GAPS BETWEEN PRIME NUMBERS THAT SATISFY THE GOLDBACH EQUATION

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1. Introduction

In the theory of prime numbers, we are interested in the twin prime conjecture which states that there exist infinitely many prime numbers p such that $p+2$ is also prime. If p_n denotes the n-th prime, the prime number theorem says the average of gaps between consecutive primes $p_{n+1}-p_n$ is $\log p_n$. Thus the earlier approach to the twin prime conjecture involved comparing gaps to the average. The first non-trivial result that the inequality

$$p_{n+1}-p_n < (1-\delta)\log p_n \quad (\text{for some } \delta > 0) \tag{1.1}$$

holds for infinitely many p_n was given by Erdős [3]. His proof is given by elementary argument with the sieve method. Bombieri and Davenport [1] proved that we can take $\delta = 1/2$ in the inequality (1.1) by using the circle method. Eventually Goldston, Pintz, and Yıldırım [4] proved that, for any $\varepsilon > 0$, we can take $\delta = 1-\varepsilon$ by using the sieve method. One can find more details of the history in [4].

On the other hand, the Goldbach conjecture is also an important problem in the theory of prime numbers. If $\mathbb{P}(N)$ denotes the set of primes $p \leq N$ which $N-p$ is also prime for a given positive integer N, The Goldbach conjecture says $\#\mathbb{P}(N) > 0$ for any even integer $N \geq 4$. Furthermore, Hardy and Littlewood conjectured the asymptotic formula

$$\#\mathbb{P}(N) = \frac{N}{(\log N)^2}\left(\mathfrak{S}(N)+o(1)\right) \quad (N \to \infty).$$

Here $\mathfrak{S}(N)$ is called the singular series for the Goldbach conjecture and it is defined by

$$\mathfrak{S}(N) = \mathfrak{S}\prod_{\substack{p|N\\ p>2}}\frac{p-1}{p-2}, \quad \mathfrak{S} = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

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for even N and $\mathfrak{S}(N) = 0$ for odd N. One can prove Hardy and Littlewood's conjecture for almost all even integers (cf.[8]). Thus we can say the average of gaps between consecutive elements in $\mathbb{P}(N)$ is $\mathfrak{S}(N)^{-1}(\log N)^2$ for almost all N. Mikawa and Nakamura proved (1.1) type result for the set $\mathbb{P}(N)$ as

$$\min_{p,p' \in \mathbb{P}(N)} |p - p'| \leq \left(\frac{5}{6} + \varepsilon\right) \mathfrak{S}(N)^{-1}(\log N)^2 \tag{1.2}$$

holds for almost all even integers and $\varepsilon > 0$.[1] The author recently proved an improvement of (1.2) as follows.

Theorem 1.1. *Let $\varepsilon > 0$ be an arbitrarily positive. Then, for any $A > 0$, we have*

$$\min_{p,p' \in \mathbb{P}(N)} |p - p'| < \left(\frac{20\sqrt{3} + 15\sqrt{5}}{48\sqrt{3}} + \varepsilon\right) \mathfrak{S}(N)^{-1}(\log N)^2$$

for large enough $X > 0$ and all but $O(X(\log X)^{-A})$ even integers N in the range $X/2 < N \leq X$. Here the constant

$$\frac{20\sqrt{3} + 15\sqrt{5}}{48\sqrt{3}} = 0.8201\ldots.$$

is slightly smaller than $5/6$.

This note provides a short introduction to the proof of Theorem 1.1. As a remark we haven't managed to apply the method of [4] at the moment. Therefore we use Mikawa and Nakamura's idea and Rosser-Iwaniec's linear sieve method.

2. Outline of the proof

We define

$$Z(N; 2n) = \sum_{\substack{p,p' \in \mathbb{P}(N) \\ p' = p + 2n}} (\log p)(\log(N - p))(\log p')(\log(N - p'))$$

for an integer n. If we prove the positivity for $Z(N; 2n)$, there exist primes p, p' that satisfy the Goldbach equation and $p - p' = 2n$. Mikawa and Nakamura proved the following average result.

[1]This result has not been published.

Proposition 2.1. *Let $\varepsilon > 0$ be an arbitrarily positive and $X > 0$ be large enough. We assume an integer k satisfies $\log k \ll \log\log X$. Then we have*[2]

$$\sum_{n=1}^{k}(k-n)Z(N;2n) > k^2\mathfrak{S}(N)^2N - \frac{5}{12}k\mathfrak{S}(N)N(\log N)^2 + O\left(k^{3/2+\varepsilon}\mathfrak{S}(N)^2N(\log X)^{\varepsilon}\right) + \Delta(N).$$

We easily deduce (1.2) from Proposition 2.1 for almost all N. The proof of Proposition 2.1 is an application of the circle method inspired by Bombieri and Davenport [1]. Similar to some other cases, the proof depends on the famous Bombieri's prime number theorem.

One of the ways to improve Mikawa and Nakamura's result (1.2) is calculating an upper bound for $Z(N;2n)$. The singular series for $Z(N;2n)$ is defined as follows. Let $\nu_{N,n}(p)$ be the size of the set $\{0, -2n, N, N-2n\}$ modulo p. Then we define the singular series $\mathfrak{T}(N,n)$ for $Z(N;2n)$ such that

$$\mathfrak{T}(N,n) = \prod_p \left(1 - \frac{\nu_{N,n}(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-4}.$$

The author proved the following upper bound for $Z(N;2n)$.

Theorem 2.2. *Let $\varepsilon > 0$ be arbitrary and $X > 0$ be large enough. We assume an integer k satisfies $\log k \ll \log\log X$. Then we have*

$$Z(N;2n) \le (16+\varepsilon)\mathfrak{T}(N,n)N + \Delta(N)$$

for uniformly in $n \le k$.

By combining Proposition 2.1 and Theorem 2.2, we can prove Theorem 1.1. The proof of Theorem 1.1 almost follows the argument of Bombieri and Davenport [1].

3. Application of the linear sieve to $Z(N;2n)$

We use

$$Z_1(N;2n) = \sum_{\substack{p_1+p_2=N \\ p_1+2n\in\mathbb{P}(N)}} (\log p_1)(\log p_2)$$

instead of $Z(N;2n)$. The upper bound for $Z_1(N;2n)$ is calculated by the combination of the sieve method and the circle method. If λ_d is an

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[2] When we write $f(N) = \Delta(N)$ for a function $f(N)$, then $f(N)$ is small in average namely $\sum_{N\le X}|f(N)|^2 \ll_A X^3(\log X)^{-A}$ for any $A > 0$.

upper bound sieve weight of level $D > 0$, then one can see

$$Z_1(N; 2n) \leq \sum_{\substack{p_1+p_2=N \\ (p_1+2n.P(z))=1 \\ (p_2-2n,P(z))=1}} (\log p_1)(\log p_2) + O\left(z(\log X)^2\right)$$

$$\leq \sum_{p_1+p_2=N} (\log p_1) \left(\sum_{\substack{d|P(z) \\ d|p_1+2n}} \lambda_d \right) (\log p_2) \left(\sum_{\substack{d|P(z) \\ d|p_2-2n}} \lambda_d \right)$$

where $P(z) = \prod_{p<z} p$ and $2 \leq z \leq X^{1/2}$. Thus we apply the circle method to the last sum and then we have to estimate the type of trigonometric sum with sieve weight

$$\sum_{p \leq X} (\log p) \left(\sum_{\substack{d|P(z) \\ d|p+h}} \lambda_d \right) e(\alpha p)$$
$$= \sum_{\substack{d|P(z) \\ d \leq D}} \lambda_d \sum_{\substack{p \leq X \\ p \equiv -h \pmod d}} (\log p) e(\alpha p)$$

for $h \in \mathbb{Z}, \alpha \in \mathbb{R}$. The main term that occurs from the major arc integration is

$$\mathfrak{T}(N, n) \frac{N}{(\log D)^2}(4 + o(1)),$$

thus we need to find large D which we can deal with the minor arc integral. A good estimate of our trigonometric sum on the minor arc was given by Matomäki when λ_d is *well-factorable.*

If we say an arithmetic function λ is *well-factorable of level* D, then for any decomposition $MN = D$ with $M, N \geq 1$ we can find arithmetic functions α and β supported on $[1, M]$ and $[1, N]$ respectively that satisfy

$$\lambda = \alpha * \beta, \ |\alpha(m)| \leq 1, \ |\beta(n)| \leq 1$$

where $\alpha * \beta$ is the Dirichlet convolution of α and β. This property was first introduced by Iwaniec [5] and he found a well-factorable structure for the linear sieve. The strength of the well-factorable property is the following Matomäki's minor arc estimate.

Proposition 3.1 (K. Matomäki, [6]). *Let c be an integer. Then, for any $A > 0$, there exists $B > 0$ such that if λ is a well-factorable function* *of level* $X^{1/2}(\log X)^{-B}$ *and* α *satisfies* $|\alpha - a/q| \le q^{-2}$ *for*

$$(\log X)^B \le q \le \frac{X}{(\log X)^B},\ 1 \le a \le q,\ (a,q) = 1$$

we have

$$\sum_{\substack{d \le X^{1/2}(\log X)^{-B} \\ (d,c)=1}} \lambda(d) \sum_{\substack{p \le X \\ p \equiv c \ (\mathrm{mod}\ d)}} (\log p) e(\alpha p) \ll \frac{X}{(\log X)^A}.$$

The minor arc estimate of such trigonometric sum was first given by Mikawa [7] with a well-factorable function of level $X^{4/9}(\log X)^{-B}$. Thus we employ the linear sieve to estimate $Z_1(N; 2n)$ and Matomäki's result allows us to take $D \sim X^{1/2}(\log X)^{-B}$. As a technical remark, we need some asymptotic formula and an idea of the vector sieve introduced by Brüdern and Fouvry [2] to deal with the major and minor arc calculation. These arguments give a basis for the proof of Theorem 2.2.

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