb{R}^d)$ with variable exponent in [18]. That is, $H^{p(\cdot)}(\mathbb{R}^d)$ is a predual of $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$. In general the predual is not unique. It was proven in [15] that an atomic Hardy-type space $H^{[\phi,q]}(\mathbb{R}^d)$ is also a predual of $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$.

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In this report we extend the first duality in (1.1) to

$$\left(\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}\right)^* = H^{[\phi,q]}(\mathbb{R}^d), \tag{1.2}$$

which is an extension of the result in [29]. In this report we treat a wider class of $\phi$ than [29]. We also extend the characterization of $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\rm BMO(\mathbb{R}^d)}$ by Uchiyama [28] to $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$, which is also an extension of [2]. Moreover, using this characterization, we prove the boundedness of singular and fractional integral operators on $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$. As corollaries, for $\alpha \in (0,1)$ and $p \in (1,\infty)$, we have the boundedness of the fractional integral operator $I_\alpha$ from $L^p(\mathbb{R}^d)$ to $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\rm BMO(\mathbb{R}^d)}$ if $-d/p+\alpha=0$, and from $L^p(\mathbb{R}^d)$ to $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm Lip}_\beta(\mathbb{R}^d)}$ if $-d/p+\alpha=\beta\in(0,1)$.

In recent years, function spaces $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm Lip}_\beta(\mathbb{R}^d)}$ and related function spaces have attracted more and more attention related to the compactness of commutators. For instance, Nogayama and Sawano [21] studied the compactness of commutators $[b, I_\alpha]$ on Morrey spaces, where $b$ is a function in $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm Lip}_\beta(\mathbb{R}^d)}$ and $I_\alpha$ is the fractional integral operator. This result is extended to generalized Morrey and Orlicz-Morrey spaces using $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ by [1, 2, 31]. In addition, Guo et al. [11] studied the compactness of iterated commutators on weighted Lebesgue spaces. Torres and Xue [27] and Tao et al. [25] studied $\overline{B_\infty(\mathbb{R}^d)}^{\rm BMO(\mathbb{R}^d)}$ and commutators of bilinear operators on Lebesgue and weighted Lebesgue spaces, respectively, where $B_\infty(\mathbb{R}^d)$ is the set of all $b \in C^\infty(\mathbb{R}^d)\cap {\rm BMO}(\mathbb{R}^d)$ with $D^\gamma b(x)\to 0$ as $|x|\to\infty$ for any $\gamma\in(\mathbb{N}\cup\{0\})^d\setminus\{0\}$. Recently, Tao et al. [26] extended these results to $\overline{B_\infty(\mathbb{R}^d)}^{{\rm Lip}_\beta(\mathbb{R}^d)}$. In this report we study properties on the function space $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ and the boundedness of operators on this function space.

In this report we denote by $B(x,r)$ the open ball centered at $x\in\mathbb{R}^d$ and of radius $r$. For a function $f\in L^1_{\rm loc}(\mathbb{R}^d)$ and a ball $B$, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy,$$

where $|B|$ is the Lebesgue measure of $B$.

For a function $\phi:\mathbb{R}^d\times(0,\infty)\to(0,\infty)$ and a ball $B=B(x,r)$ we write $\phi(B)=\phi(x,r)$. The function space $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ is defined as follows:

**Definition 1.1.** For $p\in[1,\infty)$ and $\phi:\mathbb{R}^d\times(0,\infty)\to(0,\infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ be the set of all functions $f$ such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)}\left(\fint_B |f(y)-f_B|^p\,dy\right)^{1/p},$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^d$.

Then $\|f\|_{\mathcal{L}_{p,\phi}}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ is a Banach space. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathrm{BMO}(\mathbb{R}^d)$. If $p = 1$ and $\phi(x,r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^d)$. If $\phi(x,r) = r^\lambda$ $(-d/p \le \lambda < 0)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ is the same as the Morrey space modulo constant functions. In particular, if $\lambda = -d/p$, then it is the Lebesgue space $L^p(\mathbb{R}^d)$. The results in [2, 29] are only the case that $\phi$ is almost increasing. Our results cover the case that $\phi$ is almost increasing on one area, is a constant on another area and is almost decreasing on the other area.

We denote by $\mathrm{MO}(f,B)$ the mean oscillation of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ on the ball $B$, i.e.,

$$\mathrm{MO}(f,B) = \fint_B |f(y) - f_B|\,dy = \frac{1}{|B|}\int_B |f(y) - f_B|\,dy.$$

Then Uchiyama proved the following characterization:

**Theorem 1.1** ([28]). *Let $f \in \mathrm{BMO}(\mathbb{R}^d)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{BMO}(\mathbb{R}^d)}$ if and only if $f$ satisfies the following three conditions:*

(i) $\displaystyle\lim_{r\to+0} \sup_{x\in\mathbb{R}^d} \mathrm{MO}(f, B(x,r)) = 0.$

(ii) $\displaystyle\lim_{r\to\infty} \sup_{x\in\mathbb{R}^d} \mathrm{MO}(f, B(x,r)) = 0.$

(iii) $\displaystyle\lim_{|y|\to\infty} \mathrm{MO}(f, B(x+y,r)) = 0$ *for each ball $B(x,r)$.*

The above characterization was first mentioned by Neri [20, Remark 2.6] without proof, in which the above three conditions were referred to as CMO, continuous mean oscillation. By [2] Theorem 1.1 was extended to $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ in the case that $\phi$ is almost increasing. We entend it to a wider class of $\phi$.

The organization of this report is as follows. In Section 2 we state the definitions and known properties on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and $H^{[\phi,q]}(\mathbb{R}^d)$. Then we state the main results in Section 3, that is, the duality (1.2) and the characterization of $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$. As applications of the characterization we prove the boundedness of singular and fractional integral operators on $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ in Sections 4 and 5, respectively.

At the end of this section, we make some conventions. Throughout this report, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_p$, is dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

## 2 Definitions and known properties

In this section we state the definitions and known properties on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and $H^{[\phi,q]}(\mathbb{R}^d)$.

For a ball $B = B(x,r)$ and a positive constant $k$ we denote $B(x,kr)$ by $kB$. For a measurable set $G \subset \mathbb{R}^d$, we denote by $|G|$ and $\chi_G$ the Lebesgue measure of $G$ and the characteristic function of $G$, respectively.

We say that a function $\theta : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ satisfies the doubling condition (resp. nearness condition) if there exists a positive constant $C$ such that, for all $x, y \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(x,s)} \le C, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2, \tag{DC}$$

$$\left(\text{resp. } \frac{1}{C} \le \frac{\theta(x,r)}{\theta(y,r)} \le C, \quad \text{if } |x-y| \le r\right). \tag{NC}$$

We say that $\theta$ is almost increasing (resp. almost decreasing) if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$\theta(x,r) \le C\theta(x,s), \quad \text{if } r < s \tag{AI}$$

$$(\text{resp. } C\theta(x,r) \ge \theta(x,s), \quad \text{if } r < s). \tag{AD}$$

For two functions $\theta, \kappa : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, we write $\theta \sim \kappa$ if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^d$ and $r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\kappa(x,r)} \le C. \tag{2.1}$$

Let $1 \le p < \infty$ and $\phi, \tilde{\phi} : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$. If $\phi \sim \tilde{\phi}$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathcal{L}_{p,\tilde{\phi}}(\mathbb{R}^d)$ with equivalent norms.

In this report we consider the following class of $\phi$:

**Definition 2.1.** (i) For $p \in [1,\infty)$, let $\mathcal{G}_p$ be the set of all functions $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ such that $r \mapsto r^{d/p}\phi(x,r)$ is almost increasing and that $r \mapsto \phi(x,r)/r$ is almost decreasing. That is, there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^d$ and $r, s \in (0,\infty)$,

$$r^{d/p}\phi(x,r) \le Cs^{d/p}\phi(x,s), \quad C\phi(x,r)/r \ge \phi(x,s)/s, \quad \text{if } r < s.$$

(ii) Let $\mathcal{G}^{\mathrm{inc}}$ be the set of all functions $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ such that $\phi$ is almost increasing and that $r \mapsto \phi(x,r)/r$ is almost decreasing.

*Remark* 2.1. If $\phi \in \mathcal{G}_p$ or $\phi \in \mathcal{G}^{\mathrm{inc}}$, then $\phi$ satisfies the doubling condition (DC). If $1 < p < q < \infty$, then $\mathcal{G}^{\mathrm{inc}} \subsetneqq \mathcal{G}_q \subsetneqq \mathcal{G}_p \subsetneqq \mathcal{G}_1$. Actually, letting $\phi_\lambda(x,r) = r^\lambda$, we have $\phi_{-d/q} \in \mathcal{G}_q \setminus \mathcal{G}^{\mathrm{inc}}$, $\phi_{-d/p} \in \mathcal{G}_p \setminus \mathcal{G}_q$ and $\phi_{-d} \in \mathcal{G}_p \setminus \mathcal{G}_1$. We also note that $\mathcal{L}_{q,\phi_{-d/p}}(\mathbb{R}^d) = \{\text{constant functions}\}$, since, if $f \in \mathcal{L}_{q,\phi_{-d/p}}(\mathbb{R}^d)$, then

$$\int_{B(0,r)} |f - f_{B(0,r)}| \le \|f\|_{\mathcal{L}_{q,\phi_{-d/p}}} |B(0,r)|^{d/q} \phi_{-d/p}(x,r) \to 0 \quad \text{as } r \to \infty,$$

and $\lim_{r\to\infty} f_{B(0,r)}$ exists, see [14, Lemma 3.2].

*Remark* 2.2. It is known that, if $\phi$ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies (NC), then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathcal{L}_{1,\phi}(\mathbb{R}^d)$ with equivalent norms for each $p \in (1,\infty)$, see [15, Theorem 3.1].

Let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1, \\ r^{\alpha_*}, & 1 \le r < \infty, \end{cases} \tag{2.2}$$

where $\alpha(\cdot) : \mathbb{R}^d \to (-\infty,\infty)$ and $\alpha_* \in (-\infty,\infty)$. Let

$$\alpha_+ = \inf_{x\in\mathbb{R}^d} \alpha(x), \quad \alpha_- = \sup_{x\in\mathbb{R}^d} \alpha(x).$$

If $-d/p \le \alpha_- \le \alpha_+ \le 1$ and $-d/p \le \alpha_* \le 1$, then $\phi$ is in $\mathcal{G}_p$. If $0 \le \alpha_- \le \alpha_+ \le 1$ and $0 \le \alpha_* \le 1$, then $\phi$ is in $\mathcal{G}^{\mathrm{inc}}$. If $-\infty < \alpha_- \le \alpha_+ < \infty$ and $\alpha(\cdot)$ is log-Hölder continuous, that is, there exists a positive constant $C$ such that, for all $x, y \in \mathbb{R}^d$,

$$|\alpha(x) - \alpha(y)| \le \frac{C}{\log(e/|x-y|)} \quad \text{if} \quad 0 < |x-y| < 1,$$

then $\phi$ satisfies (NC), see [16, Proposition 3.3].

Moreover, for $\alpha(\cdot) : \mathbb{R}^d \to [0,\infty)$ and $\alpha_* \in [0,\infty)$, let $\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^d)$ be the set of all functions $f$ such that the following functional is finite:

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}} = \max\left\{ \sup_{0<|x-y|<1} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)} + |x-y|^{\alpha(y)}}, \sup_{|x-y|\ge 1} \frac{|f(x)-f(y)|}{|x-y|^{\alpha_*}} \right\},$$

see [16, Definition 2.2]. If $0 < \alpha_- \le \alpha_+ \le 1$, $0 < \alpha_* \le 1$, and $\alpha(\cdot)$ is log-Hölder continuous, then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathcal{L}_{1,\phi}(\mathbb{R}^d) = \mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^d) \tag{2.3}$$

with equivalent norms, see [16, Corollary 3.5].

Next we state the definition of $H^{[\phi,q]}(\mathbb{R}^d)$.

**Definition 2.2** ($[\phi,q]$-atom)**.** Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ and $1 < q \le \infty$. A function $a$ on $\mathbb{R}^d$ is called a $[\phi,q]$-atom if there exists a ball $B$ such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_{L^q} \le \dfrac{1}{|B|^{1/q'}\phi(B)}$,

(iii) $\displaystyle\int_{\mathbb{R}^d} a(x)\,dx = 0$,

where $\|a\|_{L^q}$ is the $L^q$ norm of $a$ and $1/q + 1/q' = 1$. We denote by $A[\phi, q]$ the set of all $[\phi, q]$-atoms.

If $a$ is a $[\phi, q]$-atom and a ball $B$ satisfies (i)–(iii), then

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} a(x)g(x)\,dx\right| &= \left|\int_B a(x)(g(x) - g_B)\,dx\right| \\
&\le \|a\|_{L^q}\left(\int_B |g(x) - g_B|^{q'}\,dx\right)^{1/q'} \\
&\le \frac{1}{\phi(B)}\left(\frac{1}{|B|}\int_B |g(x) - g_B|^{q'}\,dx\right)^{1/q'} \\
&\le \|g\|_{\mathcal{L}_{q',\phi}}.
\end{aligned}$$

That is, the mapping $g \mapsto \int ag$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(\mathbb{R}^d)$ with norm not exceeding 1.

**Definition 2.3** ($H^{[\phi,q]}(\mathbb{R}^d)$)**.** Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, $1 < q \le \infty$ and $1/q + 1/q' = 1$. Assume that $\mathcal{L}_{q',\phi}(\mathbb{R}^d) \ne \{0\}$. We define the space $H^{[\phi,q]}(\mathbb{R}^d) \subset (\mathcal{L}_{q',\phi}(\mathbb{R}^d))^*$ as follows:

> $f \in H^{[\phi,q]}(\mathbb{R}^d)$ if and only if there exist sequences $\{a_j\} \subset A[\phi, q]$ and positive numbers $\{\lambda_j\}$ such that
>
> $$f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(\mathbb{R}^d))^* \quad \text{and} \quad \sum_j \lambda_j < \infty. \tag{2.4}$$

In general, the expression (2.4) is not unique. We define

$$\|f\|_{H^{[\phi,q]}} = \inf\left\{\sum_j \lambda_j\right\},$$

where the infimum is taken over all expressions as in (2.4). Then $\|f\|_{H^{[\phi,q]}}$ is a norm and $H^{[\phi,q]}(\mathbb{R}^d)$ is a Banach space.

*Remark* 2.3. If $\phi$ is in $\mathcal{G}_p$ and satisfies (NC), then $C^\infty_{\mathrm{comp}}(\mathbb{R}^d) \subset \mathcal{L}_{p,\phi}(\mathbb{R}^d)$, see [29, Proposition 6.4].

*Remark* 2.4. If $\phi \equiv 1$ and $q = \infty$, then $H^{[\phi,q]}(\mathbb{R}^d)$ is the usual Hardy space $H^1(\mathbb{R}^d)$. If $\phi$ is in $\mathcal{G}^{\rm inc}$ and satisfies (NC), then $H^{[\phi,q]}(\mathbb{R}^d) = H^{[\phi,\infty]}(\mathbb{R}^d)$ with equivalent norms for each $q \in (1,\infty)$, see [15, Theorem 3.3].

At the end of this section we state the known duality.

**Definition 2.4.** Denote by $H_0^{[\phi,q]}(\mathbb{R}^d)$ the space of all finite linear combinations of $[\phi,q]$-atoms and fix $b \in \mathcal{L}_{q',\phi}(\mathbb{R}^d)$. We define a linear functional

$$L_b(g) = \int_{\mathbb{R}^d} g(x)b(x)\,dx, \quad g \in H_0^{[\phi,q]}(\mathbb{R}^d) \tag{2.5}$$

as an absolutely convergent integral.

**Theorem 2.1** ([15]). *Assume that $\phi, q$ satisfy the conditions in Definition 2.3. Then*

$$\left(H^{[\phi,q]}(\mathbb{R}^d)\right)^* = \mathcal{L}_{q',\phi}(\mathbb{R}^d).$$

*More precisely, given $b \in \mathcal{L}_{q',\phi}(\mathbb{R}^d)$, the linear functional $L_b$ defined by Definition 2.4 can be extended on the entire $H^{[\phi,q]}(\mathbb{R}^d)$. Conversely, for every bounded linear functional $L$ on $H^{[\phi,q]}(\mathbb{R}^d)$ there exists $b \in \mathcal{L}_{q',\phi}(\mathbb{R}^d)$ such that for all $f \in H_0^{[\phi,q]}(\mathbb{R}^d)$ we have $L(f) = L_b(f)$. The norm $\|L_b\|_{(H^{[\phi,q]})^*}$ is equivalent to $\|b\|_{\mathcal{L}_{q',\phi}}$.*

If $\phi \equiv 1$ and $q = \infty$, then the above duality means $\left(H^1(\mathbb{R}^d)\right)^* = \mathrm{BMO}(\mathbb{R}^d)$. Zorko [**?**] considered the above duality in the case that $\phi : (0,\infty) \to (0,\infty)$ is decreasing.

# 3 Main results

To state the main results we need the following conditions:

$$\lim_{r\to+0} \inf_{x\in B(0,M)} \frac{\phi(x,r)}{r} = \infty \quad \text{for each } M > 0, \tag{3.1}$$

$$\lim_{r\to\infty} r^{d/p}\phi(0,r) = \infty. \tag{3.2}$$

By the condition (NC) on $\phi$, the condition (3.2) is equivalent to

$$\lim_{r\to\infty} r^{d/p}\phi(x,r) = \infty \quad \text{for each } x \in \mathbb{R}^d.$$

Then our first result is the following:

**Theorem 3.1.** *Let $p \in [1,\infty)$, $q \in (1,\infty]$ and $1/p + 1/q = 1$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC), (3.1) *and* (3.2). *Then*

$$\left(\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}\right)^* = H^{[\phi,q]}(\mathbb{R}^d). \tag{3.3}$$

*More precisely, for $f \in H^{[\phi,q]}(\mathbb{R}^d)$, the linear functional*

$$\langle f, v\rangle = \int_{\mathbb{R}^d} f(x)v(x)\,dx, \quad v \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d) \tag{3.4}$$

*can be extended on $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$. Conversely, each bounded linear functional on $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ has the form* (3.4)*, for some $f \in H^{[\phi,q]}(\mathbb{R}^d)$. The linear functional norm is equivalent to $\|f\|_{H^{[\phi,q]}}$.*

If $p = 1$, $q = \infty$ and $\phi \equiv 1$, then the above duality means

$$\left(\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{BMO}(\mathbb{R}^d)}\right)^* = H^1(\mathbb{R}^d).$$

Let $\phi$ be defined by (2.2) with $-d/p < \alpha_- \le \alpha_+ < 1$ and $-d/p < \alpha_* < 1$. If $\alpha(\cdot)$ is log-Hölder continuous also, then $\phi$ is in $\mathcal{G}_p$ and satisfies (NC), (3.1) and (3.2). In this case we have the duality (3.3) as a corollary of Theorem 3.1. Moreover, if $\alpha_- > 0$ and $\alpha_* > 0$, then

$$\left(\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^d)}\right)^* = H^{[\phi,\infty]}(\mathbb{R}^d).$$

For a measurable function $f$ and a ball $B$, we denote by $\mathrm{MO}_p(f,B)$ the $p$-th mean oscillation of $f$ on $B$, that is,

$$\mathrm{MO}_p(f,B) = \left(\fint_B |f(y) - f_B|^p\,dy\right)^{1/p}. \tag{3.5}$$

Then our second result is the following:

**Theorem 3.2.** *Let $p \in [1,\infty)$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC)*,* (3.1) *and* (3.2)*. Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ if and only if $f$ satisfies the following three conditions:*

(i) $\displaystyle\lim_{r\to+0}\sup_{x\in\mathbb{R}^d}\frac{\mathrm{MO}_p(f,B(x,r))}{\phi(x,r)} = 0.$

(ii) $\displaystyle\lim_{r\to\infty}\sup_{x\in\mathbb{R}^d}\frac{\mathrm{MO}_p(f,B(x,r))}{\phi(x,r)} = 0.$

(iii) $\displaystyle\lim_{|x|\to\infty}\frac{\mathrm{MO}_p(f,B(x,r))}{\phi(x,r)} = 0$ *for each $r > 0$.*

*Remark* 3.1. We do not need (3.1) and (3.2) to prove that, if $f$ satisfies (i)–(iii), then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$. Conversely, if $\phi(x,r) = r$ which does not satisfy (3.1), then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = \mathrm{Lip}_1(\mathbb{R}^d)$ and (i) of Theorem 3.2 fails for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R})$ such that $f(x) = x$ near the origin, since $\int_{-r}^r f = 0$ and $r^{-1}\left(\fint_{-r}^r |f|^p\right)^{1/p} \sim 1$ as $r \to +0$, see [11, An example on page 39]. Also, if $\phi(x,r) = r^{-d/p}$ which does not satisfy (3.2), then $\mathcal{L}_{p,\phi}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ modulo constant functions and (ii) of Theorem 3.2 fails for any odd function $f \neq 0$ in $L^p(\mathbb{R}) = \overline{C^\infty_{\mathrm{comp}}(\mathbb{R})}^{L^p(\mathbb{R})}$, since $\int_{-r}^r f = 0$ and $\left(\int_{-r}^r |f|^p\right)^{1/p} \to \|f\|_{L^p}$ as $r \to \infty$.

If $p = 1$ and $\phi \equiv 1$, then Theorem 3.2 is the same as Theorem 1.1. If $\phi(x,r) = r^\alpha$, $\alpha > 0$, then we have the characterization of $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{Lip}_\alpha(\mathbb{R}^d)}$ as a corollary of Theorem 3.2, which was proven by [21].

As another corollary, we consider the case of (2.2) (see (2.3) also).

**Corollary 3.3.** *Let $p \in [1,\infty)$. Let $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ be defined by* (2.2)*. Assume that $-d/p < \alpha_- \le \alpha_+ < 1$, $-d/p < \alpha_* < 1$ and that $\alpha(\cdot)$ is log-Hölder continuous. Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ if and only if $f$ satisfies the following three conditions:*

(i) $\displaystyle \lim_{r\to+0} \sup_{x\in\mathbb{R}^d} \frac{\mathrm{MO}_p(f,B(x,r))}{r^{\alpha(x)}} = 0.$

(ii) $\displaystyle \lim_{r\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\mathrm{MO}_p(f,B(x,r))}{r^{\alpha_*}} = 0.$

(iii) $\displaystyle \lim_{|x|\to\infty} \mathrm{MO}_p(f,B(x,r)) = 0$ *for each $r > 0$.*

*Moreover, if $\alpha_- > 0$ and $\alpha_* > 0$, then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^d)}$ if and only if $f$ satisfies the above three conditions.*

Under the assumptions that $\phi \in \mathcal{G}^{\mathrm{inc}}$, $p = 1$ and that

$$\lim_{r\to\infty} \inf_{x\in\mathbb{R}^d} r^d \phi(x,r) = \infty, \tag{3.6}$$

Theorems 3.1 and 3.2 were proven in [29] and [2], respectively. In this report we base on the same methods to prove Theorems 3.1 and 3.2. However, since $\mathcal{G}^{\mathrm{inc}} \subsetneqq \mathcal{G}_p$ (see Remark 2.1) and the condition (3.2) is weaker than (3.6) (see Remark 3.3 below), we need to check precisely whether the methods in [29] and [2] are adaptable or not to our cases. Actually, several parts are not adaptable and we need to modify them. Therefore, we state precisely the proofs of Theorems 3.1 and 3.2 in the following sections. Moreover, we will show the following two propositions to prove Theorem 3.2. Proposition 3.5 is useful to prove the boundedness of operators on $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$.

**Proposition 3.4.** *Let $p \in [1,\infty)$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC), (3.1) *and* (3.2)*. If $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$, then $f$ satisfies* (i)–(iii) in Theorem 3.2.

**Proposition 3.5.** *Let $p \in [1, \infty)$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC). *Let $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$. Assume that, for some constant $A \in [0, \infty)$, $f$ satisfies the following three conditions:*

(A-1) $\displaystyle \limsup_{r \to +0} \sup_{x \in \mathbb{R}^d} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} \le A.$

(A-2) $\displaystyle \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} \le A.$

(A-3) $\displaystyle \limsup_{|x| \to \infty} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} \le A$ *for each* $r > 0$.

*Then*

$$\inf_{g \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)} \|f - g\|_{\mathcal{L}_{p,\phi}} \le CA, \tag{3.7}$$

*where the constant $C$ is independent of $f$ and $A$.*

*Remark* 3.2. From Proposition 3.5 we see that

$$\begin{aligned} d(f, C^\infty_{\mathrm{comp}}(\mathbb{R}^d)) \le C\Bigg( &\limsup_{r \to +0} \sup_{x \in \mathbb{R}^d} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} \\ &+ \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} + \sup_{r>0} \limsup_{|x| \to \infty} \frac{\mathrm{MO}_p(f, B(x,r))}{\phi(x,r)} \Bigg), \end{aligned}$$

where $d(f, C^\infty_{\mathrm{comp}}(\mathbb{R}^d))$ is the distance between $f$ and $C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$.

*Remark* 3.3. Let $\phi(x,r) = r^\theta/(r + |x|)^{\theta/2}$, $0 < \theta < 1$. Then $\phi$ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies (NC), (3.1) and (3.2). However $\phi$ does not satisfy (3.6).

# 4 Singular integral operators

In this section we prove the boundedness of the singular integral operators. We denote by $L^p_{\mathrm{comp}}(\mathbb{R}^d)$ the set of all $f \in L^p(\mathbb{R}^d)$ with compact support. Let $0 < \kappa \le 1$. We shall consider a singular integral operator $T$ with measurable kernel $K$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following properties:

$$|K(x,y)| \le \frac{C}{|x-y|^d} \quad \text{for} \quad x \ne y, \tag{4.1}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \frac{C}{|x-y|^d} \left( \frac{|x-z|}{|x-y|} \right)^\kappa \quad \text{for} \quad |x-y| \ge 2|x-z|, \tag{4.2}$$

$$\int_{r \le |x-y| < R} K(x,y)\, dy = \int_{r \le |x-y| < R} K(y,x)\, dy = 0 \quad \text{for } 0 < r < R < \infty \text{ and } x \in \mathbb{R}^d, \tag{4.3}$$

where $C$ is a positive constant independent of $x, y, z \in \mathbb{R}^d$. For $\eta > 0$, let

$$T_\eta f(x) = \int_{|x-y|\geq\eta} K(x,y) f(y)\, dy.$$

Then $T_\eta f(x)$ is well defined for $f \in L^p_{\mathrm{comp}}(\mathbb{R}^d)$, $1 < p < \infty$. We assume that, for all $1 < p < \infty$, there exists positive constant $C_p$ independently $\eta > 0$ such that,

$$\|T_\eta f\|_{L^p} \leq C_p \|f\|_{L^p} \quad \text{for} \quad f \in L^p_{\mathrm{comp}}(\mathbb{R}^d),$$

and $T_\eta f$ converges to $Tf$ in $L^p(\mathbb{R}^d)$ as $\eta \to +0$. By this assumption, the operator $T$ can be extended as a continuous linear operator on $L^p(\mathbb{R}^d)$. We shall say the operator $T$ satisfying the above conditions is a singular integral operator of type $\kappa$. For example, Riesz transforms are singular integral operators of type 1.

Now, to define $T$ for functions $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$, we first define the modified version of $T_\eta$ by

$$\tilde{T}_\eta f(x) = \int_{|x-y|\geq\eta} f(y)\big[K(x,y) - K(0,y)(1 - \chi_{B(0,1)}(y))\big]\, dy. \tag{4.4}$$

Then we can show that the integral in the definition above converges absolutely for each $x$ and that $\tilde{T}_\eta f$ converges in $L^p(B)$ as $\eta \to +0$ for each ball $B$ (see the proof of [16, Theorem 4.1]). We denote the limit by $\tilde{T}f$. If $f$ is the constant function 1, then $\tilde{T}1 \equiv 0$. Actually,

$$\begin{aligned}
\tilde{T}_\eta 1(x) &= \int_{\mathbb{R}^d \setminus B(x,\eta)} \big[K(x,y)\chi_{B(x,1)}(y)\big]\, dy \\
&\quad + \int_{\mathbb{R}^d \setminus B(x,\eta)} \big[K(x,y)(1 - \chi_{B(x,1)}(y)) - K(0,y)(1 - \chi_{B(0,1)}(y))\big]\, dy \\
&= \int_{B(x,\eta)} \big[-K(0,y)(1 - \chi_{B(0,1)}(y))\big]\, dy \to 0 \quad \text{as} \quad \eta \to +0,
\end{aligned}$$

since

$$\int_{\mathbb{R}^d \setminus B(x,\eta)} \big[K(x,y)\chi_{B(x,1)}(y)\big]\, dy = \int_{B(x,1)\setminus B(x,\eta)} K(x,y)\, dy = 0,$$

and

$$\int_{\mathbb{R}^d} \big[K(x,y)(1 - \chi_{B(x,1)}(y)) - K(0,y)(1 - \chi_{B(0,1)}(y))\big]\, dy = 0.$$

Therefore, $\tilde{T}$ is well defined on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ which is a space modulo constant functions. Further, if both $Tf$ and $\tilde{T}f$ are well defined, then $\tilde{T}f - Tf$ is a constant function. Then it is enough to consider only $Tf$ instead of $\tilde{T}f$ for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$.

The following theorem is known:

**Theorem 4.1** ([16])**.** *Let $T$ be a singular integral operator of type $\kappa \in (0,1]$, $p \in [1,\infty)$ and $\phi$ satisfy* (DC)*. Assume that there exists a positive constant $A$ such that, for all $x \in \mathbb{R}^d$ and $r > 0$,*

$$r^\kappa \int_r^\infty \frac{\phi(x,t)}{t^{1+\kappa}}\,dt \le A\phi(x,r). \tag{4.5}$$

*Then $\tilde{T}$ is bounded on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ for $p \in (1,\infty)$. Moreover, if $\phi$ is almost increasing and satisfies the nearness condition* (NC)*, then $\tilde{T}$ is bounded on $\mathcal{L}_{1,\phi}(\mathbb{R}^d)$.*

We have the following theorem:

**Theorem 4.2.** *Let $T$ be a singular integral operator of type $\kappa \in (0,1]$. Let $p \in [1,\infty)$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC) *and* (4.5)*. If $p = 1$, then assume that $\phi \in \mathcal{G}^{\mathrm{inc}}$ also. Assume that*

$$\inf_{x \in \mathbb{R}^d} \phi(x,1) > 0. \tag{4.6}$$

*Then $T$ is bounded on $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$.*

In a similar way to [29, 30] we can apply Theorem 4.2 to the dual and bidual operators of $T$. In general, if a linear operator $T$ is bounded from a normed linear space $X$ to a normed linear space $Y$, then the dual operator $T^*$ is bounded from $Y^*$ to $X^*$, where $X^*$ and $Y^*$ are the dual spaces of $X$ and $Y$, respectively, see [33, Theorem 2' (p. 195)]. This idea was used by [22, 23, 24] for Morrey spaces.

**Theorem 4.3.** *Let $p \in [1,\infty)$, and let $\phi$ be in $\mathcal{G}_p$ and satisfy* (NC)*,* (4.5) *and* (4.6)*. If $p = 1$, then assume that $\phi \in \mathcal{G}^{\mathrm{inc}}$ also. Let the kernel $K$ satisfy* (4.1)*–*(4.3)*, and let $K^t(x,y) = K(y,x)$. Assume that $T$ and $T^t$ are singular integral operators with kernel $K$ and $K^t$ of type $\kappa \in (0,1]$, respectively.*

(i) *The dual operator $T^*$ of $T$ coincides with $T^t$ on $H^{[\phi,p']}(\mathbb{R}^d)$. Consequently, singular integral operators of type $\kappa \in (0,1]$ are bounded on $H^{[\phi,p']}(\mathbb{R}^d)$.*

(ii) *The bidual operator $T^{**}$ of $T$ coincides with $\tilde{T}$ on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$. Consequently, $\tilde{T}$ is a bounded linear operator on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$.*

The boundedness of the singular integral operators on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and its predual, see [16] and [17], respectively. See also [3] for Campanato-type spaces based on so-called ball Banach function spaces.

# 5 Fractional integral operators

Let $I_\alpha$ be the fractional integral operator of order $\alpha > 0$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,dy, \quad x \in \mathbb{R}^d.$$

It is well known as the Hardy-Littlewood-Sobolev theorem that $I_\alpha$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if $\alpha \in (0,d)$, $p,q \in (1,\infty)$ and $-d/p + \alpha = -d/q$.

To define the fractional integral operator $I_\alpha$ on the generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$, we define the modified version of $I_\alpha$ by

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^d} f(y)\left(\frac{1}{|x-y|^{d-\alpha}} - \frac{1-\chi_{B(0,1)}(y)}{|y|^{d-\alpha}}\right) dy.$$

If $\alpha \in (0,1)$, then the integral above converges for each $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^d)$. Moreover, if $f$ is the constant function 1, then $\tilde{I}_\alpha 1$ is a constant function, since

$$\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|z-y|^{d-\alpha}}$$

is integrable on $\mathbb{R}^d$ as a function of $y$, for every choice of $x$ and $z$, and

$$\int_{\mathbb{R}^d} \left(\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|z-y|^{d-\alpha}}\right) dy = 0,$$

see [13, Lemma 4,2] or [30, Lemma 4.1], for example. Therefore, $\tilde{I}_\alpha$ is well defined on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ which is a space modulo constant functions, see [16] for details. Further, if both $I_\alpha f$ and $\tilde{I}_\alpha f$ are well defined, then $\tilde{I}_\alpha f - I_\alpha f$ is a constant function. Then it is enough to consider only $I_\alpha f$ instead of $\tilde{I}_\alpha f$ for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$.

The following theorem is known:

**Theorem 5.1** ([16])**.** *Let $\alpha \in (0,1)$, $p,q \in [1,\infty)$, and let $\phi$ and $\psi$ satisfy* (DC). *Assume that there exists a positive constant $A$ such that, for all $x \in \mathbb{R}^d$ and $r \in (0,\infty)$,*

$$\int_r^\infty \frac{t^\alpha \phi(x,t)}{t^2}\,dt \le A\frac{\psi(x,r)}{r}. \tag{5.1}$$

*If $p = 1$ and $1 \le q < d/(d-\alpha)$, if $1 < p < d/\alpha$ and $1 \le q \le dp/(d-\alpha p)$, or if $d/\alpha \le p < \infty$ and $1 \le q < \infty$, then $\tilde{I}_\alpha$ is bounded from $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ to $\mathcal{L}_{q,\psi}(\mathbb{R}^d)$.*

We have the following theorem:

**Theorem 5.2.** *Let $\alpha \in (0,1)$ and $p,q \in [1,\infty)$, let $\phi \in \mathcal{G}_p$ and $\psi \in \mathcal{G}_q$, and let $\phi$ and $\psi$ satisfy* (NC) *and* (4.5) *with $\kappa = 1$. If $p = 1$, then assume also that $\phi \in \mathcal{G}_{p_0}$*

*for some* $p_0 \in (1,\infty)$*. Assume* (4.6) *and* (5.1)*. If* $p = 1$ *and* $1 \le q < d/(d-\alpha)$*, if* $1 < p < d/\alpha$ *and* $1 \le q \le dp/(d-\alpha p)$*, or if* $d/\alpha \le p < \infty$ *and* $1 \le q < \infty$*, then* $I_\alpha$ *is bounded from* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{q,\psi}(\mathbb{R}^d)}$*. Moreover, if* $\phi, \psi \in \mathcal{G}^{\rm inc}$*, then, for any* $p, q \in [1,\infty)$*,* $I_\alpha$ *is bounded from* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{q,\psi}(\mathbb{R}^d)}$*.*

**Corollary 5.3.** *Let* $\alpha \in (0,1)$ *and*

$$\phi(x,r) = \begin{cases} r^{\beta(x)}, & 0 < r < 1, \\ r^{\beta_*}, & 1 \le r < \infty, \end{cases} \qquad \psi(x,r) = \begin{cases} r^{\gamma(x)}, & 0 < r < 1, \\ r^{\gamma_*}, & 1 \le r < \infty, \end{cases} \tag{5.2}$$

*where* $\beta(\cdot), \gamma(\cdot) : \mathbb{R}^d \to (-\infty,\infty)$ *and* $\beta_*, \gamma_* \in (-\infty,\infty)$*. Assume that* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *are log-Hölder continuous. Let* $p, q \in [1,\infty)$*, and let* $\beta_-, \beta_+, \beta_* \in [-d/p, 1-\alpha)$*. If* $p = 1$*, then we assume also that* $\beta_-, \beta_+, \beta_* \in [-d/p_0, 1-\alpha)$ *for some* $p_0 \in (1,\infty)$*. Assume that* $\beta(x) + \alpha = \gamma(x)$ *and that* $\beta_* + \alpha = \gamma_*$*. If* $p = 1$ *and* $1 \le q < d/(d-\alpha)$*, if* $1 < p < d/\alpha$ *and* $1 \le q \le dp/(d-\alpha p)$*, or if* $d/\alpha \le p < \infty$ *and* $1 \le q < \infty$*, then* $I_\alpha$ *is bounded from* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{q,\psi}(\mathbb{R}^d)}$*. Moreover, if* $\gamma_-, \gamma_* > 0$*, then* $I_\alpha$ *is bounded from* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{\mathcal{L}_{p,\phi}(\mathbb{R}^d)}$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm Lip}^{\gamma_*}_{\gamma(\cdot)}}$*.*

**Corollary 5.4.** *Let* $\alpha \in (0,1)$*,* $\beta \in [0,1)$*,* $p \in (1,\infty)$ *and* $-d/p + \alpha = \beta$*. Then* $I_\alpha$ *is bounded from* $L^p(\mathbb{R}^d)$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm BMO}(\mathbb{R}^d)}$ *if* $\beta = 0$*, and from* $L^p(\mathbb{R}^d)$ *to* $\overline{C^\infty_{\rm comp}(\mathbb{R}^d)}^{{\rm Lip}_\beta(\mathbb{R}^d)}$ *if* $\beta \in (0,1)$*.*

**Theorem 5.5.** *Let* $\alpha \in (0,1)$ *and* $p, q \in [1,\infty)$*, let* $\phi \in \mathcal{G}_p$ *and* $\psi \in \mathcal{G}_q$*, and let* $\phi$ *and* $\psi$ *satisfy* (NC) *and* (4.5) *with* $\kappa = 1$*. If* $p = 1$*, then assume also that* $\phi \in \mathcal{G}_{p_0}$ *for some* $p_0 \in (1,\infty)$*. Assume* (4.6) *and* (5.1)*. Let* $p = 1$ *and* $1 \le q < d/(d-\alpha)$*, let* $1 < p < d/\alpha$ *and* $1 \le q \le dp/(d-\alpha p)$*, or let* $d/\alpha \le p < \infty$ *and* $1 \le q < \infty$*.*

(i) *The dual operator* $(I_\alpha)^*$ *of* $I_\alpha$ *coincides with* $I_\alpha$ *from* $H^{[\psi,q']}(\mathbb{R}^n)$ *to* $H^{[\phi,p']}(\mathbb{R}^n)$*. Consequently,* $I_\alpha$ *is a bounded linear operator from* $H^{[\psi,q']}(\mathbb{R}^n)$ *to* $H^{[\phi,p']}(\mathbb{R}^n)$*.*

(ii) *The bidual operator* $(I_\alpha)^{**}$ *of* $I_\alpha$ *coincides with* $\tilde{I}_\alpha$ *from* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$*. Consequently,* $\tilde{I}_\alpha$ *is a bounded linear operator from* $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$*.*

The boundedness of the fractional integral operators on $\mathcal{L}_{p,\phi}(\mathbb{R}^d)$ and its predual, see [16] and [17], respectively. See also [4] for Campanato-type spaces based on so-called ball Banach function spaces.

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## References


[1] R. Arai and E. Nakai, Compact commutators of Calderón-Zygmund and generalized fractional integral operators with a function in generalized Campanato spaces on generalized Morrey spaces, Tokyo J. Math. 42 (2019), No. 2, 471–496.

[2] R. Arai and E. Nakai, An extension of the characterization of CMO and its application to compact commutators on Morrey spaces, J. Math. Soc. Japan 72 (2020), No. 2, 507–539.

[3] Y. Chen, H. Jia and D. Yang, Boundedness of Calderón-Zygmund operators on ball Campanato-type function spaces, Anal. Math. Phys. 12 (2022), No. 5, Paper No. 118, 35 pp.

[4] Y. Chen, H. Jia and D. Yang, Boundedness of fractional integrals on ball Campanato-type function spaces, Bull. Sci. Math. 182 (2023), Paper No. 103210, 59 pp.

[5] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis. Bull. Amer, Math. Soc. 83(1977), 569–645.

[6] G. Dafni, Local VMO and weak convergence in $h^1$, Canad. Math. Bull. 45 (2002), No. 1, 46–59.

[7] C. Fefferman, Characterizations of bounded mean oscillation, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[8] C. Fefferman and E. M. Stein, $H^p$ spaces of several variables, Acta Math. 129 (1972), No. 3–4, 137–193.

[9] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Math. Stud. 116, North-Holland, Amsterdam, 1985.

[10] L. Grafakos, Modern Fourier analysis. Third edition. Graduate Texts in Mathematics, 250. Springer, New York, 2014.

[11] W. Guo, J. He, H. Wu and D. Yang, Boundedness and compactness of commutators associated with Lipschitz functions, Anal. Appl. (Singap.) 20 (2022), No. 1, 35–71.

[12] E. Nakai, Pointwise multipliers for functions of weighted bounded mean oscillation, Studia Math. 105 (1993), No. 2, 105–119.

[13] E. Nakai, On generalized fractional integrals, Taiwanese J. Math. 5 (2001), No. 3, 587–602.

[14] E. Nakai, The Campanato, Morrey and Hölder spaces on spaces of homogeneous type, Studia Math. 176 (2006), No. 1, 1–19.

[15] E. Nakai, A generalization of Hardy spaces $H^p$ by using atoms, Acta Math. Sin. (Engl. Ser.) 24 (2008), No. 8, 1243–1268.

[16] E. Nakai, Singular and fractional integral operators on Campanato spaces with variable growth conditions, Rev. Mat. Complut. 23 (2010), No. 2, 355–381.

[17] E. Nakai, Singular and fractional integral operators on preduals of Campanato spaces with variable growth condition, Sci. China Math. 60, No. 11, 2219–2240.

[18] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal. 262 (2012), No. 9, 3665–3748.

[19] E. Nakai and K. Yabuta, Pointwise multipliers for functions of bounded mean oscillation, J. Math. Soc. Japan, 37 (1985), 207–218.

[20] U. Neri, Fractional integration on the space $H^1$ and its dual. Studia Math. 53 (1975), No. 2, 175–189.

[21] T. Nogayama and Y. Sawano, Compactness of the commutators generated by Lipschitz functions and fractional integral operators, Mat. Zametki 102 (2017), No. 5, 749–760; translation in Math. Notes 102 (2017), No. 5-6, 687–697.

[22] M. Rosenthal and H. Triebel, Calderŕn-Zygmund operators in Morrey spaces, Rev. Mat. Complut. 27 (2014), No. 1, 1–11.

[23] M. Rosenthal and H. Triebel, Morrey spaces, their duals and preduals, Rev. Mat. Complut. 28 (2015), No. 1, 1–30.

[24] Y. Sawano and S. R. El-Shabrawy, Weak Morrey spaces with applications. Math. Nachr. 291 (2018), No. 1, 178–186.

[25] J. Tao, Q. Xue, D. Yang and W. Yuan, XMO and weighted compact bilinear commutators J. Fourier Anal. Appl. 27 (2021), no. 3, Paper No. 60, 34 pp.

[26] J. Tao, D. Yang and D. Yang, A new vanishing Lipschitz-type subspace of BMO and compactness of bilinear commutators, Math. Ann. 386 (2023), No. 1-2, 495–531.

[27] R. Torres and Q. Xue, On compactness of commutators of multiplication and bilinear pseudodifferential operators and a new subspace of BMO, Rev. Mat. Iberoam. 36 (2020), No. 3, 939–956.

[28] A. Uchiyama, On the compactness of operators of Hankel type, Tôhoku Math. J. (2) 30 (1978), No. 1, 163–171.

[29] S. Yamaguchi, An extension of the VMO-$H^1$ duality, J. Math. Soc. Japan 75 (2023), No. 1, 1–19.

[30] S. Yamaguchi and E. Nakai, Generalized fractional integral operators on Campanato spaces and their bi-preduals, Math. J. Ibaraki Univ. 53 (2021), 17–34.

[31] S. Yamaguchi and E. Nakai, Compactness of commutators of integral operators with functions in Campanato spaces on Orlicz-Morrey spaces, J. Fourier Anal. Appl. 28 (2022), No. 2, Paper No. 33, 32 pp.

[32] S. Yamaguchi, E. Nakai and K. Shimomura, Bi-predual spaces of generalized Campanato spaces with variable growth condition, to appear in Acta Math. Sin. (Engl. Ser.)

[33] K. Yosida, Functional analysis. Sixth edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 123. Springer-Verlag, Berlin-New York, 1980.