CONJECTURES OF DIOPHANTINE EQUATIONS ON PIATETSKI-SHAPIRO SEQUENCES

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ABSTRACT. Let $\alpha > 1$ be a non-integral real number. Let $PS(\alpha)$ be the set of positive integers of the form $\lfloor n^{\alpha} \rfloor$ for some $n \in \mathbb{N}$. In this article, we discuss the equation x + y = z, where $(x, y, z) \in PS(\alpha)^3$. The author conjectures that for almost all or all $2 < \alpha < 3$ the equation x + y = z has infinitely many solutions $(x, y, z) \in PS(\alpha)^3$. In this article, we aim to present heuristic and numerical evidence of the conjecture.

1. INTRODUCTION

Let \mathbb{N} be the set of all positive integers. We say that an integral sequence $(a_n)_{n \in \mathbb{N}}$ is a Piatetski-Shapiro sequence if there exists a non-integral $\alpha > 1$ such that $a_n = \lfloor n^{\alpha} \rfloor$ for all $n \in \mathbb{N}$. Further, we define $PS(\alpha) = \{ \lfloor n^{\alpha} \rfloor : n \in \mathbb{N} \}$. In this article, we discuss the following equation:

$$(1.1) x+y=z,$$

where $(x, y, z) \in PS(\alpha)^3$. The equation (1.1) is an extension of Fermat's equation $(x^n + y^n = z^n)$ from the integral powers to real powers of integers. By Fermat's last theorem, the equation (1.1) does not have any solutions $(x, y, z) \in PS(\alpha)^3$ if $\alpha \ge 3$ and $\alpha \in \mathbb{N}$.

By using the result given by Frantzikinakis and Wierdl [FW09, Proposition 5.1], for every $\alpha \in (1, 2)$, there are infinitely many solutions $(x, y, z) \in PS(\alpha)^3$ to (1.1). Further, the author [Sai] showed that for almost all $\alpha > 3$ in the sense of the 1-dimensional Lebesgue measure, the equation (1.1) has at most finitely many solutions $(x, y, z) \in PS(\alpha)^3$. We describe these results in the following figure.

For all real numbers s, t with 1 < s < t, we define

(1.2) $\mathcal{A}(s,t) = \{ \alpha \in [s,t] : (1.1) \text{ has infinitely many solutions } (x,y,z) \in \mathrm{PS}(\alpha)^3 \}.$

Matsusaka and the author [MS21, Theorem 1.1] showed that for all real numbers s, t with 2 < s < t, the Hausdorff dimension of $\mathcal{A}(s,t)$ is not less than $1/s^3$. In particular, $\mathcal{A}(s,t)$ is dense in [s,t]. However, we do not know whether $\mathcal{A}(2,3)$ has positive Lebesgue measure or not. Here, the author conjectures the following.

Conjecture 1.1. For almost all or all $\alpha \in (2,3)$, the equation (1.1) would have infinitely many solutions $(x, y, z) \in PS(\alpha)^3$.

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The goal of this article is to give heuristic and numerical observations on Conjecture 1.1. This problem was first proposed by Glasscock [Gla17]. He investigated the equation

$$(1.3) y = ax + b$$

where a and b are fixed real numbers satisfying that (1.3) has infinitely many solutions $(x, y) \in \mathbb{N}^2$. In [Gla17, Gla20], he showed that for almost all $\alpha > 1$

- (1) if $\alpha < 2$, then (1.3) has infinitely many solutions $(x, y) \in PS(\alpha)^2$;
- (2) if $\alpha > 2$, then (1.3) has at most finitely many solutions $(x, y) \in PS(\alpha)^2$.

The author also contributed to the solvability of (1.3) on $PS(\alpha)$. In [Sai22], if we additionally assume $0 \le b < a$, then the author showed that for all $1 < \alpha < 2$, we find infinitely many solutions $(x, y) \in PS(\alpha)^2$ to (1.3). Thus, he improved (1) of Glasscock's result from "for almost all" to "for all" in the case $1 < \alpha < 2$. Moreover, in the same article, the author showed that for all 2 < s < t, the Hausdorff dimension of the set

 $\{\alpha \in [s,t] : (1.3) \text{ has infinitely many } (x,y) \in \mathrm{PS}(\alpha)^2\}$

is coincident with 2/s. Without the assumption $0 \le b < a$, it is unknown whether his improvements are true or not.

Glasssock successfully discovered that $\alpha = 2$ is a threshold for the infiniteness or finiteness of solutions to (1.3) on $PS(\alpha)$. Further, he proposed the following interesting question.

Question 1.2 ([Gla17, Question 6]). Does there exist $\mathscr{G} > 1$ with the property that for almost every or all $\alpha > 1$, the equation x + y = z has infinitely many solutions $(x, y, z) \in PS(\alpha)^3$ or not, according as $\alpha < \mathscr{G}$ or $\alpha > \mathscr{G}$?

The result of Matsusaka and the author [MS21, Theorem 1.1] leads to a negative answer to the "all" version of Question 1.2. The "almost all" version is still open, which is directly connected with Conjecture 1.1

Notation 1.3. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. For all intervals $I \subset \mathbb{R}$, let $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. We write f(x) = g(x) + o(h(x)) if $(f(x) - g(x))/h(x) \to 0$ as $x \to \infty$, where h(x) is a positive-valued function.

2. HEURISTIC OBSERVATIONS

For every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$s(\alpha, N) = \{ (\lfloor p^{\alpha} \rfloor, \lfloor q^{\alpha} \rfloor) \colon p, q \in \mathbb{N}, \ p \le q \le N, \ \lfloor p^{\alpha} \rfloor + \lfloor q^{\alpha} \rfloor = \lfloor r^{\alpha} \rfloor \text{ for some } r \in \mathbb{N} \}.$$

In this section, by heuristic calculation, we propose the following conjecture. If Conjecture 2.1 was true for every $2 < \alpha < 3$, then Conjecture 1.1 would be also true.

Conjecture 2.1. For almost all or all $\alpha \in (2,3)$, we would have

$$#s(\alpha, N) = \left(\frac{C(\alpha)}{\alpha(3-\alpha)} + o(1)\right) N^{3-\alpha} \quad (as \ N \to \infty),$$

where $C(\alpha) \coloneqq \int_0^1 (1+x^{\alpha})^{1/\alpha-1} dx$.

Yoshida [Yos, Corollary 1.2] gave a quantitative result on the number of solutions (1.1). For all $1 < \alpha < (\sqrt{21} + 4)/5 = 1.7165 \cdots$, he showed that

$$(2.1) \qquad \#\{(\ell, m, n) \in \mathbb{N}^3 \colon \ell < x, \lfloor \ell^{\alpha} \rfloor + \lfloor m^{\alpha} \rfloor = \lfloor n^{\alpha} \rfloor\} = (\beta \alpha^{-\beta} \zeta(\beta) + o(1)) x^{\alpha(\beta-1)+1}$$

as $x \to \infty$, where $\beta = 1/(\alpha - 1)$ and $\zeta(s)$ denotes the Riemann zeta function. Note that the left-hand side of (2.1) is slightly different from $\#s(\alpha, N)$.

Heuristic evidence of Conjecture 2.1. By the definition of $s(\alpha, N)$, it follows that

$$\#s(\alpha, N) = \sum_{1 \le q \le N} \sum_{1 \le p \le q} \#[A, B)_{\mathbb{Z}},$$

where $A = (\lfloor p^{\alpha} \rfloor + \lfloor q^{\alpha} \rfloor)^{1/\alpha}$ and $B = (\lfloor p^{\alpha} \rfloor + \lfloor q^{\alpha} \rfloor + 1)^{1/\alpha}$. We now assume that

By 0 < B - A < 1, the assumption (2.2) never holds. We ignore this problem to calculate heuristically, which is not a small gap.

By ignoring the integral parts and applying Taylor's expansion, we see that

$$B - A \sim \frac{1}{\alpha} (p^{\alpha} + q^{\alpha})^{1/\alpha - 1} = \frac{q^{2-\alpha}}{\alpha} \left(1 + \left(\frac{p}{q}\right)^{\alpha} \right)^{1/\alpha - 1} \frac{1}{q}$$

Remark that we do not give the precise definition of " $X \sim Y$ ". The symbol means that X and Y share the almost same magnitude. By Riemann sum, we obtain

$$\lim_{q \to \infty} \sum_{1 \le p \le q} \left(1 + \left(\frac{p}{q}\right)^{\alpha} \right)^{1/\alpha - 1} \frac{1}{q} = \int_0^1 \left(1 + x^{\alpha} \right)^{1/\alpha - 1} dx = C(\alpha).$$

Therefore, we can expect that

$$\#s(\alpha, N) \sim \frac{1}{\alpha} \sum_{1 \le q \le N} \sum_{1 \le p \le q} (p^{\alpha} + q^{\alpha})^{1/\alpha - 1} \sim \frac{C(\alpha)}{\alpha} \sum_{1 \le q \le N} q^{2-\alpha} \sim \frac{C(\alpha)}{\alpha(3-\alpha)} N^{3-\alpha}.$$

By the above calculation, we should consider the set of $(p,q) \in \mathbb{N}^2$ such that $[A, B)_{\mathbb{Z}}$ is non-empty. However, the interval [A, B) would be too short to be investigated.

3. Numerical observations

In this section, we give numerical evidence on Conjectures 1.1 and 2.1. The author calculates $\#s(\alpha, N)$ for N = 50000 and $\alpha = 2.00 + 0.05i$ (i = 1, 2, ..., 19) by using Mathematica. The used program will be described at the end of this article. Mathematica outputs the following.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\#s(\alpha, 5\cdot 10^5)$	13035	7754	4763	2739	1653	1060	647	356	247	153
α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	_
$\#s(\alpha, 5\cdot 10^5)$	93	69	41	22	17	7	4	0	1	-

From the table, we do not guess the existence of a threshold like Glasscock's result on (1.3). Let us note that N = 50000 may be still small. Indeed by numerical calculation, for every $\alpha \in \{2.8, 2.85, 2.9, 2.95\}$, we obtain

(3.1)
$$\frac{C(\alpha)}{\alpha(3-\alpha)} 50000^{3-\alpha} \le \frac{C(2.8)}{2.8(3-2.8)} 50000^{3-2.8} = 13.70\cdots$$

Thus, it is too small to guess whether Conjecture 1.1 is true or not. In the case $\alpha = 2.95$, if we chose $N \ge 1.671 \times 10^{59}$, then the most left-hand side of (3.1) would be greater than 1000.

On the other hand, we expect that Conjecture 2.1 is true for small $\alpha > 2$. Indeed, for every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$\delta(\alpha, N) = \#s(\alpha, N) \cdot \frac{\alpha(3-\alpha)}{C(\alpha)} N^{\alpha-3} - 1.$$

If $\delta(\alpha, N) \to 0$ as $N \to \infty$, then Conjecture 2.1 would be true for α . The following figure describes $\{(N, \delta(\alpha, N)) \mid N = 1, 2, \dots, 50000 \text{ and } \delta(\alpha, N) \in [-0.10, 0.02]\}$ for every $\alpha \in \{2.05, 2.10, 2.15, 2.20, 2.25, 2.30\}$.



Figure 1. Graphs of $\delta(\alpha, 50000)$

We also give the following table of $\delta(\alpha, 5 \cdot 10^5) 10^2$ to exhibit the calculation more clearly. Remark that it is not $\delta(\alpha, 5 \cdot 10^5)$, but $\delta(\alpha, 5 \cdot 10^5) \times 10^2$.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\delta(\alpha, 5 \cdot 10^5) 10^2$	-0.970	-1.73	0.303	-4.55	-5.10	-0.258	-0.781	-0.116	-1.43	-2.73
α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	-
$\delta(\alpha, 5\cdot 10^5)10^2$	-6.79	7.62	-2.07	-21.2	-11.2	-48.9	-61.7	-100	-90.3	-

Thus, $\delta(\alpha, N)$ seems small for several α 's. This supports Conjecture 2.1. Moreover, we observe that almost all values in the table are negative. Thus, we expect that almost every (or every) $\alpha \in (2, 3)$ would satisfy that

$$\#s(\alpha, N) \le \frac{C(\alpha)}{\alpha(3-\alpha)} N^{3-\alpha}$$

for sufficiently large N. At last, we give a program to calculate $\#s(\alpha, N)$ in Mathematica.

```
ps[alpha_,n_]:= Floor[n^alpha];
.....
For[alpha = 2.05, alpha <= 2.95, alpha = alpha + 0.05,
sols = {}; gamma = 1/alpha;
For[p = 1, p <= 50000, p = p + 1, For[q = p, q <= 50000, q = q + 1,
sum = ps[alpha,p] + ps[alpha,q]; r = Ceiling[sum^gamma];
If[sum == ps[alpha,r],
sols = Join[sols, {{p,q}}];
];];];
Print[{alpha, Length[sols]}]; Print[sols];]
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