On the order estimation of the double L-function

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Abstract

Double L-function is a generalization of classical Dirichlet L-function to two variable functions. We give upper bounds of double L-functions, and Our upper bounds are explicit in conductor aspect. This article a review of the author's talk given at the "RIMS workshop 2023 Analytic Number Theory and Related Topics".

§1. Introduction

Let $s = \sigma + it$ be a complex variable. The study of moments of the Riemann zeta-function is an active research area in analytic number theory. Classically, it is known that

$$\int_1^T |\zeta(1/2+it)|^2 dt = T\log T - T(\log(2\pi) - 2\gamma + 1) + E. \tag{1}$$

In 1949, Atkinson [2] obtained the precise formula of the error term E in (1). First, Atkinson divided the product of two Riemann zeta-functions into

$$\zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2),$$

where

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{s_1}(m+n)^{s_2}}.$$

By using the Poisson summation formula, he proved a meromorphic continuation of $\zeta_2(s_1, s_2)$ to a certain domain. Nowadays $\zeta_2(s_1, s_2)$ is called the double zeta-function and is widely studied. So Atkinson's work is the first result of analytic aspects of $\zeta_2(s_1, s_2)$. It is known that mean squares are closely related to the Lindelöf hypothesis. So the double zeta-function is not only a generalization of the Riemann zeta-function, but also has a correspondence to the moment problem.

In this direction, we want to know the magnitude of the double zeta-function. This was first done by Ishikawa and Matsumoto [8] and they showed the upper bounds of the double zeta-function. Their result was improved by several works by Kiuchi and Tanigawa [10],

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Kiuchi, Tanigawa and Zhai [11] and Banerjee, Minamide and Tanigawa [3]. In the latest work of [3], they showed

$$\zeta_2(1/2+it_1, 1/2+it_2) \ll |t_1|^{\mu(\frac{1}{2})+\varepsilon} |t_2|^{\mu(\frac{1}{2})+\varepsilon}$$

for $|t_2|^{\frac{\frac{1}{2}-\mu(\frac{1}{2})}{1+\mu(\frac{1}{2})}} \ll |t_1| \ll |t_2|^{\frac{1+\mu(\frac{1}{2})}{\frac{1}{2}-\mu(\frac{1}{2})}}$, where

$$\mu(\sigma) := \inf\{c \geq 0 \mid \zeta(\sigma+it) \ll |t|^c\}. \tag{2}$$

Since it is conjectured that $\mu(1/2) = 0$ (the Lindelöf Hypothesis). So [3] implies that under the Lindelöf Hypothesis, the double zeta-function can become small when the condition (2) holds. Unconditionally, Bourgain [4] showed that $\mu(1/2) \leq 13/84$.

§2. Main results

Let χ be a Dirichlet character modulo q and let $L(s,\chi)$ denote the corresponding Dirichlet L-function defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This series converges absolutely for $\sigma > 1$ and can be extended analytically to all $\mathbb{C}$-plane, except for a simple pole at $s = 1$ if χ is the principal character. For the case of Dirichlet L-functions, the order on the critical line is also conjectured (the generalized Lindelöf Hypothesis). It asserts that for $L(s,\chi)$ and for all $\varepsilon > 0$, it holds that

$$L(1/2+it,\chi) \ll_\varepsilon ((|t|+1)q)^\varepsilon.$$

As a generalization of Dirichlet L-functions, we define double L-functions. Let χ_1, χ_2 be Dirichlet characters modulo q. Then the corresponding double Dirichlet L-function is defined by

$$L_2(s_1,s_2;\chi_1,\chi_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)}{m^{s_1}(m+n)^{s_2}}. \tag{3}$$

The double L-function can be continued meromorphically to $\mathbb{C}^2$, and if χ_1, χ_2 are both non-principal, then $L_2(s_1,s_2;\chi_1,\chi_2)$ is entire on $\mathbb{C}^2$. We prove the following result.

Theorem 1. *Let $(s_1,s_2) \in \mathbb{C}^2$ with $\sigma_1+\sigma_2 > 0$ and $0 < \sigma_2 < 1$. Suppose that χ_1, χ_2 are primitive Dirichlet characters modulo q. Then if $\chi_2(-1) = 1$ we have*

$$\begin{aligned} L_2(s_1,s_2;\chi_1,\chi_2) = {} & \frac{2(2\pi)^{s_2-1}\tau(\chi_2)\Gamma(1-s_2)}{q^{s_2}} \sum_{mn \leq \frac{q|t_2|}{2\pi}} \frac{\chi_1(m)\overline{\chi_2}(n)\sin(\frac{\pi s_2}{2}+\frac{2\pi mn}{q})}{m^{s_1}n^{1-s_2}} \\ & + O\left(q^{\frac{1}{2}}(q|t_2|)^{\delta+\varepsilon}\right), \end{aligned}$$

and if $\chi_1(-1)=-1$ *we have*

$$L_2(s_1,s_2;\chi_1,\chi_2)=\frac{2(2\pi)^{s_2-1}\tau(\chi_2)\Gamma(1-s_2)}{iq^{s_2}}\sum_{mn\leq\frac{q|t_2|}{2\pi}}\frac{\chi_1(m)\overline{\chi_2}(n)\cos(\frac{\pi s_2}{2}+\frac{2\pi mn}{q})}{m^{s_1}n^{1-s_2}}$$
$$+O\left((1+|t_1+t_2|)q^{\frac{3}{2}+\varepsilon}(q|t_2|)^{-\min\{1,\frac{\sigma_1+\sigma_2}{2}\}+\varepsilon}\right)+O\left(q^{\frac{1}{2}}(q|t_2|)^{\delta+\varepsilon}\right),$$

where $\tau(\chi_2)$ *is the Gauss sum,* $\delta=\max\{0,1-\sigma_1-\sigma_2\}$ *and implicit constants are independent on* q,t_1 *and* t_2*.*

Remark 2. *It is not necessary to restrict the conductor of the characters to the same one, but for simplicity, we fix the same conductor.*

The above formulas can be regarded as approximation formulas for double L-functions. Since $2(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s/2), 2(2\pi)^{s-1}\Gamma(1-s)\cos(\pi s/2)\asymp t^{\frac{1}{2}-\sigma}$, Theorem 1 implies the following result.

Theorem 3. *Let* $(s_1,s_2)\in\mathbb{C}^2$ *with* $\sigma_1+\sigma_2>0, 0<\sigma_2<1$ *and* $|t_2|\geq 2$*. Suppose that* χ_1,χ_2 *are primitive Dirichlet characters modulo q. Then if* $\chi_2(-1)=1$ *we have*

$$L_2(s_1,s_2;\chi_1,\chi_2)\ll(q\,|t_2|)^{\frac{1}{2}+\delta+\varepsilon},$$

and if $\chi_2(-1)=-1$ *we have*

$$L_2(s_1,s_2;\chi_1,\chi_2)\ll(q\,|t_2|)^{\frac{1}{2}+\delta+\varepsilon}+(1+|t_1+t_2|)q^{\frac{3}{2}+\varepsilon}(q\,|t_2|)^{-\min\{1,\frac{\sigma_1+\sigma_2}{2}\}},$$

where $\delta=\max\{0,1-\sigma_1-\sigma_2\}$ *and the implicit constants are independent on* q,t_1 *and* t_2*.*

When χ_1,χ_2 are both primitive, we succeed in extending the region where the order estimates hold to $\sigma_1+\sigma_2>0$ if $0<\sigma_2<1$. Moreover, when $0<\sigma_2<1$, our bounds are better than [13], even including the conductor aspect when $q\ll|t_2|^{2\theta_2(\eta)-\varepsilon}$ if $\chi_2(-1)=1$, and when $q\ll|t_2|^{\theta_2(\eta)-\varepsilon}/|t_1+t_2|$ if $\chi_2(-1)=-1$, where $\theta_2(\sigma)=\inf\{\xi\geq 0\mid L(\sigma+it,\chi_2)\ll|t|^{\xi}\}$.

We omit the proofs of Theorem 1 and Theorem 3. For details, see [15]. We describe ideas of the proofs in the next section.

§3. Confluent hypergeometric functions

We consider the Kummer's equation

$$z\frac{d^2w}{dz^2}+(c-z)\frac{dw}{dz}-aw=0. \tag{4}$$

It is known that Kummer's confluent hypergeometric function

$${}_1F_1(a,c;x)=\sum_{n=0}^{\infty}\frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)}\frac{x^n}{n!},$$

and Tricomi's confluent hypergeometric function

$$\Psi(a,c;x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}{}_1F_1(a,c;x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}{}_1F_1(a-c+1,2-c;x)$$

are standard solution of (4).

From [14], [9], an expression in terms of $\Psi(a,c;x)$ is known for $\zeta_2(s_1,s_2)$. Moreover, in 2011, Komori, Matsumoto and Tsumura [12] showed that the similar expression holds for $L_2(s_1,s_2;\chi_1,\chi_2)$. In this work, we prove the following expression of $L_2(s_1,s_2;\chi_1,\chi_2)$ which is slightly different from [12].

Lemma 4. *If χ_1,χ_2 are both primitive, then for $\sigma_1+\sigma_2>0$ we have*

$$L_2(s_1,s_2;\chi_1,\chi_2)=\frac{\varepsilon(\chi_2)}{\sqrt{q}}\sum_{m,n=1}^{\infty}\frac{\chi_1(m)\overline{\chi_2}(n)}{m^{s_1+s_2-1}}$$
$$\times\left\{e^{\frac{\pi i}{2}\kappa_2}\Psi(1,2-s_2;2\pi imn/q)+e^{-\frac{\pi i}{2}\kappa_2}\Psi(1,2-s_2;-2\pi imn/q)\right\},$$

where $\varepsilon(\chi_2)=\frac{\tau(\chi_2)}{\sqrt{q}i^{\kappa_2}}$ with denoting κ_2 as $\chi_2(-1)=(-1)^{\kappa_2}$.

Then, we can transform Toricomi's confluent hypergeometric function to the incomplete gamma function of second kind. By using the classical formula

$$x^a e^{-x}\Psi(1,a+1;x)=\Gamma(a,x),$$

we can calculate $L_2(s_1,s_2;\chi_1,\chi_2)$ because we can apply the estimates of integrals introduced by Hardy and Littlewood [6].

§4. Other double L-functions

We describe a similar argument for different other double L-functions. For Dirichlet characters χ_1,χ_2, the corresponding double L-functions which is different from (3) is defined by

$$\mathcal{L}_2(s_1,s_2;\chi_1,\chi_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi_1(m)\chi_2(m+n)}{m^{s_1}(m+n)^{s_2}}.$$

This double L-function was first introduced by Goncharov [5], and Akiyama and Ishikawa [1]. In [1], they showed that even though χ_1,χ_2 are both primitive, $\mathcal{L}_2(s_1,s_2;\chi_1,\chi_2)$ is not entire on $\mathbb{C}^2$. Indeed, they showed that the true singularities of $\mathcal{L}_2(s_1,s_2;\chi_1,\chi_2)$ are given by

$$\begin{cases} s_1+s_2=0,-2,-4,-6,-8,\dots & \text{if} \quad \chi_1\chi_2(-1)=1,\\ s_1+s_2=1,-1,-3,-5,-7,\dots & \text{if} \quad \chi_1\chi_2(-1)=-1.\end{cases}$$

In 2002, Ishikawa [7] considered

$$H(x)=\sum_{\substack{m_1m_2\le x\\ m_1<m_2}}\chi_1(m_1)\chi_2(m_2)=\sum_{n\le x}h(n),$$

where

$$h(n)=\sum_{\substack{n_1n_2=n\\ n_1<n_2}}\chi_1(n_1)\chi_2(n_2)$$

and obtained the Riesz mean of $h(n)$. We should remark that he considered the multiple L-function of general depth, but for simplicity, we omit details.

If we can obtain sharper upper bounds of $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$, it might be possible to obtain an asymptotic formula for $H(x)$. However, we cannot apply our method to $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$ when $\chi_1\chi_2(-1) = -1$ because in this case, the series in Lemma 4 does not converge. Moreover, when $\chi_1\chi_2(-1) = 1$, there is no pole of $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$ in the domain $\sigma > 0$. So even though we obtain tight upper bound, the main term does not appear. We expect that it is difficult to extend our method to the domain $\sigma_1, \sigma_2 < 0$. Therefore further studies are required to obtain an asymptotic formula for $H(x)$.

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