Restrictions and Extensions of RKHSs by Means of Bounded Linear Operators Using the Tikhonov Regularization

Saburou Saitoh
Institute of Reproducing Kernels,
saburou.saitoh@gmail.com

January 30, 2024

Abstract: In this note, we shall consider the estimates of the norms of reproducing kernel Hilbert spaces on a domain and on its subset, and at the same time, restrictions and extensions of the functions by means of bounded linear operators using the Tikhonov regularization. Here, the Bergman spaces and Sobolev Hilbert spaces may be considered as typical examples and the method will have a general nature for reproducing kernel Hilbert spaces (RKHSs). We see many open problems and the up-to-date related information.

Key Words: Extension and restriction of a function, Bergman space, Sobolev space, Fourier transform, reproducing kernel, bounded linear operator, analytic extension, Tikhonov regularization, norm estimate, norm inequality.

2010 Mathematics Subject Classification: Primary 30C40, 44A05, 35A22; Secondly 46E22, 45A05

In this note:

1) Bounded linear operator equations on a reproducing kernel Hilbert space may be solved analytically and numerically. Its importance and essence are introduced.

2) Restriction and extension of functions in a reproducing kernel space are discussed with a general theory and with concrete examples.

3) Very concrete new results and open problems are introduced in Sobolev spaces and analytic functions. (**MAIN PART**)

4) Very recent results and information related are introduced.

1 Introduction

At first, we shall consider the estimates of the norms of Bergman spaces on a domain and on its subset and at the same time, restrictions and extensions of the Bergman functions by means of bounded linear operators using the Tikhonov regularization.

For a simple example, on the Bergman space on the entire complex plane $\mathbb{C}$ comprising of analytic functions $f(z)$ with finite integrals, for any fixed $t > 0$

$$\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(z)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy < \infty, z = x + iy,$$

from the identity ([16], [18])

$$\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 dx, \quad (1.1)$$

we have the inequality

$$\int_{\mathbb{R}} |f(x)|^2 dx \leq \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy, \quad (1.2)$$

that is, the restriction $f(x)$ of $f(z)$ to the real line is a bounded linear operator from the Bergman space to the $L_2(\mathbb{R})$ space. Our basic interests are such an inequality and operator properties for the derived bounded linear operator; in particular, we will examine the relation of the functions $f(x)$ on $\mathbb{R}$ and $f(z)$ on $\mathbb{C}$.

Here, the Bergman spaces may be considered as a typical example and the method will have a general nature for reproducing kernel Hilbert spaces (RKHSs).

As a basic tool, we use bounded linear operators using reproducing kernel Hilbert spaces and the Tikhonov regularization for bounded linear operator equations; this method is a powerful and general one in analytical and numerical viewpoints that was proved by the famous and difficult real inversion formula of the Laplace transform and many concrete examples by applying the **infinite precision method** by H. Fujiwara.

In order to state this note in a self-contained way, at the first part of the paper, we shall state their basic properties compactly that are taken from the book ([18]). Then, we shall apply them to the concrete cases like the Bergman spaces and the Sobolev spaces with some new idea and applications.

2 Moore-Penrose generalized solution

Let L be any bounded linear operator from a reproducing kernel Hilbert space $H_K(E)$ admitting a kernel $K : E \times E \to \mathbb{C}$ into a Hilbert space $\mathcal{H}$. We set $K_p = K(\cdot, p)$.

For any member $\mathbf{d}$ of $\mathcal{H}$, we consider the best approximation problem

$$\inf_{f \in H_K(E)} \|Lf - \mathbf{d}\|_{\mathcal{H}}. \tag{2.1}$$

Set

$$k(p,q) \equiv \langle L^*LK_q, L^*LK_p \rangle_{H_K(E)} = L^*LL^*L[K_q](p) \tag{2.2}$$

and

$$P = \mathrm{proj}_{H_K(E) \to \ker(L)^\perp} = \mathrm{proj}_{H_K(E) \to \overline{\mathrm{Ran}(L^*L)}}. \tag{2.3}$$

Theorem A ([18], 167 page): *Under the notations* (2.2) *and* (2.3), *we have*

$$H_k(E) = \{L^*Lf \ : \ f \in H_K(E)\} \tag{2.4}$$

and the inner product is given by:

$$\langle L^*Lf, L^*Lg \rangle_{H_k(E)} = \langle Pf, g \rangle_{H_K(E)} \tag{2.5}$$

for $f, g \in H_K(E)$.

Theorem B ([18], 167 page): *Problem* (2.1) *admits a solution if and only if* $L^*\mathbf{d} \in H_k(E)$. *If this is the case, then we have* $L^*\mathbf{d} = L^*L\tilde{f}$ *for some* $\tilde{f} \in H_K(E)$ *and* $\tilde{f}$ *is a solution to* (2.1).

Let $f_{\mathbf{d}} \in H_K(E)$ be the element such that

$$L^*\mathbf{d} = L^*Lf_{\mathbf{d}} \tag{2.6}$$

with $f_{\mathbf{d}} \in \ker(L)^{\perp}$.

The extremal function $f_{\mathbf{d}}(p)$ has the following representation:

Theorem C ([18], 168 page): *Keep to the same assumption as above. Then we have*

$$f_{\mathbf{d}}(p) = \langle L^*\mathbf{d}, L^*LK_p\rangle_{H_k(E)} \quad (p \in E). \tag{2.7}$$

The adjoint operator L^* of L, as we see from equality:

$$L^*\mathbf{d}(p) = \langle L^*\mathbf{d}, K_p\rangle_{H_K(E)} = \langle \mathbf{d}, LK_p\rangle_{\mathcal{H}} \quad (p \in E), \tag{2.8}$$

is represented by the known data $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From Theorems A, B, C, we see that the problem is well established by the theory of reproducing kernels. That is, the existence, the uniqueness and the representation of the solutions in the problem are well formulated. In particular, note that the adjoint operator is represented in a good way; this fact will turn out very important in our framework. The extremal function $f_{\mathbf{d}}$ is the **Moore-Penrose generalized inverse** $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria in Theorem A is involved and the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is, in general, not good, but abstract and an ideal one, in general.

Furthermore, we note that

Theorem D ([18], 178 page): *The following are equivalent*:

(1) *L is injective*;

(2) *L^*L is injective*;

(3) *$\{L^*LK_x\}_{x\in E}$ is complete in $H_K(E)$*;

(4) *$L^*L : H_K(E) \to H_k(E)$ is isometry.*

In particular, note that even the simple case, L^* is still, in general, not injective, and so we can not say that from $L^*\mathbf{d} = L^*Lf$, $Lf = \mathbf{d}$, the classical solution.

3 By the Tikhonov regularization

When the data contain error or noise in some practical cases, the exact theory by the Moore-Penrose generalized solutions is not applicable, therefore, we shall introduce the concept of the Tikhonov regularization with general data g.

Theorem E ([18], 182 page): *Let $\alpha > 0$. For a bounded linear operator L for a reproducing kernel Hilbert space $H_K(E)$ into a Hilbert space $\mathcal{H}$, the following minimizing problem admits a unique solution;*

$$\min_{f \in H_K(E)} \left(\|Lf - \mathbf{d}\|_{\mathcal{H}}{}^2 + \alpha \|f\|_{H_K(E)}{}^2 \right). \tag{3.1}$$

Furthermore, the minimum is attained by

$$f_{\mathbf{d},\alpha} = (L^*L + \alpha)^{-1} L^* \mathbf{d} = \left(\int_{\mathbb{R}} \frac{1}{\lambda + \alpha} dE_\lambda \right) L^* \mathbf{d} \tag{3.2}$$

by using the spectral decomposition. Furthermore, $\mathbf{d} \mapsto f_{\mathbf{d},\alpha}$ is almost the inverse of L in the following sense:

$$\lim_{\alpha \downarrow 0} f_{Lg,\alpha} = g \tag{3.3}$$

in $H_K(E)$ for all $g \in H_K(E)$ and when there exists the Moore-Penrose generalized solution,

$$\lim_{\alpha \downarrow 0} L f_{\mathbf{d},\alpha} = \mathbf{d} \tag{3.4}$$

in $\mathcal{H}$.

Theorem F ([18], 183 page): *Let $L : H_K(E) \to \mathcal{H}$ be a bounded linear operator. Then define an inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\alpha}(E)} = \alpha \langle f_1, f_2 \rangle_{H_K(E)} + \langle L f_1, L f_2 \rangle_{\mathcal{H}} \tag{3.5}$$

for $f_1, f_2 \in H_K(E)$. Then $(H_K(E), \langle \cdot, \cdot \rangle_{H_{K_\alpha}(E)})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by:

$$K_\alpha(p, q) = [(\alpha + L^*L)^{-1} K_q](p). \tag{3.6}$$

Here, $K_\alpha(p, q)$ satisfies

$$K_\alpha(p, q) + \frac{1}{\alpha} \langle L[(K_\alpha)_q], L[K_p] \rangle_{\mathcal{H}} = \frac{1}{\alpha} K(p, q), \tag{3.7}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases.

Theorem G ([18], 184-185 pages): *Under the same assumption as Theorems E and F,*

$$f \in H_K \mapsto \alpha \|f\|_{H_K(E)}{}^2 + \|Lf - \mathbf{d}\|_{\mathcal{H}}{}^2 \in \mathbb{R}$$

attains the minimum only at $f_{\mathbf{d},\alpha} \in H_K(E)$ *which satisfies*

$$f_{\mathbf{d},\alpha}(p) = \langle \mathbf{d}, L[(K_\alpha)_p] \rangle_{\mathcal{H}}. \tag{3.8}$$

Furthermore, $f_{\mathbf{d},\alpha}(p)$ *satisfies*

$$|f_{\mathbf{d},\alpha}(p)| \leq \|L\|_{H_K(E) \to \mathcal{H}} \sqrt{\frac{K(p,p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{3.9}$$

The representation (3.8) is not direct by using the solution of (3.7). However, the equation (3.7) is the Fredholm integral type in the second kind and so, the solutions are effective and numerically stable, as we see from the real inversion formula of the Laplace transform by taking a small α. See Chapter 4 of [18].

In particular, H. Fujiwara solved the integral equation corresponding to (3.7) for the real inversion formula of the Laplace transform with 6000 points discretization with **600 digits precision** based on the concept of **infinite precision**. Then, the regularization parameters were $\alpha = 10^{-100}, 10^{-400}$ surprisingly. H. Fujiwara was successful in deriving numerically the real inversion for the Laplace transform of the distribution delta which was proposed by V. V. Kryzhniy as a difficult case. This fact will mean that the above results are valid for very general functions approximated by the functions of the reproducing kernel Hilbert space.

See also [2, 3] for another typical applications.

4 Restriction of reproducing kernel Hilbert spaces

In order to consider the restriction and extension of reproducing kernel Hilbert spaces, we shall recall the fundamental general property.

Suppose that we are given a positive definite quadratic form function $K : E \times E \to \mathbb{C}$. The main concern here is to consider restriction of K to $E_0 \times E_0$, where E_0 is a subset of E. Of course, the restriction is again a positive definite quadratic form function on the subset $E_0 \times E_0$. We shall consider the relation between the two reproducing kernel Hilbert spaces.

Theorem H ([18], 78-80 pages): *Let E_0 be a subset of E. Then the Hilbert space that $K|E_0 \times E_0 : E_0 \times E_0 \to \mathbb{C}$ defines is given by:*

$$H_{K|E_0\times E_0}(E_0) = \{f \in \mathcal{F}(E_0) \,:\, f = \tilde{f}|E_0 \textit{ for some } \tilde{f} \in H_K(E)\}. \tag{4.1}$$

Furthermore, the norm is expressed in terms of the one of $H_K(E)$:

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} \,:\, \tilde{f} \in H_K(E),\, f = \tilde{f}|E_0\}. \tag{4.2}$$

In Theorem H, note that the inequality, for any function $f \in H_K(E)$

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} \leq \|f\|_{H_K(E)}, \tag{4.3}$$

that is, the restriction map is a bounded linear operator.

5 A general inequality to the restriction

The restriction of reproducing kernel Hilbert spaces may be stated by Theorem H, however, as we see from the concrete example of (1.1), in general, its construction is involved. In our method, meanwhile, we need a bounded linear operator stated by the norm inequality as in (1.2). There, of course, we are interested in its precise norm of the bounded linear operator, however, for many cases we do not need its precise norm for our applications.

Therefore, we first introduce a general norm inequality of type (1.2) with the typical Bergman space.

Let D be a bounded regular domain on the complex $z = x + iy$ plane whoes boundary is composed of a finite number of disjoint analytic Jordan curves. Let $AL_2(D)$ be a Hilbert space (Bergman space) comprising analytic functions $f(z)$ on D and with finite norms $\|f\|_{AL_2(D)} = \left\{\iint_D |f(z)|^2 dxdy\right\}^{\frac{1}{2}}$. As we see simply, $f \to f(z)(z \in D)$ is a bounded linear functional on $AL_2(D)$. Therefore, there exists a reproducing kernel $K(z, u)$ such that for any $u \in D$ and for any function $f \in AL_2(D)$,

$$f(u) = \iint_D f(z)\overline{K(z,u)}dxdy. \tag{5.1}$$

From $K(z,u) = \overline{K(u,z)}$, $K(z,u)$ is analytic in $\overline{u}$ (complex conjugate). So, we shall denote it as $K(z,\overline{u})$. This is the Bergman kernel of D or on D.

In general, we have the inequality

$$|f(z)| \leq \|f\|_{AL_2(D)} K(z,\overline{z})^{(1/2)}. \tag{5.2}$$

Therefore, for any subset D_0 of D, we can take a positive continuous function ρ on D_0 satisfying the norm inequality

$$\int_{D_0} |f(z)|^2 \rho(z) dm(z) \leq \|f\|^2_{AL_2(D)} \int_{D_0} K(z,\overline{z})\rho(z) dm(z) \tag{5.3}$$

with

$$\int_{D_0} K(z,\overline{z})\rho(z) dm(z) < \infty,$$

for some positive measure dm. For dm, we consider the simple line integral or surface measure for some simple cases of D_0.

With (5.3), we shall consider it as a bounded linear operator of the Bergman space $AL_2(D)$ into the space $AL_2(D_0, \rho)$ satisfying

$$\int_{D_0} |f(z)|^2 \rho(z) dm(z) < \infty.$$

In order to simply the situation, in the sequel we assume that D_0 is a rectifiable line in the domain D and dm is the line element $|dz|$.

Now as in ([16], [18]), for a general

$$\int_{D_0} |g(z)|^2 \rho(z) dm(z) < \infty.$$

we can discus approximations and analytic extension problem in this framework by the operator theory. However, here, we shall consider the bounded linear operator in the framework of analytic functions.

For the bounded linear operator L from the Bergman space $AL_2(D)$ into the space $AL_2(D_0, \rho|dz|)$

$$\int_{D_0} |f(z)|^2 \rho(z) |dz| < \infty,$$

for the sake of analyticity of functions,

L is injective

and

for the image Lf of L of the Bergman space $AL_2(D)$, its inverse f of L is, of course, uniquely, determined. For $F \in AL_2(D_0, \rho|dz|)$, from

$$(L^*F)(z) = (L^*F(\zeta), K(\zeta, \overline{z}))_{AL_2(D)} = (F, LK(\zeta, \overline{z}))_{AL_2(D_0,\rho|dz|)}, \qquad (5.4)$$

if the family $\{LK(\zeta, \overline{z}); z \in D\}$ of functions is complete, then the adjoint operator L^* is injective.

With the basic assumption that is clear in the representation of concrete cases, we can obtain the basic result:

Theorem 5.1 ([18], page 168): *The analytic extension operator L^{-1} from $AL_2(D_0, \rho|dz|)$ onto $AL_2(D)$ that is the inverse of the restriction operator L and is uniquely determined is represented by using (5.4) with*

$$f(z) = \langle L^*F, L^*LK_z \rangle_{H_k} \quad (z \in D) \qquad (5.5)$$

by using Theorem C.

6 Bergman norms and Szegö norms

For two functions φ and ψ of $H_2(D)$ for any regular domain D and for the analytic Hardy 2 space (Szegö space), we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\iint_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2|dz| \frac{1}{2\pi}\int_{\partial D} |\psi(z)|^2|dz|, \qquad (6.1)$$

and so, we obtain the bounded linear operator from the Szegö space to the Bergman space

$$\iint_D |f(z)|^2 dxdy \leq \frac{l(\partial D)}{4\pi}\int_{\partial D} |f(z)|^2|dz|, \qquad (6.2)$$

for the length $l(\partial D)$ of the boundary ∂D ([13]).

Note that the inequality (6.2) is a curious one in the sense that the Bergman norm is for analytic differentials, but the Szegö norm is for half order differentials. In connection with this inequality, we recall the interesting best possible norm inequality:

$$\iint_D |f'(z)|^2 dxdy \leq \frac{1}{2}\int_{\partial D} \frac{|f'(z)dz|^2}{idW(z,t)} \qquad (6.3)$$

$$= \frac{1}{2\pi}\int_{\partial D}|f'(z)|^2\left(\frac{\partial G(z,t)}{\partial \nu_z}\right)^{-1}|dz|,$$

that means the relation between the Bergman norms and the weighted Szegö norm for (exact) differentials. Here, for the conjugate harmonic function $G^*(z,t)$ of the Green function $G(z,t)$ of D, $W(z,t) = G(z,t) + iG^*(z,t)$, and $idW(z,t)$ is a single-valued meromorphic differential and positive along the boundary ∂D and $\partial/\partial\nu_z$ is the inner normal derivative with respect to D. $\partial G(z,t)/\partial\nu_z$ is a positive continuous function on ∂D ([14]).

This inequality is not so simple to derive and for its proof we must examine deeply the relations among the Hardy reproducing kernel, its conjugate kernel and the Bergman kernel.

The conjugate Hardy space which is given by the right hand side of (6.3) was surprisingly generalized as the Ohsawa-Saitoh-Hardy space on n-dimensional complex manifolds by Q. Guan and Z. Yuan ([9]) through ([6, 7, 8]) with many concrete and deep results.

Meanwhile, for the weighted Szegö space, some general spaces containing discontinuous weights were considered in T. Ł. Zynda, Z. P. Winiarski, J. J. Sadowski, and S. G. Krantz, [20].

In connection with the isometric equality (1.1) and the inequalities (5.3) and (6.2), we may consider, in (1.1) and (1.2)

Open problem 6.1: Does there exist a weight function ρ satisfying the inequality

$$\frac{1}{\sqrt{2\pi t}}\iint_{\mathbb{C}}|f(x+iy)|^2\exp\left(-\frac{y^2}{2t}\right)dxdy \le \int_{\mathbb{R}}|f(x)|^2\rho(x)dx. \tag{6.4}$$

7 Sobolev Hilbert spaces

Example.

The space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}}(|f(x)|^2 + |f'(x)|^2)dx}. \tag{7.1}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel

$$K(x,y) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}\exp(i(x-y)\xi)d\xi = \frac{1}{2}e^{-|x-y|} \quad (x,y \in \mathbb{R}). \tag{7.2}$$

Its restriction to the closed interval $[a,b]$ is the reproducing kernel Hilbert space $H_S[a,b] = W^{1,2}[a,b]$ as a set of functions, and the norm is given by

$$\|f\|_{H_S[a,b]} \equiv \sqrt{\left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\,dx\right) + |f(a)|^2 + |f(b)|^2}. \tag{7.3}$$

([18], pages 10-16).

The representation (7.2) means that the functions $f(x)$ of $H_S(\mathbb{R})$ are represented in the form

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}\exp(ix\xi)F(\xi)d\xi$$

with the functions $F(\xi)$ satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}|F(\xi)|^2 d\xi < \infty$$

and the norm is represented by

$$\|f\|_{H_S(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}|F(\xi)|^2 d\xi}.$$

The restriction mapping L from the space $H_S(\mathbb{R})$ to the space $H_S[a,b]$ is, of course, not injective and so, in particular, we obtain the norm inequality

$$\|f\|_{H_S(\mathbb{R})} \geq \|f\|_{H_S[a,b]};$$

that is,

$$\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2)dx \geq \left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\,dx\right) + |f(a)|^2 + |f(b)|^2. \tag{7.4}$$

By our general theory, we can give the precise correspondence of the two spaces; that is,

$$f|_{[a,b]}(x) = (f(\xi), K(\xi, x))_{H_S(\mathbb{R})} \tag{7.5}$$

and

$$f(x) = (f_{[a,b]}(\xi), K(\xi, x))_{H_S[a,b]}, \tag{7.6}$$

with the minimum extension f of $f_{[a,b]}$ in $H_S[a,b]$ to $H_S(\mathbb{R})$. Indeed, we can derive directly the identity (7.6) for the minimum extension f of $f_{[a,b]}$ in $H_S[a,b]$ to $H_S(\mathbb{R})$. See the following Theorem 7.1 for the space $W^{2,2}(\mathbb{R})$.

However, for the minimum extension formula we have the general formula in Theorem H,

$$f(p) = (f|E_0(\cdot), K(\cdot, p))_{H_{K|E_0 \times E_0}(E_0)},$$

for the minimum extension f of $f|E_0$. See the proof of Proposition 2.5 in [18] (pages 79-80), in particular, (2.48).

We obtained several realizations of restricted reproducing kernel Hilbert spaces as in (7.3), however, they are, in general, involved. See [16], [18]. The formula (7.3) is a simple result, however, the realization of the restricted reproducing kernel spaces is, in general, complicated in this sense.

Open Problem 7.1: *Let $m > \frac{n}{2}$ be an integer. Denote by ${}_mC_\nu$ the binomial coefficient and by $W^{m,2}(\mathbb{R}^n)$ the Sobolev space whose norm is given by*

$$\|F\|_{W^{m,2}(\mathbb{R}^n)} = \sqrt{\sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha F(x)}{\partial x^\alpha} \right|^2 dx \right)}. \tag{7.7}$$

Then, the reproducing kernel K is given by

$$K(x, y) \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\exp(i(x-y)\cdot\xi)}{(1+|\xi|^2)^m} d\xi \quad (x, y \in \mathbb{R}^n) \tag{7.8}$$

([18], page 22). How will be the realization of the norm for the restricted reproducing kernel Hilbert space to some nontrivial subset (the typical case is a sphere $\{r < a\}$) of $\mathbb{R}^n$ as in the case of one dimensional way (7.4)?

The typical case for the space $W^{2,2}(\mathbb{R})$

For the Sobolev Hilbert space $W^{2,2}(\mathbb{R})$ defined to be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm:

$$\|f\|_{W^{2,2}(\mathbb{R})} = \sqrt{\|f''\|_{L^2(\mathbb{R})}{}^2 + 2\|f'\|_{L^2(\mathbb{R})}{}^2 + \|f\|_{L^2(\mathbb{R})}{}^2},$$

we have the reproducing kernel

$$G(s,t) \equiv \frac{1}{4}e^{-|s-t|}(1+|s-t|) \quad (s,t \in \mathbb{R})$$

([18], pages 21-22).

In order to look for the reproducing kernel Hilbert space $W_S([a,b]), (a < b)$ admitting the restricted reproducing kernel $G(s,t)$ to the interval $[a,b]$, we calculate the integral, for any function $f \in W^{2,2}(\mathbb{R})$

$$(f(s), G(s,t))_{W^{2,2}([a,b])}.$$

By setting $G_t(s) = G(s,t)$, we note that

$$\begin{aligned}
G_t(s) &= \frac{t-s+1}{4}\exp(s-t)\chi_{(a,t)}(s) + \frac{s-t+1}{4}\exp(t-s)\chi_{[t,b)}(s),\\
\frac{dG_t}{ds}(s) &= \frac{t-s}{4}\exp(s-t)\chi_{(a,t)}(s) - \frac{s-t}{4}\exp(t-s)\chi_{[t,b)}(s),\\
\frac{d^2G_t}{ds^2}(s) &= \frac{t-s-1}{4}\exp(s-t)\chi_{(a,t)}(s) + \frac{s-t-1}{4}\exp(t-s)\chi_{[t,b)}(s). \quad (7.9)
\end{aligned}$$

Then, by integration by parts repeatedly, we have

$$\begin{aligned}
&(f(s), G(s,t))_{W^{2,2}([a,b])}\\
&= f(t)\\
&+f(a)\frac{-t+a-2}{4}\exp(a-t) - f'(a)\frac{t-a-1}{4}\exp(a-t)\\
&-f(b)\frac{b-t+2}{4}\exp(t-b) + f'(b)\frac{b-t-1}{4}\exp(t-b). \quad (7.10)
\end{aligned}$$

That is

$$\begin{aligned}
&(f(s), G(s,t))_{W_S([a,b])}\\
&= f(t)
\end{aligned}$$

$$+f(a)G(a,t)\left(-1+\frac{-1}{1+t-a}\right)$$
$$+f'(a)G'(a,t)\left(\frac{1}{t-a}-1\right)$$
$$+f(b)G(b,t)\left(-1+\frac{-1}{1+b-t}\right)$$
$$+f'(b)G'(b,t)\left(\frac{1}{b-t}-1\right). \tag{7.11}$$

For this formula, see, [18], pages 15-16 for the one dimensional case.

We thus have the desired identity admitting the restricted reproducing kernel of $G(s,t)$ to the interval $[a,b]$

$$(f,G(\cdot,t))_{W_S([a,b])} = (f,G(\cdot,t))_{W^{2,2}(\mathbf{R})} \tag{7.12}$$
$$-f(a)\frac{-t+a-2}{4}\exp(a-t)+f'(a)\frac{t-a-1}{4}\exp(a-t)$$
$$+f(b)\frac{b-t+2}{4}\exp(t-b)-f'(b)\frac{b-t-1}{4}\exp(t-b).$$

We can see that this identity is right, indeed, we shall give another natural method in order to see it.

Another method

In order to look for the norm admitting the restricted reproducing kernel of $G(s,t)$ to the interval $[a,b]$, note that the integral

$$\int_{-\infty}^{a}\left(f''(x)^2+2f'(x)^2+f(x)^2\right)dx \tag{7.13}$$

is identical with its integral of the function

$$f(x)=4f(a)G_a(x)-4f'(a)G'_a(x) \tag{7.14}$$

that is the minimum integral over $(-\infty,a)$ of the functions $W^{2,2}(\mathbf{R})$ taking the values $f(a)$ and $f'(a)$.

The function is given by

$$f(x)=[(A+B)(a-x)+A]\exp(x-a)$$

with

$$A = f(a), \quad B = -f'(a).$$

Then, by direct calculations, we have

$$\int_{-\infty}^{a} \left(f''(x)^2 + 2f'(x)^2 + f(x)^2\right) dx = 2(A^2 + AB + B^2) \tag{7.15}$$

$$= 2(f(a)^2 - f(a)f'(a) + f'(a)^2).$$

From this result, we see that the corresponding inner product over $(-\infty, a)$ is represented by

$$(f_1, f_2)_{W^{2,2}(-\infty,a)} = 2(f_1(a)f_2(a) + f_1'(a)f_2'(a)) \tag{7.16}$$

$$+\frac{1}{2}(f_1(a) - f_1'(a))(f_2(a) - f_2'(a)) - \frac{1}{2}(f_1(a) + f_1'(a))(f_2(a) + f_2'(a)).$$

The situation for the integrals over $(b, +\infty)$ is similar and so we obtain the desired isometric identity

$$\|f\|^2_{W^{2,2}(\mathbf{R})} = \|f\|^2_{W^{2,2}([a,b])} \tag{7.17}$$

$$+2(f(a)^2 - f(a)f'(a) + f'(a)^2)$$

$$+2(f(b)^2 - f(b)f'(b) + f'(b)^2).$$

Therefore, the inner product relation is given by

$$(f_1, f_2)_{W^{2,2}(\mathbf{R})} = (f_1, f_2)_{W^{2,2}([a,b])}$$

$$+2(f_1(a)f_2(a) + f_1'(a)f_2'(a)) + \frac{1}{2}(f_1(a) - f_1'(a))(f_2(a) - f_2'(a))$$

$$-\frac{1}{2}(f_1(a) + f_1'(a))(f_2(a) + f_2'(a))$$

$$+2(f_1(b)f_2(b) + f_1'(b)f_2'(b)) + \frac{1}{2}(f_1(b) - f_1'(b))(f_2(b) - f_2'(b))$$

$$-\frac{1}{2}(f_1(b) + f_1'(b))(f_2(b) + f_2'(b)).$$

We can confirm that (7.10) and (7.16) are consistent, directly. Indeed,

$$2(f(a)G_t(a) + f'(a)G_t'(a)) + \frac{1}{2}(f(a) - f'(a))(G_t(a) - G_t'(a))$$

$$-\frac{1}{2}(f(a)+f'(a))(G_t(a)+G_t'(a))$$

is identical with

$$+f(a)\frac{-t+a-2}{4}\exp{(a-t)}+f'(a)\frac{t-a-1}{4}\exp{(a-t)}.$$

For the point b, the result is similar.

In particular, we have

Theorem 7.1: *The extension of the functions f in $W_S([a,b])$ to $W^{2,2}(\mathbf{R})$ with the minimum norm is given by*

$$f(t)=(f,G(\cdot,t))_{W^{2,2}([a,b])}$$

$$+2(f(a)G_t(a)+f'(a)G_t'(a))+\frac{1}{2}(f(a)-f'(a))(G_t(a)-G_t'(a))$$

$$-\frac{1}{2}(f(a)+f'(a))(G_t(a)+G_t'(a))$$

$$+2(f(b)G_t(b)+f'(b)G_t'(b))+\frac{1}{2}(f(b)-f'(b))(G_t(b)-G_t'(b))$$

$$-\frac{1}{2}(f(b)+f'(b))(G_t(b)+G_t'(b)).$$

Related versions

By the similar method or directly we have the following results.

Let

$$K(s,t)\equiv\int_0^\infty\frac{\cos(s\,u)\cos(t\,u)}{u^2+1}\,du=\frac{\pi}{4}\left(\exp(-|s-t|)+\exp(-s-t)\right) \tag{7.18}$$

for $s,t>0$. Then $H_K(0,\infty)=W^{1,2}(0,\infty)$ as a set of functions and the norm is given by:

$$\|f\|_{H_K(0,\infty)}=\sqrt{\frac{2}{\pi}\int_0^\infty(|f'(u)|^2+|f(u)|^2)\,du} \tag{7.19}$$

([18], pages 12-13). From the restriction of the kernel $K(s,t)$ to $[a,b],a>0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)}\geq\frac{2}{\pi}\frac{1-\exp(-2a)}{1+\exp(-2a)}|f(a)|^2 \tag{7.20}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du+\frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t)\equiv\int_0^\infty \frac{\sin(s\,u)\sin(t\,u)}{u^2+1}\,du=\frac{\pi}{4}\left(\exp(-|s-t|)-\exp(-s-t)\right) \quad (7.21)$$

for $s,t>0$. Then we have

$$H_K(0,\infty)=\{f\in \mathrm{AC}(0,\infty)\ :\ f(0)=0\} \quad (7.22)$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)}=\sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2+|f(u)|^2)\,du} \quad (7.23)$$

([18], pages 13-14). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)}\geq\frac{2}{\pi}\frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2 \quad (7.24)$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du+\frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t)\equiv\min(s,t)\quad (s,t>0). \quad (7.25)$$

Then we have

$$H_K(0,\infty)=\left\{f\in W^{1,2}(0,\infty)\ :\ \lim_{\varepsilon\downarrow 0}f(\varepsilon)=0\right\} \quad (7.26)$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)}=\sqrt{\int_0^\infty |f'(u)|^2\,du} \quad (7.27)$$

([18], pages 14-15). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)}\geq\frac{1}{a}|f(a)|^2+\int_a^b |f'(u)|^2\,du. \quad (7.28)$$

We have many type Sobolev Hilbert spaces. For example,
for $\omega^2 = \gamma^2 - \alpha^2 > 0$, the kernel

$$K(s,t) = \frac{\exp(-\alpha|s-t|)}{4\alpha\gamma^2}\{\cos(\omega|s-t|) + \frac{\alpha}{\omega}\sin(\omega|s-t|)\}$$

is the reproducing kernel for the Sobolev Hilbert space admitting the norm

$$||u||^2 = 4\alpha\gamma^2 u(a)^2 + 4\alpha u'(a)^2$$

$$+\int_a^b \left(u''(t) + 2\alpha u'(t) + \gamma^2 u(t)\right)^2 dt$$

(E. Parzen, [12]).

See also the recent paper A. Yamada ([19]).

Basic applications of the realization of the restricted reproducing kernel Hilbert space

The identity (7.6) and other derived identities show that the extension of the function with the minimum norm to the whole (half) space from a closed interval $[a, b]$ is given simply. This means that in the related Fourier transform, the inversion that corresponds to the function with the minimum norm may be calculated in terms of the values on the interval $[a, b]$.

A typical problem for the Bergman space

The identity (7.3) creates the new problem and concept. We shall state the prototype problem:

On the unit disc $|z| < 1$, look for the identity

$$\iint_{\{|z|<1\}} |f(z)|^2 dxdxy = \iint_{\{|z|<a\}} |f(z)|^2 dxdy \qquad (7.29)$$

$$+?$$

with some information of f on $|z| = a, 0 < a < 1$; that is, the integral

$$\iint_{\{a<|z|<1\}} |f(z)|^2 dxdy$$

is represented in terms of f on $|z| = a, 0 < a < 1$.

8 On the Bergman space

On the above line, we can obtain the following simple and general result:

Lemma 8.1: *For a general domain D existing the Green function $G(z,t)$ on the complex plane, an analytic function $f(z)$ of the Bergman space on D is represented by the Green function and the Bergman kernel $K(z,\overline{u})$ on a simply connected subdomain D_s of D with a piecewise smooth closed Jordan boundary curve in D in the following way, for $z \in D_s$*

$$f(z) = (f(\zeta), K(\zeta,\overline{z}))_{L_2(D_s)} + \frac{1}{\pi i}\int_{\partial D_s} f(\zeta)\frac{\partial G(\zeta,z)}{\partial z}d\zeta. \tag{8.1}$$

Proof: First, recall the trivial identity

$$\begin{aligned} f(z) &= (f(\zeta), K(\zeta,\overline{z}))_{L_2(D)} \\ &= (f(\zeta), K(\zeta,\overline{z}))_{L_2(D_s)} + (f(\zeta), K(\zeta,\overline{z}))_{L_2(D\setminus D_s)}. \end{aligned}$$

We calculate the last integral. By using the identity

$$K(z,\overline{u}) = -\frac{2}{\pi}\frac{\partial^2 G(z,u)}{\partial z\partial\overline{u}} \tag{8.2}$$

and the Green formula, we have

$$(f(\zeta), K(\zeta,\overline{z}))_{L_2(D\setminus D_s)} = \frac{1}{\pi i}\int_{\partial D_s} f(\zeta)\frac{\partial G(\zeta,z)}{\partial z}d\zeta.$$

Note that the derivative of the Green function is zero on the boundary of the domain D. We thus obtain the desired result.

Note the identity, for $z \in D_s$

$$(f(\zeta), K(\zeta,\overline{z}))_{L_2(D\setminus D_s)} = \frac{1}{\pi i}\int_{\partial D_s} f(\zeta)\frac{\partial G(\zeta,z)}{\partial z}d\zeta. \tag{8.3}$$

The inner product on $L_2(D \setminus D_s)$ will have the curious representation. We wish to represent it in some inner product form in the symmetric form on the boundary ∂D_s in order to look for the representation of the norm on the boundary.

For (7.29), we have

$$\iint_{\{|z|<a\}} f(z)\frac{1}{\pi}\overline{\frac{1}{(1-\overline{u}z)^2}}dxdy = a^2 f(a^2 u).$$

It will be a curious operator. For the Szegö kernel case, we have the correspondent result as in

$$\int_{\{|z|=a\}} f(z)\frac{1}{2\pi}\overline{\frac{1}{(1-\overline{u}z)}}|dz| = af(au),$$

however, some correspondent result to Lemma 8.1 is unclear and curious.

Of course, we can apply Theorem 1, (5.5), however, the structure of the related reproducing kernel Hilbert space H_k is involved in this situation, because the small kernel $k(z,\overline{u})$ is given in the present situation by

$$k(z,\overline{u})$$
$$= \iint_{D_s}\left(\iint_{D_s} K(z_1,\overline{u})\overline{K(z_1,\overline{z_2})}dm(z_1)\right)\overline{K(z_2,\overline{z})}dm(z_2)$$

with the L_2 surface measure. We can not use the isometric mapping L^*L from the Bergman space on D into the Bergman space on D for the realization of the space H_k .

However, with a numerical sense, we can obtain the reasonable result for the concrete case.

A remark for extension and restriction

In connection with restriction and extension of functions in reproducing kernel Hilbert spaces, recall the discritization principle; that is, the norms in reproducing kernels are represented in terms of a discrete countable uniqueness set ([18], page 127-139). Then, restriction and extension problems are reduced to those of reproducing kernels themselves.

9 From Ohsawa-Takegoshi's type norm inequalities

As well-known, the converse norm inequality of (5.3)

$$\|f\rho^{1/2}\|^2_{AL_2(D)} \le C \int_{D_0} |f(z)|^2 \rho(z) dm(z) \tag{9.1}$$

is known with some positive function ρ and some constant C on some complex domain in a general dimension. See for example, [10] and [11]. Surprisingly enough, the best constant C is even discovered, see [1, 4, 5]. The author wonders whether can we apply our idea in this paper to their great and deep results.

In this section, we assume this norm inequality

$$\|f\|^2_{AL_2(D)} \le C \int_{D_0} |f(z)|^2 \rho(z) |dz| \tag{9.2}$$

with a rectifiable curve D_0 as a typical case and we shall consider the bounded linear operator L from the space with finite norm

$$\int_{D_0} |f(z)|^2 \rho(z) |dz| < \infty$$

into the Bergman space $AL_2(D)$; that is for $Lf = f$, (9.2) is valid for analytic extension operator L in the framework of analytic functions. Here we write the same f as an analytic function and its restriction, because by the identity theorem we can assume that the correspondence is one-to-one.

Anyhow, it seems that the general theory for bounded linear operator equations on reproducing kernel Hilbert spaces in the first part in this note is a general and powerful theory. It creates many deep and concrete problems.

Acknowledgements

The author wishes to express his deep thanks Professors Y. Sawano, M. Seto and A. Yamada for their great contributions to the meeting and the paper.

This meeting and paper were supported by the RIMS.

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