On some analytic properties of a function associated with the Selberg class satisfying certain special conditions

Hideto Iwata
Graduate School of Mathematics,*
Nagoya University

1 Introduction

J.Kaczorowski defined the associated Euler totient function for a class of generalized L-functions including the Riemann zeta function, Dirichlet L-functions and obtained an asymptotic formula (see [4]) : By a polynomial Euler product expressions we mean a function $F(s)$ of a complex variable $s = \sigma + it$ which for $\sigma > 1$ is defined by a Euler product expressions of the form

$$F(s) = \prod_p F_p(s) = \prod_p \prod_{j=1}^d \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \tag{1.1}$$

where p runs over primes and $|\alpha_j(p)| \le 1$ for all p and $1 \le j \le d$. We take the smallest $d \in \mathbb{Z}_{>0}$ such that there is at least one prime p_0 satisfying

$$\prod_{j=1}^d \alpha_j(p_0) \neq 0,$$

where d is called the *Euler degree* of F. Note that the L-functions from number theory including the Riemann zeta function $\zeta(s)$ and Dirichlet L-function $L(s,\chi)$ and Dedekind zeta and Hecke L-functions of algebraic number fields, as well as the (normalized) L-functions of holomorphic modular forms and, conjecturally, many other L-functions are polynomial Euler products expressions.

M.Rękoś described the analytic property of some function connected with the Euler totient function (see [6]) : Let $\mathbb{H}$ be the upper half-plane. We describe basic analytic properties of the function $f(z)$ defined for $z \in \mathbb{H}$ as follows :

$$f(z) = \lim_{n\to\infty} \sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_n}} \frac{e^{\rho z}\zeta(\rho-1)}{\zeta'(\rho)}, \tag{1.2}$$

where $\{T_n\}$ denotes a real sequence yielding appropriate groupings of the zeros, and the summation is over non-trivial zeros of $\zeta(s)$ with positive imaginary part. For simplicity we assume here that all the zeros of $\zeta(s)$ have simple. M.Rękoś showed the holomorphy of $f(z)$ for $\operatorname{Im} z > 0$, meromorphic continuation to the whole z-plane with its principal part, and a functional relation containing its reflection property. The functional equation for $f(z)$ connects the values of the function f at the points z and $\bar{z}$. Define a smooth curve $\tau : [0,1] \ni t \mapsto \tau(t) \in \mathbb{C}$ such that $\tau(0) = -1/4$, $\tau(1) = 5/2$ and $0 < \operatorname{Im} \tau(t) < 1$ for $t \in (0,1)$, and define it by $\ell(-1/4, 5/2)$. The analytic property of $f(z)$ is described by the following theorems:

Furocho, Chikusa-ku, Nagoya, 464-8602, Japan.

Theorem 1.1 (Theorem 1.in [6]). *The function $f(z)$ is holomorphic on $\mathbb{H}$ and for $z \in \mathbb{H}$ we have*

$$2\pi i f(z) = f_1(z) + f_2(z) - e^{5z/2} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{5/2}(z - \log n)}, \tag{1.3}$$

where the functions

$$f_1(z) = \int_{-1/4+i\infty}^{-1/4} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds, \tag{1.4}$$

$$f_2(z) = \int_{\ell(-1/4,5/2)} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds \tag{1.5}$$

are holomorphic on $\mathbb{H}$ and on the whole z-plane respectively, and the last term is a meromorphic function on the whole z-plane, whose poles are at $z = \log n$ of the second order with the residues $-\varphi(n)/2\pi i$ $(n = 1, 2, \ldots)$. Here $\varphi(n)$ denotes Euler's totient function.

Theorem 1.2 (Theorem 2.in [6]). *The function $f(z)$ can be continued analytically to a meromorphic function on the whole z-plane, which satisfies the functional equation*

$$f(z) + \overline{f(\overline{z})} = B(z) \tag{1.6}$$

and

$$B(z) = -\frac{6}{\pi^2} e^{2z} + \frac{1}{2\pi^2} \sum_{k,n=1}^{\infty} \frac{\mu(k)}{n^2 k} \left[\frac{1}{(nke^z - 1)^2} + \frac{2}{nke^z - 1} + \frac{1}{(nke^z + 1)^2} - \frac{2}{nke^z + 1} \right], \tag{1.7}$$

where $B(z)$ is a meromorphic function on the whole z-plane with the poles of the second order at $z = -\log nk$, $n, k = 1, 2, \ldots$ and $\mu(k)$ is the Möbius function.

We now provide the *Selberg class* $\mathcal{S}$ defined as follows : $f \in \mathcal{S}$ if

(i) (*ordinary Dirichlet series*) $f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$, converges absolutely for $\sigma > 1$;
(ii) (*analytic continuation*) there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^m f(s)$ is entire of finite order;
(iii) (*functional equation*) $f(s)$ satisfies a functional equation of the form $\Phi(s) = \omega \overline{\Phi(1 - \overline{s})}$, where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) f(s) = \gamma(s) f(s), \tag{1.8}$$

say, with $r \geq 0, Q > 0, \lambda_j > 0, \operatorname{Re} \mu_j \geq 0$ and $|\omega| = 1$;
(iv) (*Ramanujan conjecture*) $a_f(n) \ll n^{\epsilon}$ holds for any $\epsilon > 0$;
(v) (*Euler product*) $f(s) = \prod_p \exp\left(\sum_{\ell=0}^{\infty} \frac{b_f(p^\ell)}{p^{\ell s}} \right)$ for $\sigma > 1$, where $b_f(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_f(n) \ll n^{\vartheta}$ for some $\vartheta < 1/2$.

The empty products are hereafter to be equal to 1.

2 The extension of $f(z)$ to the subclass of $\mathcal{S}$

If a functions $F \in \mathcal{S}$ has a polynomial Euler product expressions (1.1), the subclass of $\mathcal{S}$ of the functions with polynomial Euler product expressions is denoted by $\mathcal{S}^{\text{poly}}$. Establishing the results which extend Theorem1.1 and 1.2 to the function $F \in \mathcal{S}^{\text{poly}}$ are the main aim of the present paper. Let $\{\rho\}$ denote the non-trivial zeros of F with positive imaginary parts, and assume that each ρ is simple. Moreover, let $\{T_n\}$ denote a real sequence yielding appropriate groupings of the zeros, where its precise definition is to be given by (3.2) below. For $\operatorname{Im} z > 0$ and $F \in \mathcal{S}^{\text{poly}}$, we consider the function defined by

$$f(z,F)=\lim_{n\to\infty}\sum_{\substack{\rho\\0<\mathrm{Im}\,\rho<T_n}}\frac{e^{\rho z}\zeta(\rho-1)}{F'(\rho)}. \tag{2.1}$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}$, they are to be incorporated in the summation. The reason why $\zeta(s)$ appears in the numerator on the right hand side is that the Barnes type integral (5.3) below for the Whittaker function can then be applied under the hypothesis $(r,\lambda_j)=(1,1)$ for all j in (1.8) (see Lemma 5.1 and Section 6).

Fact 2.1 (Fact 2.1 in [3])**.** *The limit in* (2.1) *exists for all* $z\in\mathbb{H}$.

We will prove Fact 2.1 in the next section.

3 Proof of Fact 2.1.

We prove Fact 2.1, for which the following Lemma is used.

Lemma 3.1 (Lemma 4. in [8])**.** *Let* $F\in\mathcal{S}$ *and let* T *be sufficiently large, and fix* $H=D\log\log T$ *with a large constant* $D>0$. *We take any subinterval* $[n,n+1]$ *with n chosen such that* $n\in\mathbb{Z}_{>0}\cap[T-H,T+H]$. *Then, there are the lines* $t=t_0$ *such that*

$$|F(\sigma+it_0)|^{-1}=O(\exp(C(\log T)^2)), \tag{3.1}$$

uniformly in $\sigma\geq-2$, *where C is a positive constant.*

Let T be sufficiently large. We fix $H=D\log\log T$, where D is a large positive constant. We take any subinterval $[n,n+1]$, where n is a positive integer in $[T-H,T+H]$. Then, by Lemma 3.1 there are the lines $t=T_n$ such that

$$|F(\sigma+iT_n)|^{-1}=O(\exp(C_1(\log T)^2)) \tag{3.2}$$

uniformly for $\sigma\geq-2$, where C_1 is a positive constant. Since T_n is contained in the interval $[T-H,T+H]$, we can see that $T_n\sim T$ as n tends to infinity. Let $\alpha=\min\{\mathrm{Im}\,\rho>0\}/2$ and $\mathscr{L}$ denote the contour consisting of the line segments

$$[b,b+iT_n],\ [b+iT_n,a+iT_n],\ [a+iT_n,a],\ [a,(a+b)/2+i\alpha],\ [(a+b)/2+i\alpha,b],$$

where $\max\{-3/2,\max\{\mathrm{Re}\,\rho<0\}/2\}<a<0, b>5/2$. We assume that the real part of $s=a+it$ $(t\in\mathbb{R})$ does not coincide the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$. We consider the following contour integral round $\mathscr{L}$:

$$\int_{\mathscr{L}}\frac{\zeta(s-1)}{F(s)}e^{sz}ds. \tag{3.3}$$

Since we assume the order of ρ is simple, we have by residue theorem

$$\begin{aligned}\int_{\mathscr{L}}\frac{\zeta(s-1)}{F(s)}e^{sz}ds&=\int_{a+iT_n}^{a}\frac{\zeta(s-1)}{F(s)}e^{zs}ds+\int_{L}\frac{\zeta(s-1)}{F(s)}e^{zs}ds\\&\quad+\int_{b}^{b+iT_n}\frac{\zeta(s-1)}{F(s)}e^{zs}ds+\int_{b+iT_n}^{a+iT_n}\frac{\zeta(s-1)}{F(s)}e^{zs}ds\\&=2\pi i\sum_{\substack{\rho\\0<\mathrm{Im}\,\rho<T_n}}\frac{e^{\rho z}\zeta(\rho-1)}{F'(\rho)},\end{aligned} \tag{3.4}$$

where the path L above consists of joining the two line segments $[a,(a+b)/2+i\alpha]$ and $[(a+b)/2+i\alpha,b]$. We now estimate the integral along the line segment $[b+iT_n,a+iT_n]$. For $a\leq\sigma\leq b$, we have by (3.2),

$$|F(\sigma+iT_n)|^{-1}=O(\exp(C(\log T)^2)),$$

which, with the vertical estimate for $\zeta(s-1)$, shows with $z = x + iy$ that

$$\left| \int_{a+iT_n}^{b+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds \right| \ll (b-a) T_n^c \exp\{C(\log T)^2 - yT_n + |x|(|a| + |b|)\} \tag{3.5}$$

for $T, T_n \geq 1$ with a constant $c = c(a, b) > 0$, where the last bound tends to zero as $n \to \infty$. By Theorem 4.1 below, the convergence of the other integrals in (3.4) are ensured (see (4.5)-(4.7)). The limit in (2.1) therefore exists. □

4 Main theorems

Letting $n \to \infty$, we have

$$\int_{a+i\infty}^{a} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{L} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{b}^{b+i\infty} \frac{\zeta(s-1)}{F(s)} e^{zs} ds = 2\pi i f(z, F), \tag{4.1}$$

with $f(z, F)$ in (2.1). To evaluate the integral along the vertical line with $s = b + it$ ($t \geq 0$), we prepare the Dirichlet series expansion of $\zeta(s-1)/F(s)$ for $\sigma > 2$.

Definition 4.1 (p.34 in [4])**.** For $\sigma > 1$ and $F \in \mathcal{S}^{\text{poly}}$, we define the Dirichlet coefficients μ_F as follows :

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^s} = \prod_{p} \prod_{j=1}^{d} \left(1 - \frac{\alpha_j(p)}{p^s}\right). \tag{4.2}$$

Remark 4.2 (p.34 in [4])**.** By (4.2), $|\mu_F(n)| \leq \tau_d(n)$, where $\tau_d(n)$ is the divisor function of order d, so that $\zeta^d(s) = \sum_{n=1}^{\infty} \tau_d(n)/n^s$ for $\sigma > 1$. In particular $\tau_1(n) = 1$ for all n.

Using (4.2) in $\sigma > 2$, we obtain

$$\frac{\zeta(s-1)}{F(s)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{4.3}$$

where

$$g(n) = \sum_{d|n} \mu_F(d) \frac{n}{d}. \tag{4.4}$$

Theorem 4.1 (Theorem 4.1 in [3])**.** *Let* $\max\{-3/2, \max\{Re\ \rho < 0\}/2\} < a < 0, b > 5/2$*. Then, the function* (2.1) *is holomorphic on* $\mathbb{H}$*, and for* $z \in \mathbb{H}$ *the formula*

$$2\pi i f(z, F) = f_1(z, F) + f_2(z, F) - e^{bz} \sum_{n=1}^{\infty} \frac{g(n)}{n^b (z - \log n)}, \tag{4.5}$$

is valid, where the functions

$$f_1(z, F) = \int_{a+i\infty}^{a} \frac{\zeta(s-1)}{F(s)} e^{sz} ds, \tag{4.6}$$

$$f_2(z, F) = \int_{L} \frac{\zeta(s-1)}{F(s)} e^{sz} ds \tag{4.7}$$

are holomorphic on $\mathbb{H}$ *and on the whole z-plane, and the last term on the right is a meromorphic function on the whole z-plane with the poles at* $z = \log n$ ($n = 1, 2, \ldots$).

We need not use the condition of a which does not coincide the poles of $\Gamma(s + \mu)\Gamma(s - \mu)$ in proving of Theorem 4.1.

Theorem 4.2 (Theorem 4.2 in [3])**.** *For any* $F \in \mathcal{S}^{poly}$ *with* $(r, \lambda_j) = (1, 1)$ *for all* j *in* (1.8) *and* $0 \leq \mu < 1$*, the function* (2.1) *has a meromorphic continuation to* $y > -\pi$*.*

The complex number μ_1 when $r = 1$ in (1.8) is hereafter referred to as μ. The L-functions associated with holomorphic cusp forms and Dedekind zeta functions of the imaginary quadratic fields are the examples of F considered in Theorem 4.2. Letting

$$\mathbb{H}^- = \{z \in \mathbb{C} : \text{Im}\ z < 0\}, \tag{4.8}$$

we next study the function, for $z \in \mathbb{H}^-$,

$$f^-(z,F) = \lim_{n\to\infty} \sum_{\substack{\rho \\ -T_n<\operatorname{Im}\rho<0}} \frac{e^{\rho z}\zeta(\rho-1)}{F'(\rho)}. \tag{4.9}$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}^-$, they are incorporated in the summation. The existence of the limit on the right hand side of (4.9) is proved similarly to Fact 2.1.

Corollary 4.3 (Corollary 4.3 in [3]). *For any $F \in \mathcal{S}^{poly}$ satisfying the same conditions as in Theorem 4.2, the function* (4.9) *has a meromorphic continuation to $y < \pi$.*

Theorem 4.4 (Theorem 4.4 in [3]). *For any $F \in \mathcal{S}^{poly}$ satisfying the same conditions as in Theorem 4.2, the function* (2.1) *can be continued analytically to the whole z- plane. In addition to the condition as in Theorem 4.2, we assume that the Dirichlet series coefficients $a_F(n)$ of $F \in \mathcal{S}^{poly}$ is real-valued for all n. Then, the function* (2.1) *satisfies the functional equation*

$$f(z,F) + \overline{f(\overline{z},F)} = B(z,F), \tag{4.10}$$

where

$$B(z,F) = \frac{1}{2\pi i}(f_1(z,F) + f_1^-(z,F)) - \frac{e^{2z}}{F(2)} \tag{4.11}$$

for all $z \in \mathbb{C}$ and the function $f_1^-(z,F)$ is holomorphic on $\mathbb{H}^-$.

5 Some auxiliary results on the Whittaker function

We first introduce the Whittaker function $W_{\kappa,\mu}(z)$ (via the confluent hypergeometric function $\Psi(\alpha,\gamma;z)$ below), which is necessary to prove our main theorems, and then prepare some auxiliary results, i.e.its integral expression and asymptotic expansions.

Definition 5.1 (The confluent hypergeometric function of the second kind ([1])). Let $\Psi(\alpha,\gamma;z)$ be the *confluent hypergeometric function of the second kind* defined by

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)(e^{2\pi i\alpha}-1)} \int_{\infty}^{(0+)} e^{-zw} w^{\alpha-1}(1+w)^{\gamma-\alpha-1} dw \tag{5.1}$$

for any $(\alpha,\gamma) \in \mathbb{C}^2$ and for $|\arg z| < \pi$. Here the path of integration is a contour in the w-plane which consists of the upper real axis from $e^{0\pi i}\infty$ to $e^{0\pi i}\delta$ with a small $\delta > 0$, the circle with the center $w = 0$ and the radius δ through which $\arg w$ varies from 0 to 2π, and the lower real axis from $e^{2\pi i}\delta$ to $e^{2\pi i}\infty$.

Definition 5.2 (The Whittaker function ([5])). The *Whittaker function* $W_{\kappa,\mu}(z)$, which has large applicability, e.g. in number theory and physics, is defined by

$$W_{\kappa,\mu}(z) = z^{\mu+1/2} e^{-z/2} \Psi\left(\frac{1}{2} - \kappa + \mu, 2\mu+1; z\right) \quad (|\arg z| < \pi), \tag{5.2}$$

where (to avoid many-valuedness) the domain of z is to be restricted in the z-plane cut along the negative real axis with $|\arg z| < \pi$.

Lemma 5.1 (Barnes type integral for the Whittaker function ([7], [9])). *The Barnes integral for $W_{\kappa,\mu}(z)$ asserts*

$$W_{\kappa,\mu}(z) = \frac{e^{-z/2}z^{\kappa}}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s)\Gamma\left(-s-\kappa-\mu+\frac{1}{2}\right)\Gamma\left(-s-\kappa+\mu+\frac{1}{2}\right)}{\Gamma\left(-\kappa-\mu+\frac{1}{2}\right)\Gamma\left(-\kappa+\mu+\frac{1}{2}\right)} z^s ds, \tag{5.3}$$

for $|\arg z| < 3\pi/2$, and $\kappa \pm \mu + 1/2 \neq 0, 1, 2, \ldots$; the contour has loops if necessary so that the poles of $\Gamma(s)$ from those of $\Gamma(-s-\kappa-\mu+1/2) \times \Gamma(-s-\kappa+\mu+1/2)$ are on opposite sides of it.

In (5.3), it holds for all finite values of c provided that the contour of integration can always be deformed so as to separate the poles $\Gamma(s)$ and those of the other Γ-factors. By Stirling's formula for $\Gamma(s)$ (cf. [2]), the integral in (5.3) represents a function of z which is holomorphic at all points in the domain $|\arg z| \le 3\pi/2 - \alpha$ with any small $\alpha > 0$. The asymptotic expansions for $\Psi(\alpha;\gamma;z)$ as $z \to 0$ readily asserts the following proposition.

Proposition 5.2 (The asymptotic expansions for $\Psi(a;b;z)$ as $z \to 0$ ([7])). *We have the asymptotic expansions, as $z \to 0$ through $|\arg z| < \pi$,*

$$\Psi(a;b;z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|z|^{\mathrm{Re}b-2}) \quad (\mathrm{Re}\, b \ge 2, b \ne 2), \tag{5.4}$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|\log z|) \quad (b = 2), \tag{5.5}$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(1) \quad (1 < \mathrm{Re}\, b < 2), \tag{5.6}$$

$$= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|z|) \quad (\mathrm{Re}\, b = 1, b \ne 1), \tag{5.7}$$

$$= -\frac{1}{\Gamma(a)} \left\{ \log z + \frac{\Gamma'}{\Gamma}(a) + 2C_0 \right\} + O(|z \log z|) \quad (b = 1), \tag{5.8}$$

where C_0 is Euler's constant.

By the definition (5.2) and Proposition 5.2, we have the following asymptotic expansions as $z \to 0$ for $W_{\kappa,\mu}(z)$.

Proposition 5.3 (The asymptotic expansions in $z \to 0$ for $W_{\kappa,\mu}(z)$)**.**

$$W_{\kappa,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} z^{1/2-\mu} + O(z^{3/2-\mathrm{Re}\mu}) \quad (\mathrm{Re}\,\mu \ge 1/2, \mu \ne 1/2), \tag{5.9}$$

$$= \frac{1}{\Gamma(1-\kappa)} + O(|z \log z|) \quad (\mu = 1/2), \tag{5.10}$$

$$= \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} z^{1/2-\mu} + O(|z|^{\mathrm{Re}\mu+1/2}) \quad (0 < \mathrm{Re}\,\mu < 1/2), \tag{5.11}$$

$$= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)} z^{\mu+1/2} + \frac{\Gamma(2\mu)}{\Gamma\left(\mu+\frac{1}{2}-\kappa\right)} z^{-\mu+1/2} + O(|z|^{\mathrm{Re}\mu+3/2}) \quad (\mathrm{Re}\,\mu = 0, \mu \ne 0), \tag{5.12}$$

$$= -\frac{z^{1/2}}{\Gamma\left(\frac{1}{2}-\kappa\right)} \left(\log z + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-\kappa\right) + 2C_0 \right) + O(|z|^{3/2} |\log z|) \quad (\mu = 0). \tag{5.13}$$

6 Proof of Theorem 4.2

We prove that the function $f(z,F)$ ($z = x + iy$) has a meromorphic continuation to $y > -\pi$. By Theorem 4.1, the function

$$f_1(z,F) = \int_{a+i\infty}^{a} \frac{\zeta(s-1)}{F(s)} e^{zs} ds = -\int_{a}^{a+i\infty} \frac{\zeta(s-1)}{F(s)} e^{zs} ds$$

is convergent for $y > 0$. We recall the hypotheses that $(r, \lambda_j) = (1,1)$ for all j in (1.8) and $0 \le \mu < 1$. We rewrite the functional equation (1.8) under these hypotheses as follows :

$$\begin{aligned} Q^s \Gamma(s+\mu) F(s) &= \omega \overline{Q^{1-\overline{s}} \Gamma(1-\overline{s}+\mu) F(1-\overline{s})} \\ &= \omega Q^{1-s} \Gamma(1-s+\mu) \overline{F(1-\overline{s})}, \end{aligned}$$

where the conditions of Q and ω are the same as noted in (1.8). Hence

$$\frac{1}{F(s)} = \overline{\omega} Q^{2s-1} \frac{\Gamma(s+\mu)}{\Gamma(1-s+\mu)} \frac{1}{\overline{F(1-\overline{s})}}, \tag{6.1}$$

which yields, from the reciprocal relation $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$, that

$$\frac{1}{F(s)} = \frac{\overline{\omega}}{\pi} Q^{2s-1} \sin\pi(s-\mu)\Gamma(s+\mu)\Gamma(s-\mu)\frac{1}{\overline{F(1-\overline{s})}}. \tag{6.2}$$

By (6.2) and the functional equation for $\zeta(s)$, on setting

$$\begin{aligned} H(s,w) &= (2\pi Q^2)^s \zeta(2-s)\Gamma(s-\mu)\Gamma(s+\mu)\Gamma(2-s)e^{ws}, \\ K(s,w) &= \frac{H(s,w)}{\overline{F(1-\overline{s})}}, \end{aligned} \tag{6.3}$$

we have

$$\begin{aligned} f_1(z,F) &= J_-\left(z+\frac{3\pi i}{2}\right) + J_+\left(z-\frac{\pi i}{2}\right) + J_-\left(z+\frac{\pi i}{2}\right) + J_+\left(z-\frac{3\pi i}{2}\right) \\ &= \sum_{j=1}^{4} f_{1j}(z.F), \end{aligned} \tag{6.4}$$

say, where

$$J_\pm(w) = \frac{\pm i e^{\pm\pi i\mu}\overline{\omega}}{(2\pi)^3 Q} \int_a^{a+i\infty} K(s,w)ds. \tag{6.5}$$

It follows from, by Stirling's formula,

$$\Gamma(s+\mu)\Gamma(s-\mu)\Gamma(2-s) \asymp e^{-3\pi|t|/2}|t|^{a+1/2} \quad (\text{as } t \to \pm\infty), \tag{6.6}$$

$\max\{-3/2, \max\{\mathrm{Re}\,\rho < 0\}/2\} < a < 0$, and the vertical estimates for $\zeta(2-s)$ and $\overline{F(1-\overline{s})}$ that $f_{1j}(z,F)$ $(j = 1,\ldots,4)$ are holomorphic for $y > -3\pi$, for $y > -\pi$, for $y > -2\pi$ and for $y > 0$ respectively. We next show that $f_{14}(z,F)$ can be continued to $0 < y < 3\pi$. Splitting the integral in $f_{14}(z,F)$, we have

$$\begin{aligned} f_{14}(z,F) &= J_+\left(z-\frac{3\pi i}{2}\right) \\ &= \frac{\overline{\omega}e^{\mu\pi i} i}{(2\pi)^3 Q}\left\{\int_{a-i\infty}^{a+i\infty} - \int_{a-i\infty}^{a}\right\} K\left(s, z-\frac{3\pi i}{2}\right)ds \\ &= I_1(z,F) + I_2(z,F), \end{aligned}$$

say, where

$$I_1(z,F) = \frac{\overline{\omega}e^{\mu\pi i} i}{(2\pi)^3 Q}\int_{a-i\infty}^{a+i\infty} K\left(s, z-\frac{3\pi i}{2}\right)ds, \tag{6.7}$$

$$I_2(z,F) = \frac{\overline{\omega}e^{\mu\pi i}}{(2\pi)^3 Qi}\int_{a-i\infty}^{a} K\left(s, z-\frac{3\pi i}{2}\right)ds. \tag{6.8}$$

Here, we recall the hypothesis that the real part of $s = a+it$ $(t \in \mathbb{R})$ does not coincide with the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$. From (6.6), we can see that the integral $I_2(z,F)$ is convergent for $y < 3\pi$. From (6.6) again, $I_1(z,F)$ converges absolutely for $0 < y < 3\pi$. Substituting the Dirichlet series expansions in (4.2) and $\zeta(2-s)$, and then integrating term-by-term, we have

$$I_1(z,F) = \frac{\overline{\omega}e^{\mu\pi i} i}{(2\pi)^3 Q}\sum_{k,n=1}^{\infty}\frac{\overline{\mu_F(k)}}{kn^2}\int_{a-i\infty}^{a+i\infty} e^{\{\log(2\pi nkQ^2)+z-3\pi i/2\}s}\Gamma(s+\mu)\Gamma(s-\mu)\Gamma(2-s)ds, \tag{6.9}$$

where the interchange of the order of integration and summation can be justified by absolute convergence as follows : For $0 < y < 3\pi$, each s-integral in (6.9) is bounded as

$$\ll (nk)^a e^{ax}\int_{-\infty}^{\infty} e^{-(y-3\pi/2)t-3\pi|t|/2}(|t|+1)^{a+1/2}dt \ll (nk)^a,$$

and hence the series in (6.9) is bounded above as

$$\ll \sum_{k,n=1}^{\infty} \frac{|\overline{\mu_F(k)}|}{k^{1-a}n^{2-a}} < +\infty.$$

by (4.2) and $a < 0$. We next prepare the following notations which is used to describe the residues of the integrand in (6.9) and also used later in similar situations : for $\mu \notin \{0, 1/2\}$,

$$S_{k,n,m}(w,\mu) = \frac{(-1)^m}{m!}\Gamma(2\mu - m)\Gamma(2-\mu+m)(2\pi nkQ^2)^{\mu-m}e^{w(\mu-m)} \tag{6.10}$$

and for $\mu = 0$,

$$S_{k,n,m}(w,0) = \frac{m+1}{m!}\frac{e^{-mw}}{(2\pi nkQ^2)^m}\left\{\log(2\pi nkQ^2) + w + \sum_{h=1}^{m}\frac{1}{h} - C_0 - \frac{1}{m+1}\right\} \tag{6.11}$$

with Euler's constant C_0, and also for $\mu = 1/2$,

$$S_{k,n,m}\left(w,\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}+m\right)}{(m!)^2}\frac{e^{(1/2-m)w}}{(2\pi nkQ^2)^{m-1/2}}\left\{m\left(\psi\left(\frac{3}{2}+m\right) - 2\psi(m+1)\right)\right.$$
$$\left. - m\left(\log(2\pi nkQ^2) + w\right) + 1\right\} \tag{6.12}$$

with Euler's psi function $\psi(s) = (\Gamma'/\Gamma)(s)$. Then we have the following lemma.

Lemma 6.1 (Lemma 7.1 in [3])**.** *The residues of the integrand in* (6.9) *are given as follows* :

(i) *for* $\mu \notin \{0, 1/2\}$*, those at* $s = \pm\mu - m$ $(m = 0, 1, \ldots)$ *equal*

$$R^{(1)}_{k,n,m}(z, \pm\mu) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, \pm\mu\right);$$

(ii) *for* $\mu = 0$*, those at* $s = -m$ $(m = 0, 1, \ldots)$ *equal*

$$R^{(1)}_{k,n,m}(z, 0) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, 0\right);$$

(iii) *for* $\mu = 1/2$*, those at* $s = 1/2 - m$ $(m = 0, 1, \ldots)$ *equal*

$$R^{(1)}_{k,n,m}(z, 1/2) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, \frac{1}{2}\right).$$

Proof. The reciprocal relation $\Gamma(s)$ shows that $\Gamma(-m+\epsilon) = (-1)^m\pi/\Gamma(1+m-\epsilon)\sin(\pi\epsilon)$ $(m = 0, 1, \ldots)$, and this implies the Laurent series expansion, as $\epsilon \to 0$,

$$\Gamma(-m+\epsilon) = \frac{(-1)^m}{m!}\epsilon^{-1}\{1 + \psi(1+m)\epsilon + O(\epsilon^2)\}, \tag{6.13}$$

which with the Taylor series expansion

$$\Gamma(2+m-\epsilon) = (m+1)!\{1 - \psi(2+m)\epsilon + O(\epsilon^2)\}. \tag{6.14}$$

Readily asserts the cases (i) and (ii), by substituting (6.13) and (6.14) into the integrand in (6.9), and by noting $\psi(1+m) = \sum_{h=1}^{m}(1/h) + \psi(1)$ with $\psi(1) = -C_0$, while (iii) by noting $\Gamma(s-1/2)\Gamma(s+1/2) = (s-1/2)\Gamma(s-1/2)^2$ and by substituting (6.13), instead of (6.14), the Taylor series expansion, as $\epsilon \to 0$,

$$\Gamma\left(\frac{3}{2}+m-\epsilon\right) = \Gamma\left(\frac{3}{2}+m\right)\left\{1 - \psi\left(\frac{3}{2}+m\right)\epsilon + O(\epsilon^2)\right\}.$$

□

Since $\lim_{T\to+\infty}\int_{a\pm iT}^{a\pm i\infty}=0$ for $0<y<3\pi$ in (6.9), we obtain if $\mu\notin\{0,1/2\}$, by the residue theorem,

$$\int_{a-i\infty}^{a+i\infty} K\left(s,z-\frac{3\pi i}{2}\right)ds=-\int_C K\left(s,z-\frac{3\pi i}{2}\right)ds \\ -2\pi i\left\{\sum_{m_1=0}^{M_1}R^{(1)}_{k,n,m_1}(z,\mu)+\sum_{m_2=0}^{M_2}R^{(1)}_{k,n,m_2}(z,-\mu)\right\}, \tag{6.15}$$

where C is a contour separating the poles of $\Gamma(2-s)$ from those of $\Gamma(s+\mu)\Gamma(s-\mu)$ in opposite sides, and M_1 and M_2 are the integers chosen correspondingly . Here the split residual terms in (6.15) when $\mu\in\{0,1/2\}$ are to be understood as

$$\sum_{m=0}^{M'}R^{(1)}_{k,n,m}(z,0)\quad(\text{ if }\mu=0\text{ });\qquad\sum_{m=0}^{M''}R^{(1)}_{k,n,m}(z,1/2)\quad(\text{ if }\mu=1/2\text{ }), \tag{6.16}$$

where the same convention is used hereafter. The integral over C in (6.15) can be evaluated, on replacing s with $2-s$, by the formula (5.3), and this shows that it equals

$$T_{k,n}\left(z-3\pi i/2\right),$$

where and in the sequel

$$T_{k,n}(w)=2\pi i(2\pi nkQ^2)^{1/2}\exp\left(\frac{e^{-w}}{4\pi nkQ^2}+\frac{w}{2}\right)\Gamma(2-\mu)\Gamma(2+\mu)W_{-3/2,\mu}\left(\frac{e^{-w}}{2\pi nkQ^2}\right). \tag{6.17}$$

Therefore, we have

$$I_1(z,F)=\frac{\overline{\omega}e^{\mu\pi i}i}{(2\pi)^3Q}\sum_{k,n=1}^{\infty}\frac{\overline{\mu_F(k)}}{kn^2}\left[T_{k,n}\left(z-\frac{3}{2}\pi i\right)-2\pi i\left\{\sum_{m_1=0}^{M_1}R^{(1)}_{k,n,m_1}(z,\mu)\right.\right. \\ \left.\left.+\sum_{m_2=0}^{M_2}R^{(1)}_{k,n,m_2}(z,-\mu)\right\}\right]. \tag{6.18}$$

The following lemma ensures the convergence for the series on the right hand side in (6.18).

Lemma 6.2 (Lemma 7.2 in [3]). *For $F\in\mathcal{S}^{poly}$ with $(r,\lambda_j)=(1,1)$ for all j and $0\le\mu<1$ in* (1.8)*, the series on the right hand side in* (6.18) *is absolutely and uniformly convergent on every compact subset on the whole z-plane.*

By Lemma 6.2, for any $F\in\mathcal{S}^{\rm poly}$ with $(r,\lambda_j)=(1,1)$ for all j in (1.8) and $0\le\mu<1$, we have the following analytic continuation of $f_1(z,F)$ for $y>-\pi$:

$$f_1(z,F)=J_-\left(z+\frac{3\pi i}{2}\right)+J_+\left(z-\frac{\pi i}{2}\right)+J_-\left(z+\frac{\pi i}{2}\right)+\frac{\overline{\omega}e^{\mu\pi i}i}{(2\pi)^3Q} \\ \times\sum_{k,n=1}^{\infty}\frac{\overline{\mu_F(k)}}{kn^2}\left[T_{k,n}\left(z-\frac{3}{2}\pi i\right)-2\pi i\left\{\sum_{m_1=0}^{M_1}R^{(1)}_{k,n,m_1}(z,\mu)\right.\right. \\ \left.\left.+\sum_{m_2=0}^{M_2}R^{(1)}_{k,n,m_2}(z,-\mu)\right\}\right]+\frac{\overline{\omega}e^{\mu\pi i}}{(2\pi)^3Qi}\int_{a-i\infty}^{a}K\left(s,z-\frac{3\pi i}{2}\right)ds, \tag{6.19}$$

where the first integral is holomorphic for $y>-3\pi$, the second for $y>-\pi$, the third for $y>-2\pi$, the fourth double sum for $y>-\infty$ by Lemma 6.2, and the last for $y<3\pi$. Therefore, (6.19) completes the proof of the continuation of $f(z,F)$ to the region $y>-\pi$. □

Acknowledgment. The author would like to thank Professor Yu Yasufuku and Professor Maki Nakasuji for this opportunity to give a talk in RIMS Workshop 2023 ”Analytic Number Theory and Related Topics”.

References

[1] A. Erdélyi (ed.), W. Magnas, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.

[2] A. Ivić, The Riemann-Zeta function, Dover Publ. Inc., Mineloa, New York, 1985.

[3] H. Iwata, On some analytic properties of a function associated with the Selberg class satisfying certain special conditions, Tsukuba J. Math. (to appear).

[4] J. Kaczorowski, On a generalization of the Euler totient function, Monatsh. Math 170 (2013), 27-48.

[5] N. N. Lebedev, special Functions & Their Applications, Dover Publ. Inc., Mineola, New York, 1972.

[6] M.Rękoś, On some complex explicit formulae connected with the Euler's φ function. I, Funct. Approx. Comment. Math 29 (2001), 113-124.

[7] L. J. Slater, Confluent Hypergeometric Functions, Cambridge University Press,1960.

[8] K. Srinivas, Distinct zeros of functions in the Selberg class, Acta Arith. 103.3 (2002), 201-207.

[9] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th ed., Cambridge Univ. Press, 1927.