SELBERG TRACE FORMULA FOR MAASS FORMS OF WEIGHT 0 AND 1

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1. Maass forms

For a fundamental discriminant d, we define a quadratic character of conductor $|d|$ by the Kronecker symbol

$$\psi_d(n) = \left(\frac{d}{n}\right) \qquad \text{for } n \in \mathbb{Z}.$$

The L-function

$$L(s,\psi_d) = \sum_{n=1}^{\infty} \frac{\psi_d(n)}{n^s} \qquad \text{for } \Re(s) > 1$$

continues to an entire function and satisfies the functional equation

$$\Lambda(s,\psi_d) = |d|^{\frac{s}{2}} \pi^{-\frac{s+\epsilon_d}{2}} \Gamma\left(\frac{s+\epsilon_d}{2}\right) L(s,\psi_d) = i^{-\epsilon_d} \frac{\tau(\psi_d)}{\sqrt{|d|}} \Lambda(1-s,\psi_d)$$

where $\epsilon_d = \frac{1-\psi_d(-1)}{2}$. We now let $F = \mathbb{Q}(\sqrt{d})$, a quadratic extension of $\mathbb{Q}$ and the discriminant of F is d. Then the Dedekind zeta function of F can be written as a product of $\zeta(s)$ and $L(s,\psi_d)$:

$$\zeta_F(s) = \sum_{\mathfrak{n} \subset \mathfrak{o}_F} N(\mathfrak{n})^{-s} = \zeta(s) L(s,\psi_d).$$

Here $\mathfrak{o}_F$ is the ring of integers of F and $N(\mathfrak{n})$ is the norm of the ideal $\mathfrak{n} \subset \mathfrak{o}_F$. Moreover,

$$\zeta(s+it)L(s-it,\psi_d) = \sum_{m \geq 1} \frac{m^{it} \sum_{n|m} \psi_d(n) n^{2it}}{m^s}$$

is the L-function of the Eisenstein series with the quadratic character ψ_d.

The Dedekind zeta function is the L-function of the trivial character for $\mathfrak{o}_F$. In the 1920s, Hecke proved that the L-functions associated with Hecke characters have analytic continuation and satisfy functional equations. He also demonstrated that when F is an imaginary quadratic field, the L-functions associated with Hecke characters correspond to the L-functions of modular forms of weight 1. Maass [12] constructed weight 0 non-holomorphic modular forms from the L-functions of Hecke characters of real quadratic fields.

Let N be a positive integer and χ be a Dirichlet character modulo N. Take $k \in \{0,1\}$ such that $\chi(-1) = (-1)^k$. A Maass form of weight k, level N with nebentypus character χ is a smooth function φ on the Poincaré upper half plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ which satisfies the following conditions:

- $\varphi\left(\frac{az+b}{cz+d}\right) = \chi(d) \left(\frac{cz+d}{|cz+d|}\right)^k \varphi(z)$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$;
- $\int_{\Gamma\backslash\mathbb{H}} |\varphi|^2 \, d\mu < \infty$ where $d\mu = \frac{dx\, dy}{y^2}$;
- φ is an eigenfunction of the classical Laplace-Beltrami operator of weight k with the eigenvalue $\lambda \geq 0$

$$\Delta\varphi = \lambda \cdot \varphi$$

where $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$ is the negative of the classical Laplace-Beltrami operator.

A Maass form φ is cuspidal if $\varphi(z)$ is vanishing as z is approaching to any cusp of $\Gamma_0(N)\backslash\mathbb{H}$.

For $\lambda \geq 0$, we let $\mathcal{A}_k(\lambda;\chi)$ be the vector space of Maass forms of weight k, level N with nebentypus χ. A constant function is a Maass form when χ is the trivial character modulo N. We have $\mathcal{A}_0(\lambda;\chi) = 0$ when $\lambda < 0$. Selberg eigenvalue conjecture implies that $\mathcal{A}_0(\lambda;\chi) = 0$ for $0 < \lambda < \frac{1}{4}$. The conjecture is still widely open. Selberg showed that [14] $\mathcal{A}_0(\lambda;\chi) = 0$ for $0 < \lambda < \frac{1}{4} - \delta$ with $\delta = \frac{1}{16}$. The best known result towards the Selberg eigenvalue conjecture is due to H. Kim and Sarnak [10] which pushed δ to $\left(\frac{7}{64}\right)^2$. When $k = 1$, due to the local representation theory (see [5, Theorem 2.6.3]), we have $\mathcal{A}_1(\lambda;\chi) = 0$ when $\lambda < \frac{1}{4}$. When $\lambda = \frac{1}{4}$, weight 1 holomorphic modular forms also lie in $\mathcal{A}_1(\frac{1}{4};\chi)$.

By a theorem of Freidrichs the Laplace-Beltrami operator Δ admits a self-adjoint extension and by a theorem of von Neumann the space $L^2(\Gamma_0(N)\backslash\mathbb{H},\chi,k)$ has a complete spectral resolution with respect to Δ. The space decomposes into the space of Maass cusp forms and continuous spectrum which is generated by Eisenstein series (see [6, §4]).

For $n \in \mathbb{Z}_{\geq 1}$, $\gcd(n,N) = 1$, the Hecke operator $T_{\pm n} : \mathcal{A}_k(\lambda;\chi) \to \mathcal{A}_k(\lambda;\chi)$ can be defined by

$$(T_n\varphi)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \bmod d} \varphi\left(\frac{az+b}{d}\right)$$

and

$$(T_{-n}\varphi)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(-a) \sum_{b \bmod d} \varphi\left(\frac{a\bar{z}+b}{d}\right).$$

Then the space $\mathcal{A}_k(\lambda;\chi)$ can be further diagonalised into the space of simultaneous Hecke eigenfunctions.

2. Selberg trace formula

The knowledge about the weight 0 Maass forms has been accumulated greatly for the last 80 years after Maass' paper on the Maass forms [12]. Many mysteries still remain unsolved, but we have developed several methods to study them; one of them is the Selberg trace formula for Maass forms [13]. The Selberg trace formula, which was introduced by Selberg in 1956 [13], is a concrete tool to study Maass forms. Although Maass [12] constructed few examples of Maass forms with the L-functions for Hecke characters as explained in §1, the existence of Maass forms of level N in general is a highly non-trivial problem, even for $N = 1$. One consequence of the Selberg trace formula is an estimate for the density of eigenvalues, the Weyl's law, which implies that there are infinitely many Maass forms.

There are few numerical approaches towards implementing trace formulas computationally. The obstacle mainly lies in the complexity of the trace formulas. Selberg trace formulas for $\mathrm{SL}_2(\mathbb{R})$ are deeply studied in [8], but to implement the trace formulas in computer code, we should describe Selberg trace formulas explicitly.

In [3] and [2], Selberg trace formulas for square-free level and any level (for twist-minimal Maass forms) were obtained respectively. These trace formulas are explicit, and with them, conjectures regarding Maass forms were studied. One of the conjectures is the Selberg eigenvalue conjecture. Booker and Strömbergsson [3] showed that the Selberg eigenvalue conjecture is true for $\Gamma = \Gamma_0(N)$ for square-free $N \leq 854$. Note that $\Gamma(N) = \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma_1(N^2) \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$. So the result of [3] cannot recover Huxley's earlier result (with a different method) [9] that the Selberg eigenvalue conjecture is true for $\Gamma = \Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I_2 \bmod N\}$ for $N \leq 18$. With Booker and Strömbergsson [2], we showed that the Selberg eigenvalue conjecture is true for $\Gamma = \Gamma_0(N)$ for $N \leq 880$ and for $\Gamma = \Gamma(N)$ for $N \leq 226$. Comparing to Huxley's geometric method, the benefit of using the Selberg trace formula is that it actually identifies N when there is a possible Maass form with the eigenvalue $\frac{1}{4}$.

There are few algorithms available for numerical computation of Maass forms. Here, 'computing Maass forms' refers to the computation of Laplace eigenvalues and Hecke eigenvalues.

One of them is Hejhal's algorithm which was introduced in the 1970s. Based on this algorithm, Booker, Strömbergsson and Venkatesh computed the Laplacian and Hecke eigenvalues for Maass wave forms on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ to more than 1000 decimal places [4]. Seymour-Howell [15] developed an algorithm to compute the weight 0 Maass forms of square-free level with the trivial nebentypus character based on the Selberg trace formula in Strömbergsson's unpublished notes. The trace formula in Strömbergsson's notes is explicit enough to implement in computer code.

Several aspects of Maass forms remain in the dark. Moreover, the literature on weight 1 Maass forms is even more limited. In [1], we provide the Selberg trace formula for weight 1 Maass forms explicitly, exploring various applications including numerical ones. One such application is to develop an algorithm to compute weight 1 Maass forms efficiently, addressing the computational challenges inherent in their study. Following the method outlined in [15], we construct an algorithm to compute weight 1 Maass forms numerically. With the numerically computed data about Maass forms, we aim to uncover previously unknown features of Maass forms. For example, we will analyse the statistics of small eigenvalues [1].

In the next section, we present the Selberg trace formulas for weight 0 and 1 Maass forms for odd prime levels in (2.2) and (2.3).

2.1. **Selbeg trace formula of weight** 0 **and** 1**.** The Selberg trace formula for general weight is studied in [8, Chapter Four]. In this section, we provide the Selberg trace formula for weight 0 and 1 of an odd prime level explicitly [1]. To obtain the formulas (2.2) and (2.3), we start from [7, Theorem (6.33)] which gives the Selberg trace formula for GL(2) over any number field. We compute each term explicitly refer to [8, Chapter Four] and [11]. The details of the proof and applications will appear in [1].

Let $\widehat{h}:\mathbb{R}\to\mathbb{C}$ be even, continuous and absolutely integrable, with Fourier transform $h(r) = \int_{\mathbb{R}}\widehat{h}(u)e^{2\pi iru}\,du$. We say that the pair $(\widehat{h},h)$ is of trace class if there exists $\delta>0$ such that h is analytic on the strip $\Omega=\{r\in\mathbb{C}:\ |\Im(r)|<\frac{1}{2}+\delta\}$ and satisfies $h(r)\ll(1+|r|)^{-s-\delta}$ for all $r\in\Omega$.

Let N be an odd prime and χ be a primitive Dirichlet character mod N. Let $k\in\{0,1\}$ satisfying $\chi(-1)=(-1)^k$. Let n be a positive integer such that $\gcd(n,N)=1$. The trace formula is an expression for $\sum_{\lambda=\frac{1}{4}+r^2>0}\mathrm{Tr}\left(T_{\pm n}\middle|\mathcal{A}_k(\lambda;\chi)\right)h(r)$ in terms of $\widehat{h}$ and χ. Here, we express the continuous spectrum in terms of h, which can then be further expressed in terms of $\widehat{h}$.

We first list notations for the Selberg trace formulas (2.2) and (2.3).

For $D=d\ell^2$ for $\ell\in\mathbb{Z}_{\geq 1}$, we define

$$\psi_D(m)=\left(\frac{d}{m/\gcd(m,\ell)}\right)=\psi_d(m/\gcd(m,\ell)).$$

This is not multiplicative in general. For $\Re(s)>1$,

$$\begin{aligned}L(s,\psi_D)&=\sum_{m=1}^{\infty}\frac{\psi_D(m)}{m^s}\\&=L(s,\psi_d)\prod_{p|\ell}\left(1+(1-\psi_d(p))\sum_{j=1}^{\mathrm{ord}_p(\ell)}p^{-js}\right)\end{aligned}$$

so $L(s,\psi_D)$ continues analytically to $s\in\mathbb{C}$. At $s=1$, we have

$$L(1,\psi_D)=L(1,\psi_d)\prod_{p|\ell}\left(1+(1-\psi_d(p))\sum_{j=1}^{\mathrm{ord}_p(\ell)}p^{-j}\right).$$

We define, when $D=t^2\mp 4n=d\ell^2$, $\sqrt{D}\notin\mathbb{Z}$, d is a fundamental discriminant and $\ell\in\mathbb{Z}_{\geq 1}$,

$$H_{t,\pm n}(\chi)=\begin{cases}\chi\left(\frac{t+N\omega}{2}\right)\left(2+\frac{\psi_d(N)-1}{1+(N-\psi_d(N))\frac{N^{\mathrm{ord}_N(\ell)}-1}{N-1}}\right) & \text{if } N\mid D=t^2\mp 4n=d\ell^2,\\ & \text{or } N\nmid D \text{ and } \psi_d(N)=-1,\\ \chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right) & \text{if } N\nmid D \text{ and } \psi_d(N)=1.\end{cases}\tag{2.1}$$

Here $\sqrt{d}$ denotes square-root of d modulo $4N$.

Then we have the following Selberg trace formula for Maass cusp forms of level N, nebentypus character χ and weight k [1]. For the Hecke operator T_n, we have

(2.2)

$$\sum_{\substack{\lambda>0\\ \lambda=\frac14+r^2}}\mathrm{Tr}\left(T_n\middle|\mathcal{A}_k(\lambda;\chi)\right)h(r)=-\delta_{\sqrt{n}\in\mathbb{Z}}\frac{1}{\sqrt{n}}\frac{\chi(\sqrt{n})(N+1)}{12\pi}\int_0^\infty\frac{d}{dt}\left(\frac{\widehat{h}(t)}{\cosh^k(\pi t)}\right)\frac{dt}{\sinh(\pi t)}$$

$$+\sum_{\substack{t\in\mathbb{Z}\\ D=t^2-4n\\ \sqrt{D}\notin\mathbb{Q}}}H_{t,n}(\chi)L(1,\psi_D)\begin{cases}\frac{1}{2\pi}\frac{\sqrt{|D|}}{\sqrt{n}}\int_0^\infty\widehat{h}(u)\left(\frac{t}{2\sqrt{n}}\frac{1}{\cosh(\pi u)}\right)^k\frac{\cosh(\pi u)}{\frac{|D|}{4n}+\sinh^2(\pi u)}\,du & \text{if } D<0,\\ \frac{1}{2\pi}\widehat{h}\left(\frac{1}{\pi}\log\left(\frac{|t|+\sqrt{D}}{2\sqrt{n}}\right)\right) & \text{if } D>0\end{cases}$$

$$+\sum_{d|n}\left(\chi(d)+\chi(nd^{-1})\right)\frac{1}{4\pi}\widehat{h}\left(\frac{\log(nd^{-2})}{2\pi}\right)$$

$$\times\begin{cases}\left(\log\left(\frac{\pi|nd^{-1}-d|}{N^2}\right)-\sum_{m||nd^{-1}-d|}\frac{\Lambda(m)}{m}+\frac{\log N}{\gcd(N^\infty,nd^{-1}-d)}\right) & \text{if } d\neq\sqrt{n},\\ \left(\gamma_0+\log\left(\frac{\pi\sqrt{n}}{2N^{\frac32}}\right)\right) & \text{if } d=\sqrt{n}\end{cases}$$

$$+\sum_{d|n}\left(\chi(d)+\chi(nd^{-1})\right)\left\{-\frac14\int_{\frac{|\log(nd^{-2})|}{2\pi}}^\infty\widehat{h}(u)\left(\frac{\cosh\left(\frac{\log(nd^{-2})}{2}\right)}{\cosh(\pi u)}\right)^k\frac{\cosh(\pi u)}{\left|\sinh\left(\frac{\log(nd^{-2})}{2}\right)\right|+\sinh(\pi u)}\,du\right.$$

$$\left.+\sum_{m\geq 1}\frac{\Lambda(m)(\chi(m)+\chi^{-1}(m))}{m}\frac{1}{4\pi}\widehat{h}\left(\frac{\log\left(\frac{m^2n}{d^2}\right)}{2\pi}\right)\right\}$$

$$-\sum_{d|n}\left(\chi(d)+\chi(nd^{-1})\right)\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\Gamma'}{\Gamma}\left(\frac{k+1}{2}+iu\right)h(u)e^{2\pi iu\frac{\log(nd^{-2})}{2\pi}}\,du.$$

Here γ_0 is the Euler-Mascheroni constant. For the Hecke operator T_{-n} we have

$$\sum_{\substack{\lambda>0\\ \lambda=\frac{1}{4}+r^2}} \mathrm{Tr}(T_{-n}|\mathcal{A}_k(\lambda;\chi))h(r) = \sum_{\substack{t\in\mathbb{Z}\\ D=t^2+4n\\ \sqrt{D}\notin\mathbb{Q}}} H_{t,-n}(\chi)L(1,\psi_D)\frac{1}{2\pi}\widehat{h}\left(\frac{\log\left(\frac{|t|+\sqrt{D}}{2\sqrt{n}}\right)}{\pi}\right) \tag{2.3}$$

$$+\sum_{\substack{d|n\\ d\neq\sqrt{n}}}\left(\chi(-d)+\chi(nd^{-1})\right)\frac{1}{4\pi}\widehat{h}\left(\frac{\log(nd^{-2})}{2\pi}\right)$$

$$\times\left(\log\left(\frac{|nd^{-1}+d|}{N}\left(\frac{\pi}{N}\right)^{\frac{1+(-1)^k}{2}}\right)-\sum_{m|(nd^{-1}+d)}\frac{\Lambda(m)}{m}+\frac{\log N}{\gcd(N^\infty,nd^{-1}+d)}\right)$$

$$+\sum_{d|n}\left(\chi(-d)+\chi(nd^{-1})\right)$$

$$\times\Bigg\{-\delta_{d\neq\sqrt{n}}\frac{1}{4}\int_{\frac{|\log(nd^{-2})|}{2\pi}}^{\infty}\widehat{h}(u)\left(\frac{\cosh\left(\frac{\log(nd^{-2})}{2}\right)}{\cosh(\pi u)}\right)^k\frac{\sinh(\pi u)}{\left|\cosh\left(\frac{\log(nd^{-2})}{2}\right)\right|+\cosh(\pi u)}\,du$$

$$+\sum_{m\geq1}\frac{\Lambda(m)(\chi(-m)+\chi^{-1}(m))}{m}\frac{1}{4\pi}\widehat{h}\left(\frac{\log\left(\frac{m^2n}{d^2}\right)}{2\pi}\right)\Bigg\}$$

$$-\frac{1+(-1)^k}{2}\sum_{d|n}\left(\chi(-d)+\chi(nd^{-1})\right)\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\Gamma'}{\Gamma}\left(\frac{k+1}{2}+iu\right)h(u)e^{2\pi iu\frac{\log(nd^{-2})}{2\pi}}\,du$$

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