TOWARDS A CLASSIFICATION OF REGULAR SEQUENCES

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Abstract. In the mid-2000s, Adamczewski and Buguead proved the Cobham-Loxton-van der Poorten conjecture by using the subspace theorem to show that any automatic number (a number whose base expansion is given by an automatic sequence) is either rational or transcendental. About 10 years later, again using the subspace theorem, Bell, Bugeaud, and Coons, extended this result to regular sequences—the (possibly unbounded) generalization of automatic sequences. In this survey, we discuss a further characterization of regular sequences by defining an associated measure, the so-called ghost measure, which is governed by the underlying properties of a related finite set of matrices.

1. Introduction

One of the common themes in the area of transcendence and Diophantine approximation goes like this. You pick a real number. Say you chose $\xi \in \mathbb{R}$. Then ask, and try to answer the following questions:

1. Is ξ rational?
2. If not, is ξ even algebraic?
3. If not, is ξ the special value of some function that I know about?
4. If not, or even if so, can some base expansion of ξ be given by a process that I know or care about?

Probably the answer to at least one of these questions is 'yes.' If the answer to all of these questions were 'no', then one would have to ask how you picked ξ in the first place. In the transcendence community, a common example is to choose a number ξ that is related to an automatic sequence. For example, a number whose base-b expansion, for some positive integer $b \geqslant 2$, that is produced by a finite automaton, or, in general, a number that is the special value of the generating function of an automatic sequence. A classical example comes from the Thue–Morse sequence.

2. Automatic and regular sequences and numbers

The Thue–Morse sequence $\{t(n)\}_{n\geqslant 0}$ is defined on the alphabet $\{1,-1\}$ by $t(0) = 1$ and for $n \geqslant 1$ by the recurrences $t(2n) = t(n)$ and $t(2n+1) = -t(n)$. This sequence, which starts

$$\{t(n)\}_{n\geqslant 0} = \{1,-1,-1,1,-1,1,1,-1,-1,1,1,-1,1,-1,-1,1,\ldots\},$$

is one of the most ubiquitous integer sequences and one of central importance in various areas within number theory, combinatorics, theoretical computer science and dynamical systems theory; in particular, it is paradigmatic in the areas of

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Date: October 30, 2023.

RIMS Kôkyûroku, 共同研究 報告集, Analytic Number Theory and Related Topics 2023.

complexity and symbolic dynamics. While the sequence goes back at least to the 1851 paper of Prouhet [18], its interest in the context of complexity is usually attributed to Thue [19], who, in 1906, showed that t is cube-free; that is, viewing t as a one-sided infinite word, it contains no subword of the form www. This shows that the Thue–Morse sequence is not periodic, though the proof of non-periodicity is much less deep than the cube-freeness. This sequence is output by the deterministic finite automaton in Figure 1—here one inputs the binary expansion of n and reads off the value $t(n)$ from the final state.

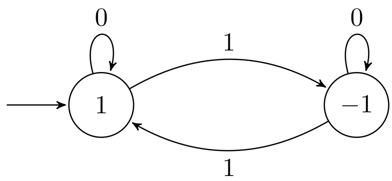

FIGURE 1. The 2-automaton producing the Thue-Morse sequence $\{t(n)\}_{n\geqslant 0}$.

Concerning transcendence, in 1930, Mahler [16] proved the following result.

Theorem 1 (Mahler, 1930)**.** *Let α be a nonzero algebraic number with $|\alpha| < 1$. Then $\sum_{n\geqslant 0} t(n)\alpha^n$ is transcendental over $\mathbb{Q}$.*

Mahler's proof is highly dependent on the generating function of t satisfying the functional equation $T(z) = \sum_{n\geqslant 0} t(n)z^n = (1-z)T(z^2)$. In fact, the generating function of any automatic sequence satisfies such a functional equation, which are now known as 'Mahler type'—a power series $F(z)$ is called a Mahler function if there are integers $d \geqslant 1$ and $k \geqslant 2$, and polynomials $a_0(z), \ldots, a_d(z)$ such that

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0.$$

In the late 1920s and early 1930s, Mahler studied the transcendental properties of these functions and special values at algebraic points. Essentially, he showed that the transcendence of the function over $\mathbb{Q}(z)$ gave the transcendence of certain special values over $\mathbb{Q}$—a property which is very special. A crowning achievement in the area of automatic sequences was the resolution of Cobham's conjecture by Adamczewski and Bugeaud [1].

Theorem 2 (Adamczewski and Bugeaud, 2007)**.** *An automatic number is either rational or transcendental.*

It is worth noting that, while a proof using Mahler functions now exists, the original proof of this result was accomplished using the Schmidt subspace theorem.

Of course, due to the dependence of automatic sequences on being produced by a finite state automaton, they necessarily can take only a finite number of values. In 1992, Allouche and Shallit [2] offered a generalisation of automatic sequences that can be unbounded, the so-called regular sequences.

Definition 3. An integer sequence f is called k-regular provided the set of subsequences

$$\ker_k(f) := \{(f(k^\ell n + r)_{n\geqslant 0} : \ell \geqslant 0, 0 \leqslant r < k^\ell\}$$

is contained in a finitely generated $\mathbb{Z}$-module.

Now, if $\ker_k(f)$ is finite, then f is automatic, so the generalisation is clear. In fact, if f is k-regular and takes a finite number of values, then f is k-automatic. So this concept is robust. A (possibly) more enlightening way to see k-regular sequences is through their linear representation.

Theorem 4 (Allouche and Shallit, 1992)**.** *An integer-valued sequence f is k-regular if, and only if, there is an integer $d \geqslant 1$, $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{d\times 1}$ and $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_k \in \mathbb{Z}^{d\times d}$ such that*

$$f(n) = \mathbf{u}^T \mathbf{A}_{i_s} \mathbf{A}_{i_{s-1}} \cdots \mathbf{A}_{i_1} \mathbf{A}_{i_0} \mathbf{v},$$

where the base-k expansion of n is $i_s i_{s-1} \cdots i_1 i_0$.

My favourite example of a regular sequence is Stern's diatomic sequence s, which is defined by $s(0) = 0$, $s(1) = 1$, and for $n \geqslant 1$ by $s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$. The sequence has many interesting properties, e.g., the ratios $s(n)/s(n+1)$ enumerate the nonnegative rational numbers, in reduced form and without repeats! The linear representation of s is given by

$$\mathbf{u}^T = \mathbf{v}^T = (1\ 0), \quad \mathbf{A}_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Following Mahler's classical method, in 2010, we showed [7] that for any nonzero algebraic number α with $|\alpha| < 1$, the number $\sum_{n\geqslant 0} s(n)\alpha^n$ is transcendental.

As an analogue to Cobham's conjecture, or more-correctly, to the theorem of Adamczewski and Bugeaud stated above, together with Bell and Bugeaud, we proved the following generalisation [5], as a consequence, again, of Schmidt's subspace theorem.

Theorem 5 (Bell, Bugeaud and Coons, 2015)**.** *Let f be a k-regular sequence and $b \geqslant 2$ be a positive integer. Then $\sum_{n\geqslant 0} f(n) b^{-n}$ is either rational or transcendental.*

All of the above results are in the area of transcendence theory called 'Mahler's Method' and are really a take-off of Mahler's original theory. But that's not how Mahler started looking at these objects.

3. Mahler and measure

Like many of us, Mahler's first paper looks a bit out of place among his other work. At least at first glance. But this takes a little explanation.

In 1926, Mahler was at Göttingen, which, as he describes it [17], "was at that time a centre of world mathematics." In that year, the renowned American applied mathematician Norbert Wiener had a Guggenheim Fellowship to work with Max Born at Göttingen. During his stay in Göttingen, Wiener was given 23-year-old Mahler as an unpaid assistant. The pair produced one (sort-of joint) paper. By 'sort of', we mean that they each wrote, individually, one part of a two-part paper with one of the longest titles I've ever encountered, which, before the addition of the subtitles for each part is "The spectrum of an array and its application to the study of the translational properties of a simple class of arithmetical functions." Before getting to Mahler's contribution, which was Part Two, let's focus on Wiener's part.

In Part One, Wiener [20] described how to use the concept of diffraction to associate a measure defined on the unit torus $\mathbb{T} = [0, 1)$ to a finite-valued integer sequence f—here, for ease, we will suppose $f(n) \in \{-1, 1\}$. We now call this the 'diffraction measure', and write $\widehat{\gamma} := \widehat{\gamma}_f$. This process is best described, these days, in reference to the Wiener diagram:

$$\begin{tikzcd} \omega := \sum\limits_{n\in\mathbb{Z}} f(n)\,\delta_n \arrow[r, "\circledast"] \arrow[d, "\mathcal{F}"'] & \gamma := \omega \circledast \omega = \sum\limits_{m\in\mathbb{Z}} \eta(m)\,\delta_m \arrow[d, "\mathcal{F}"] \\ \widehat{\omega} \arrow[r, "|\cdot|^2"] & \widehat{\gamma} = \widehat{\omega \circledast \omega} \end{tikzcd}$$

where ω is the (weighted) Dirac comb with weights w, $\circledast$ represents convolution, $\mathcal{F}$ is Fourier transformation, and the values

$$\eta(m) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} f(i)f(i+m)$$

are the autocorrelation coefficients.

With $\widehat{\gamma}_f$ in hand, one now asks, what kind of measure is it? Recall that for reasonable measures, we have the following characterisation.

Theorem 6 (Generalised Lebesgue decomposition)**.** *Any finite real Borel measure μ on $\mathbb{T}$ is the sum*

$$\mu = \mu_{\rm pp} + \mu_{\rm sc} + \mu_{\rm ac},$$

where $\mu_{\rm pp} \perp \mu_{\rm sc} \perp \mu_{\rm ac}$, and with respect to Lebesgue measure λ, $\mu_{\rm pp}$ is pure point (the so-called Bragg part), $\mu_{\rm sc}$ is singular continuous and $\mu_{\rm ac}$ is absolutely continuous.

In his part, Wiener gave some examples of sequences with values in $\{-1,1\}$ with pure spectral types. For a pure point measure, he showed that any eventually periodic sequence f has $\widehat{\gamma}_f = (\widehat{\gamma}_f)_{\rm pp}$. Now, let f be all finite sequences of the values $\{-1,1\}$ ordered lexicographically with $1 > -1$; so, the sequence of f values here is the sequence

$$1,-1,1,1,1,-1,-1,1,-1,-1,1,1,1,1,1,-1,1,-1,1,\ldots.$$

For this f, we have that $\widehat{\gamma}_f = (\widehat{\gamma}_f)_{\rm ac}$. This is a bit unsurprising as this sequence is normal—all patterns of length n occur and they occur with frequency 2^{-n}. What Wiener did not do in his part, was provide and example of a sequence that is purely singular continuous—enter Mahler.

In his part, Mahler [15] showed the following.

Theorem 7 (Mahler, 1927)**.** *If t is the Thue–Morse sequence, then $\widehat{\gamma}_t = (\widehat{\gamma}_t)_{\rm pp}$.*

The diffraction measure is only developed for sequences that take finitely many values, and is often applied to automatic sequences; see the monograph 'Aperiodic Order' by Baake and Grimm [3] for several detailed examples. In the next section, inspired by Wiener's construction, we construct a measure for an unbounded regular sequence.

4. Measures associated to regular sequences

When thinking about associated a measure to regular sequences, one thing that stood out for us, was, analogous with forming the volume averaged convolution $\omega \circledast \omega$, a need to be able to volume average. In work with Michael Baake [4], we realised that a property of the Stern sequence could be used to accomplish this—that s satisfies, for all $n \geqslant 0$, that

$$\sum_{m=0}^{2^n-1} s(2^n+m) = 3^n. \tag{1}$$

Using this we defined a sequence of pure point measures

$$\mu_{s,n} := \frac{1}{3^n} \sum_{m=0}^{2^n-1} s(2^n+m)\delta_{m/2^n},$$

where δ_x denotes the unit Dirac measure at x. We can view $(\mu_{s,n})_{n\in\mathbb{N}_0}$ as a sequence of probability measures on the 1-torus, the latter written as $\mathbb{T} = [0,1)$ with addition modulo 1. Here, we have simply re-interpreted the (normalised) values of the Stern sequence $(s(n))_{n\geqslant 0}$ between 2^n and $2^{n+1}-1$ as the weights of a pure point probability measure on $\mathbb{T}$ supported on $\{m/2^n : 0 \leqslant m < 2^n\}$. With this set up, we proved the following result [4].

Theorem 8 (Baake and Coons, 2018). *The sequence $(\mu_{s,n})_{n\in\mathbb{N}_0}$ of probability measures on $\mathbb{T}$ converges weakly to a probability measure μ_s, which is purely singular continuous.*

It turns out, an equality analogous to (1) holds for any regular sequence f. That is, if f is k-regular, the sum of f between powers of k is a linear recurrent sequence, and so, for nonnegative k-regular sequences f, this value can be used to volume average. In particular, given a linear representation $\mathbf{u}, \mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}, \mathbf{v}$ of a nonnegative k-regular sequence f, set

$$\Sigma_f(n) := \sum_{m=k^n}^{k^{n+1}-1} f(m) \quad \text{and} \quad \mu_n := \frac{1}{\Sigma_f(n)} \sum_{m=0}^{k^{n+1}-k^n-1} f(k^n+m)\delta_{m/k^n(k-1)},$$

where δ_x denotes the unit Dirac measure at x. We can view $(\mu_{f,n})_{n\in\mathbb{N}_0}$ as a sequence of probability measures on the 1-torus, the latter written as $\mathbb{T} = [0,1)$ with addition modulo 1. Here, each of these pure point probability measures on $\mathbb{T}$ is supported on $\{m/(k^n(k-1)) : 0 \leqslant m < k^n(k-1)\}$. Note that $\mu_{f,n}$ is only well-defined if $\Sigma_f(n)$ is nonzero.

To extend Theorem 8 to a larger set of regular sequences, we need a bit more notation, and a few assumptions. To consider properties of these measures, for the purpose of this short survey, we will suppose that each of $\mathbf{u}, \mathcal{A}, \mathbf{v}$ has only nonnegative entries, where $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}\}$. Also, set $\mathbf{A} := \mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_{k-1}$ and define the spectral radius $\rho := \rho(\mathbf{A})$ and the joint spectral radius ρ^* of $\mathcal{A}$ by

$$\rho^* = \rho^*(\mathcal{A}) := \lim_{n\to\infty} \max_{0\leqslant i_1,i_2,\ldots,i_n\leqslant k-1} \left\|\mathbf{A}_{i_1}\mathbf{A}_{i_2}\cdots\mathbf{A}_{i_n}\right\|^{1/n}.$$

These radii satisfy the fundamental inequality

$$\frac{\rho}{k} \leqslant \rho^* \leqslant \rho.$$

The spectral properties of the measures we will form are impacted directly by how this inequality holds, that is, which parts are strict inequalities and which are equalities. With Evans, Gohlke and Mañibo, we showed the following [8].

Theorem 9 (Coons, Evans, Gohlke and Mañibo, 2023+ε). *Let f be a regular sequence and suppose that the set $\mathcal{A}$ is irreducible and that $\mathbf{A}$ is nonnegative and primitive. Then the sequence $(\mu_{f,n})_{n\in\mathbb{N}_0}$ of probability measures on $\mathbb{T}$ converges weakly to a probability measure μ_f, which is spectrally pure. Moreover,*

(i) $\rho^* = \rho$ if, and only if, μ_f is pure point,
(ii) $\rho/k < \rho^* \neq \rho$ if, and only if, μ_f is singular continuous and
(iii) $\rho/k = \rho^*$ if, and only if, μ_f is absolutely continuous.

To see that this theorem is a generalisation of Theorem 8, one notes that in the situation of the Stern sequence s, we have that $\mathcal{A}$ is irreducible, and $\mathbf{A}$ has only positive entries, so is primitive, and as well $\rho = 3$, $\rho^* = (1+\sqrt{5})/2$, so that the fundamental inequality holds with

$$\frac{\rho}{2} = 1.5 < \rho^* = \frac{1+\sqrt{5}}{2} < \rho = 3.$$

Thus Theorem 9 directly implies Theorem 8.

5. Some final comments

In this short survey, we described some ways to associate measures to integer sequences, in particular, we described Wiener's diffraction measure construction for finite-valued sequences (such as automatic sequences), and our construction of a measure related to (unbounded) regular sequences. The measures from our construction are called *ghost measures* in the literature[1].

With these measures in hand, one asks questions about their spectral type, and those answers give us some information about the structure of the sequence. Considering our measure for regular sequences, in a very concrete way,

- a sequence with nonzero pure point component has some extremely large values comparable to the average value,
- a sequence with nonzero absolutely continuous component has some very smooth parts, and
- a sequence with nonzero singular continuous component has some self-similar or fractal behaviour.

This type of characterisation is in stark contrast to what one gains by asking the questions raised in the introduction. For example, if you know a number is transcendental (here, maybe your number is a special value of the generating function of a regular sequences), then you know it is not algebraic. And that's it. While it's great to show such a thing, it doesn't really give you any information about properties of the number. In fact, it tells you it doesn't have a certain kind of structure, which is not nothing, but is also not a lot. Our hope with associating these measures to regular sequences is to provide more information about the structure, and this method could be seen either as an alternative to the classical algebraic classification or as a complementary approach.

An immediate question is whether there is a relation between the spectral type of the measure and the algebraic properties of the number.

It turns out that regular sequences with transcendental special values can have any pure spectral type, and so also, by addition, be of any mixed spectral type. For example, we have already seen that for Stern's diatomic sequence s, the measure μ_s is purely singular continuous, and the number $\sum_{n\geqslant 1} s(n)\alpha^n$ is transcendental for any nonzero algebraic number α with $|\alpha| < 1$. For a pure point measure, we can

[1]The name *ghost measure* came from my former PhD student James Evans. For him [11], this was reminiscent of Berkeley's critique of infinitesimals in The Analyst, when he says that they are "neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them *the ghosts of departed quantities?*" The values $f(m)$ are (usually) much smaller than the sum of all terms, so the individual pure points of the μ_n disappear in the averaging as n tends to infinity. The measure μ is the ethereal imprint left behind, the ghost of the departed pure points of the μ_n.

look at the sequence

$$a(n) = \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, a classical 1916 result of Kempner [13], $A(z) = \sum_{n\geqslant 0} a(n)z^n = \sum_{n\geqslant 0} \alpha^{2^n}$ is transcendental for $\alpha = 1/b$ for any positive integer $b \geqslant 2$, and using Mahler's method, for any nonzero algebraic number α with $|\alpha| < 1$. It is quite clear as well, that $a(n)$ is 2-regular and that $\mu_a = \delta_0$. While there are many options for absolutely continuous measures, for us, one stands out. If one takes t to be the Thue–Morse sequence, but now with values in $\{0, 1\}$, then since the values 1 are extremely evenly distributed—there are never three zeros in a row, and the density of ones is $1/2$—we necessarily have that $\mu_t = (\mu_t)_{\mathrm{ac}}$ is purely absolutely continuous. Note that this is in contrast to the result for the diffraction measure $\widehat{\gamma}_t = (\widehat{\gamma}_t)_{\mathrm{sc}}$, which is purely singular continuous. Transcendence follows from Theorem 1 above. As a complementary example, the sequence of positive integers gives rise to an example of a number that is rational at rational values and irrational algebraic at irrational algebraic values, and which gives an absolutely continuous measure. It may very well be the case that every (not eventually zero) linear recurrent sequence of nonnegative integers gives rise to an absolutely continuous measure—we have yet to fully examine this question.

As a final comment, let us mention one of our favourite questions, the finiteness conjecture. Here, a finite set of matrices $\mathcal{A} = \{\mathbf{A}_0, \ldots, \mathbf{A}_{k-1}\}$ is said to satisfy the *finiteness property* provided there is a finite product $\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}$ of these matrices such that

$$\rho(\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}})^{1/m} = \rho^*(\mathcal{A}).$$

Arising from the work of Daubechies and Lagarias [9, 10], Lagarias and Wang [14] conjectured that the finiteness property holds for all finite sets of real matrices. This was shown to be false, in general, first by Bousch and Mairesse [6], then constructively by Hare, Morris, Sidorov and Theys [12]. The finiteness conjecture for rational matrices—equivalent to that for integer matrices—remains open. In our joint work with Evans, Gohlke and Mañibo [8], in relation to our work with the ghost measure, we established a new case of the finiteness conjecture.

Theorem 10. *Let $\mathcal{A}$ be a finite set of $d \times d$ nonnegative matrices with $\rho^*(\mathcal{A}) = \rho(\mathbf{A})$. Then $\mathcal{A}$ has the finiteness property.*

Acknowledgements

This paper is a slightly extended version of a talk I gave at the Institute for Research in the Mathematical Sciences (RIMS) at Kyoto University, Japan, in October 2023. I thank Yu Yasufuku and Maki Nakasuji for their kind invitation, and I thank Yohei Tachiya and Hirosaki University for their generous hospitality and support.

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