# ON THE BEEM-NAIR CONJECTURE

SHUN FURIHATA

*Research Institute for Mathematical Sciences, Kyoto University*

## Introduction

Vertex operator algebra (VOA) was defined in 1980's ([B]) so as to describe the two-dimensional conformal field theory (CFT) in physics in the algebraic framework. The study of VOAs may provide new discoveries in physics, and conversely, the research in physics can stimulate the development of new theories of VOAs. One of the physical aspects in pure mathematics via VOAs is the Higgs Branch Conjecture([BR, A3]), which implies that the Higgs branches (a certain geometrical invariant in physics) are the associated varieties of VOAs (algebraic objects in mathematics) corresponding to four-dimensional $\mathcal{N} = 2$ super CFT via the $4d/2d$ correspondence ([BLL+]). It is meaningful to find free field realization of VOAs, because the free field realization of the VOA arising from the four-dimensional super CFT mirrors the effective field theory description of the super CFT on the Higgs branch of its moduli space of vacua ([BMR]).

The universal centralizer $\mathfrak{Z}_G$ is a family of pairs $(g, s)$, where $s$ is an element in the Slodowy slice and $g \in G$ is in the stabilizer group of $s$. This is a subvariety of the equivariant Slodowy slice with a symplectic structure([Kos3]), and identified with the space associated to the sphere in the Moore-Tachikawa two-dimensional topological quantum field theory that describes the Higgs branches for the class $\mathcal{S}$ theory([MT]). We have an open immersion from a complexification of the phase space of the Kostant-Toda lattice to $\mathfrak{Z}_G$ ([C, BN]). Also, $\mathfrak{Z}_G$ is identified with the spectrum of the equivariant Borel-Moore homology of the affine Grassmannian ([BFM]) of the Langlands dual group of $G$. According to [BPRvR], the chiral counterpart of the universal centralizer $\mathfrak{Z}_G$ is the chiral universal centralizer $\mathbf{I}_G$ defined in [A1]. That is, $\mathfrak{Z}_G$ is the associated variety of $\mathbf{I}_G$. For $G = SL_2(\mathbb{C})$ and a generic level, the universal centralizer coincides with the modified regular representation of the Virasoro algebra defined in [FS]. The representation of the universal centralizer is studied in [MY] for $G = SL_2(\mathbb{C})$ and at level $-1$ and irrational levels, but for other general case, it is yet to be studied.

In [BN], the authors suggested how to construct a free field realization of a chiral universal centralizer at the critical level for simple Lie groups $G$. They conjectured that the immersion $\mathrm{KT}_G \to \mathfrak{Z}_G$ gives rise to a vertex operator algebra homomorphism between the chiralization of $\mathrm{KT}_G$ and the chiralization of $\mathfrak{Z}_G$. We will construct a free field realization of the chiral universal centralizers at any level, using an idea of [FS]. The universal centralizers are defined by performing the Kostant reduction twice to the cotangent space $T^*G$, and the chiral universal centralizers are defined analogously: by performing the quantized Drinfeld-Sokolov reduction of the chiral differential operator algebra $\mathcal{D}^{ch}_G(G)$ on $G$ twice ([MSV, BD, AG]). We take the big cell $U$ of the Bruhat decomposition of $G$, and get the free field realization by performing the quantized Drinfeld-Sokolov reduction to restriction map $\mathcal{D}^{ch}_G(G) \to \mathcal{D}^{ch}_G(U)$ twice (Theorem 9).

## 0 Notations

We let $G$ be a simple affine algebraic Lie group unless otherwise stated. Fix a Borel subgroup $B \subset G$ and a splitting $B = N \rtimes T$, where $N \subset G$ is a maximal unipotent subgroup and $T \subset G$ is a maximal torus. Set $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{n} := \mathrm{Lie}(N)$, $\mathfrak{t} := \mathrm{Lie}(T)$ and let $\Delta$ denote the set of simple roots of $\mathfrak{g}$.

Choose a root basis $\{e^{\alpha}\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ with $\alpha \in \mathfrak{g}_{\alpha}$, where $\mathfrak{g}_{\alpha}$ is the root space. We fix a principal nilpotent element $f_0 := \sum_{\alpha\in\Delta} e^{-\alpha} \in \mathfrak{n}$ and let $\chi_0 \in \mathfrak{n}^* \subset \mathfrak{g}^*$ be the Killing dual element of $f_0$. Note that $\chi_0$ is a character of $\mathfrak{n}$.

We regard complex numbers as invariant bilinear forms on $\mathfrak{g}$ by considering the bilinear form $\dfrac{k}{2h^{\vee}} \times (\cdot,\cdot)$ for $k \in \mathbb{C}$, where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$ and $(\cdot,\cdot)$ is the Killing form

# 1 Chiral Universal Centralizer

## 1.1 Kostant reduction

Let $X$ be an affine Poisson algebraic variety with a Hamiltonian $G$-action with the moment map $\mu_X$. Then we have a moment map $\mu_N$ for $N$-action;

$$\mu_N : X \to \mathfrak{g}^* \xrightarrow{\text{restriction}} \mathfrak{n}^*.$$

By the Jacobson-Morozov theorem, $f_0$ can be completed to an $\mathfrak{sl}_2$-triple $(e_0, h_0, f_0)$. Let $\mathfrak{g}^{e_0}$ be an $\mathrm{ad}_{e_0}$-invariant subspace of $\mathfrak{g}$ and $S_{f_0} := f_0 + \mathfrak{g}^{e_0} \subset \mathfrak{g}$ be the Kostant-Slodowy slice.

**Theorem 1** ([Kos1])**.** *Assume that $\chi_0$ is a regular value of $\mu_N$ and $N$-action on $\mu_N^{-1}(\chi_0)$ is free. Then*

$$\mu_N^{-1}(\chi_0)/N \cong X \times_{\mathfrak{g}^*} S_{f_0}$$

*as varieties.*

We call $\mu_N^{-1}(\chi_0)/N$ **the Kostant reduction** of $X$ with respect to $\mu_X$ and denote $X/\!/_{\chi_0}N$.

## 1.2 (Classical) universal centralizer

We have two commuting actions $\rho_L$ and $\rho_R$ of $G$ on $T^*G \cong G \times \mathfrak{g}^*$:

$$\rho_L(g)(h,\phi) = (hg^{-1}, \mathrm{Ad}_g^*(\phi)),$$
$$\rho_R(g)(h,\phi) = (gh, \phi)$$

for $g, h \in G$ and $\phi \in \mathfrak{g}^*$, where $\mathrm{Ad}^*$ means the coadjoint action of $G$ on $\mathfrak{g}^*$. In fact, $\rho_L$ and $\rho_R$ are Hamiltonian actions, which correspond to the following moment maps:

$$\mu_L(h,\phi) = \phi,$$
$$\mu_R(h,\phi) = \mathrm{Ad}_h^*(\phi).$$

We perform the Kostant reduction to $T^*G$ with respect to $\mu_L$ and get

$$T^*G/\!/_{\chi_0}N \cong T^*G \times_{\mathfrak{g}^*} S_{f_0} \cong G \times S_{f_0}.$$

Since $\rho_L$ and $\rho_R$ commute with each other, $G\times S_{f_0}$ inherits a Hamiltonian $G$-action $\rho'_R$, corresponding to a new moment map $\mu'_R$.

$$\rho'_R(g)(h,s) = (gh, s),$$
$$\mu'_R(h,s) = \mathrm{Ad}_h^*(\chi_s),$$

where $\chi_s$ is the Killing dual of $s \in S_{f_0}$. Therefore, we can take the Kostant reduction of $G \times S_{f_0}$ with respect to $\mu'_R$ and we get $\mathfrak{z}_G := \{(g,s) \in G \times S_{f_0} \mid \mathrm{Ad}_g(s) = s\}$, called **the (classical) universal centralizer** of $G$. That is,

$$\begin{aligned}(G \times S_{f_0})/\!/_{\chi_0}N &\cong (G \times S_{f_0}) \times_{\mathfrak{g}^*} S_{f_0}\\ &\cong \{(g,s) \in G \times S_{f_0} \mid \mathrm{Ad}_g(s) = s\}\\ &= \mathfrak{z}_G.\end{aligned}$$

We will follow [BN] so as to obtain a "classical free field realization". Define $\mathrm{KT}_{\mathfrak{g}} := \chi_0 + \mathfrak{t}^* + \sum_{\alpha\in\Delta} \mathfrak{g}_{\alpha}^* \setminus \{0\} \subset \mathfrak{g}^*$. Then $\mathrm{KT}_{\mathfrak{g}}$ is a symplectic variety ([Kos2]) and it is not difficult to see that $\mathrm{KT}_{\mathfrak{g}} \cong T^*(\mathbb{C}^{\times})^{\mathrm{rk}\,\mathfrak{g}}$ as a symplectic variety. We denote the projection map $\mathrm{pr} : \mathrm{KT}_{\mathfrak{g}} \to (\mathbb{C}^\times)^{\mathrm{rk}\mathfrak{g}}$. Let $Z_G : T \to (\mathbb{C}^\times)^{\mathrm{rk}\mathfrak{g}}$ be the map

$$t \mapsto (\alpha(t))_{\alpha\in\Delta},$$

where $\alpha(t)$ is defined by $\mathrm{Ad}_t(e^\alpha) = \alpha(t)e^\alpha$ for a simple root $\alpha \in \Delta$.

We define the symplectic variety $\mathrm{KT}_G$ to be the pullback of $\mathrm{pr} : \mathrm{KT}_{\mathfrak{g}} \to (\mathbb{C}^\times)^{\mathrm{rk}\mathfrak{g}}$ and $Z_G : T \to (\mathbb{C}^\times)^{\mathrm{rk}\mathfrak{g}}$.

$$\begin{array}{ccc} \mathrm{KT}_G & \xrightarrow{\pi_G} & \mathrm{KT}_{\mathfrak{g}} \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ T & \xrightarrow{Z_G} & (\mathbb{C}^\times)^{\mathrm{rk}\mathfrak{g}} \end{array}$$

Since $Z_G$ is étale, the same holds for $\pi_G$. Hence, we can pullback the symplectic form on $\mathrm{KT}_{\mathfrak{g}}$ to a symplectic form on $\mathrm{KT}_G$.

We have the following proposition, which can be regarded as giving a "classical free field realization" of $\mathcal{O}(\mathfrak{Z}_G)$, because the chiral analog of $\mathcal{O}(\mathfrak{Z}_G) \to \mathcal{O}(\mathrm{KT}_G)$ induced by $\varphi$ provides a (chiral) free field realization of the chiral universal centralizer (c.f. section 2.1).

**Proposition 2** ([C, BN]). *There exists an open immersion $\varphi : \mathrm{KT}_G \to \mathfrak{Z}_G$.*
*Moreover, $\varphi$ is symplectic.*

The image of $\varphi$ is described by the open subset $U$, which is the big cell of the Bruhat decomposition of $G$:

$$\mathrm{Im}(\varphi) = (U \times S_{f_0}) \cap \mathfrak{Z}_G =: \mathfrak{Z}_U.$$

**Remark 3.** The open subset $\mathfrak{Z}_U \subset \mathfrak{Z}_G$ is obtained by using the same method as in the $\mathfrak{Z}_G$ case. That is,

$$\begin{aligned} T^*U \times_{\mathfrak{g}^*} S_{f_0} &\cong U \times S_{f_0}, \\ (U \times S_{f_0}) \times_{\mathfrak{g}^*} S_{f_0} &\cong \{(g, s) \in U \times S_{f_0} \mid \mathrm{Ad}_g(s) = s\} \\ &= (U \times S_{f_0}) \cap \mathfrak{Z}_G = \mathfrak{Z}_U. \end{aligned}$$

## 1.3 Universal centralizer of $SL_2(\mathbb{C})$

In this section, we let $G = SL_2(\mathbb{C})$. Since

$$f_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we get

$$S_{f_0} = \left\{ \begin{pmatrix} 0 & S \\ 1 & 0 \end{pmatrix} \;\middle|\; S \in \mathbb{C} \right\}.$$

Therefore,

$$\begin{aligned} \mathfrak{Z}_G &= \left\{ \left(g, \begin{pmatrix} 0 & S \\ 1 & 0 \end{pmatrix}\right) \in G \times S_{f_0} \;\middle|\; S \in \mathbb{C}, \left[g, \begin{pmatrix} 0 & S \\ 1 & 0 \end{pmatrix}\right] = 0 \right\} \\ &= \left\{ \begin{pmatrix} Y & SX \\ X & Y \end{pmatrix} \;\middle|\; X, Y, S \in \mathbb{C}, \det \begin{pmatrix} Y & SX \\ X & Y \end{pmatrix} = 1 \right\} \end{aligned}$$

Hence, $\mathfrak{Z}_G$ is an affine variety in $\mathbb{A}^3_{\mathbb{C}}$ with the coordinate ring

$$\mathcal{O}_{\mathfrak{Z}_G}(\mathfrak{Z}_G) = \frac{\mathbb{C}[X, Y, S]}{(SX^2 - Y^2 + 1)}.$$

The nontrivial Poisson brackets among the generators of $\mathcal{O}_{\mathfrak{Z}_G}(\mathfrak{Z}_G)$ are as follows:

$$\{S, X\} = Y, \quad \{S, Y\} = SX, \quad \{X, Y\} = -\frac{1}{2}X^2.$$

Since $U = \{X \neq 0\} \subset SL_2(\mathbb{C})$, $\mathfrak{Z}_U = \{X \neq 0\} \subset \mathfrak{Z}_G$. A Poisson algebra $\mathcal{O}_{\mathfrak{Z}_G}(U_X) = \mathbb{C}[X^{\pm 1}, Y]$ is isomorphic to $\mathcal{O}_{\mathrm{KT}_G}(\mathrm{KT}_G) = \mathbb{C}[\gamma^{\pm\frac{1}{2}}, b]$ via

$$X \mapsto \gamma^{-\frac{1}{2}}, \quad Y \mapsto -b\gamma^{-\frac{1}{2}}.$$

The restriction map $\dfrac{\mathbb{C}[X, Y, S]}{(SX^2 - Y^2 + 1)} \to \mathbb{C}[\gamma^{\pm\frac{1}{2}}, b]$ is given by

$$\begin{aligned} X &\mapsto \gamma^{-\frac{1}{2}}\,, \\ Y &\mapsto -b\gamma^{-\frac{1}{2}}\,, \\ S &\mapsto b^2 + \gamma\,. \end{aligned}$$

### 1.4 Vertex algebra

In this section, we will recall some basic facts about vertex algebras. For further details, refer to [FBZ, Kac].

A **vertex algebra** is a vector space $V$ equipped with the data below:

**:** (the vacuum vector) a vector $|0\rangle \in V$,
**:** (the translation operator) a linear map $T : V \to V$,
**:** (the vertex operators) a linear map $Y(\cdot, z) : V \to \mathrm{Hom}\,(V, V((z)))$.

These data are subject to the following axioms:

**:** (the vacuum axiom)
$Y(|0\rangle, z) = \mathrm{id}_V$.
Futhermore, $Y(a, z)|0\rangle \in V[\![z]\!]$ and $\lim_{z\to 0} Y(a, z)|0\rangle = a$ ($\forall a \in V$).
**:** (the translation axiom)
$[T, Y(a, z)] = \partial_z Y(a, z)$ ($\forall a \in V$) and $T|0\rangle = 0$.
**:** (the locality axiom)
for any $a, b \in V$, $(z - w)^N [Y(a, z), Y(b, w)] = 0$ for some $N \in \mathbb{Z}_{\geq 0}$.

As a consequence of the definition, we find

$$Y(a, z)Y(b, w) = \sum_{n=0}^{N-1} \frac{Y(a_{(n)}b, w)}{(z - w)^{n+1}} + :Y(a, z)Y(b, w):$$

for any $a, b \in V$, where we write $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ and by $\dfrac{1}{z - w}$ we understand its expansion in $\mathbb{C}((z))((w))$. This is usually expressed as follows:

$$Y(a, z)Y(b, w) \sim \sum_{n=0}^{N-1} \frac{Y(a_{(n)}b, w)}{(z - w)^{n+1}}.$$

For a finite dimensional Lie algebra $\mathfrak{g}$ with a symmetric invariant bilinear form $\kappa$, we can construct a vertex algebra. Let $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}\mathbf{1}$ be the Kac-Moody affinization of $\mathfrak{g}$ associated with $\kappa$. That is, $\widehat{\mathfrak{g}}_\kappa$ is a one-dimensional central extension of $\mathfrak{g}[t^{\pm 1}]$ with the two-cocycle $c(x \otimes t^n, y \otimes t^m) = n\delta_{n+m,0}\kappa(x, y)$. We define a $\widehat{\mathfrak{g}}_\kappa$-module $V^\kappa(\mathfrak{g})$ by

$$V^\kappa(\mathfrak{g}) := U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C},$$

where $\mathbb{C}$ is a one-dimensional representation of the subalgebra $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1} \subset \widehat{\mathfrak{g}}_\kappa$ on which $\mathfrak{g}[t]$ acts 0 and $\mathbf{1}$ acts as the identity. By the Poincaré-Birkhoff-Witt theorem, $V^\kappa(\mathfrak{g})$ is isomorphic to $U(t^{-1}\mathfrak{g}[t^{-1}])$ as a vector space and has a PBW basis

$$\{x^{a_1}_{-n_1-1} \cdots x^{a_m}_{-n_m-1}|0\rangle \mid n_1 \geq \cdots \geq n_m \geq 0, \text{ and if } n_i = n_{i+1}, \text{then } a_i \leq a_{i+1}\},$$

where $\{x^a\}_{a=1}^{\dim \mathfrak{g}}$ is a basis of $\mathfrak{g}$, $x_n$ denotes $x \otimes t^n \in \mathfrak{g}[t^{\pm 1}]$ for $x \in \mathfrak{g}$ and $|0\rangle \in V^\kappa(\mathfrak{g})$ is the image of $1 \otimes 1 \in U(\widehat{\mathfrak{g}}_\kappa) \otimes \mathbb{C}$.

Let $T : V^\kappa(\mathfrak{g}) \to V^\kappa(\mathfrak{g})$ and $Y(\cdot, z) : V^\kappa(\mathfrak{g}) \to \mathrm{Hom}\big(V^\kappa(\mathfrak{g}), V^\kappa(\mathfrak{g})((z))\big)$ be linear operators defined by the relations

$$T|0\rangle = 0, \quad [T, x^a_n] = -n x^a_{n-1}$$

and

$$Y(|0\rangle, z) = \mathrm{id}_{V^\kappa(\mathfrak{g})}, \quad Y(x^a_{-1}|0\rangle, z) = x^a(z) = \sum_{n \in \mathbb{Z}} x^a_n z^{-n-1},$$

$$Y(x^{a_1}_{-n_1-1} \cdots x^{a_m}_{-n_m-1}|0\rangle, z) = \left( \prod_{j=1}^{m} \frac{1}{n_j!} \right) : \partial_z^{n_1} x^{a_1}(z) \cdots \partial_z^{n_m} x^{a_m}(z) : .$$

Then $\big(V^\kappa(\mathfrak{g}), |0\rangle, T, Y(\cdot, z)\big)$ is a vertex algebra and called **the universal affine vertex algebra** associated with $\mathfrak{g}$ and $\kappa$.

A $\mathbb{Z}$-graded vertex algebra $V = \bigoplus_{d \in \mathbb{Z}} V_d$ is called a **vertex operator algebra** if

- $\dim(V_d) < \infty$ for all $d \in \mathbb{Z}$ and $V_d = 0$ for $d \ll 0$.
- there exists a vector $\omega \in V$ which satisfies the following:
  (1) $[L_n, L_m] = (n-m)L_{n+m} + \dfrac{n^3 - n}{12}\delta_{n+m,0}c$, where $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-1}$ and $c \in \mathbb{C}$,
  (2) $L_{-1} = T$,
  (3) $L_0|_{V_d} = d\,\mathrm{id}_{V_d}$ $(\forall d \in \mathbb{Z})$.

We call $\omega$ and $c$ **the conformal vector** and **the central charge** of $V$, respectively.

## 1.5 The quantized Drinfeld-Sokolov reduction

Let $\chi : \mathfrak{n}[t^{\pm 1}] \to \mathbb{C}$ be a functional on $\mathfrak{n}[t^{\pm 1}]$ defined by the formula

$$\chi(e^\alpha t^n) = \begin{cases} 1 & (\text{if } \alpha \text{ is a simple root and } n = -1) \\ 0 & (\text{otherwise}). \end{cases}$$

Since $\chi([x, y]) = 0$ for all $x, y \in \mathfrak{n}[t^{\pm 1}]$, we get a one dimensional representation of $\mathfrak{n}[t^{\pm 1}]$, denoted by $\mathbb{C}_\chi$.

**Definition 4** ([FF, KRW]). Let $M$ be in the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ of level $k$.
Define

$$H^\bullet_{DS}(M) := H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], M \otimes \mathbb{C}_\chi),$$

where $H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], N)$ means the semi-infinite $\mathfrak{n}[t^{\pm 1}]$-cohomology with coefficients in an $\mathfrak{n}[t^{\pm 1}]$-module $N$ ([F]).

We call $H^0_{DS}(M)$ **the quantized Drinfeld-Sokolov reduction** of $M$. It follows from the definition that if $M$ is a vertex operator algebra, then so is $H^0_{DS}(M)$.

### 1.6 Chiral differential operators

The sheaf of **chiral differential operators** $\mathcal{D}_X^{ch}$ is the sheaf of conformal vertex algebras on a smooth algebraic variety $X$, defined in [MSV, BD]. We will concretely describe the structure of the global section of $\mathcal{D}_G^{ch}$, where $G$ is an affine algebraic Lie group following [AG].

We first define **the arc space** $J_\infty X$ of a finite type scheme $X$. This is a (unique) scheme defined by

$$\mathrm{Hom}_{Scheme}(\mathrm{Spec}(A), J_\infty X) \cong \mathrm{Hom}_{Scheme}(\mathrm{Spec}(A[\![z]\!]), X)$$

for any commutative $\mathbb{C}$-algebra $A$. If $X = \mathrm{Spec}(R)$ is an affine scheme, $J_\infty X = \mathrm{Spec}(J_\infty R)$, where for $R = \mathbb{C}[x_1, x_2, \dots, x_n]/(f_1, f_2, \dots, f_r)$, a differential algebra $(J_\infty R, \partial)$ is defined by

$$J_\infty R := \mathbb{C}[\partial^m x_i \mid m \geq 0, 1 \leq i \leq n]/(\partial^l f_j \mid l \geq 0, 1 \leq j \leq r).$$

As a vertex algebra, $\mathcal{D}_G^{ch}(G) \cong U(\widehat{\mathfrak{g}}_k) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathcal{O}(J_\infty G)$, where $K$ is the central element in $\widehat{\mathfrak{g}}$ and $J_\infty G$ is the arc space of $G$. Here, $\mathfrak{g}[t] \subset J_\infty \mathfrak{g}$ acts on $\mathcal{O}(J_\infty G)$ as a left invariant vector field and $K$ acts as identity. We have a following vertex algebra structure on the right hand side:
we can take $\{x, f \mid x \in \mathfrak{g}, f \in \mathcal{O}(G)\}$ as a set of strong generators with fields

$$x(z) = \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1}, \; f(z) = \sum_{n \in \mathbb{Z}} f_{(n)} z^{-n-1},$$

where $f_{(-n-1)} = \dfrac{\partial^n f}{n!}$ if $n \geq 0$ and 0 if $n < 0$. These are mutually local fields and satisfy the OPEs

$$x(z)y(w) \sim \frac{[x,y](w)}{z-w} + \frac{k \cdot (x,y)}{(z-w)^2},$$
$$f(z)g(w) \sim 0,$$
$$x(z)f(w) \sim \frac{(x_L f)(w)}{z-w},$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathcal{O}(G)$, where $x_L$ is a left invariant vector field corresponding to $x \in \mathfrak{g}$. Moreover, $U(\widehat{\mathfrak{g}}_k) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathcal{O}(J_\infty G)$ is a $\mathbb{Z}_{\geq 0}$-graded vertex algebra by setting $\deg(x) = 1, \deg(f) = 0$.

From this fact, the following proposition holds.

**Proposition 5.** *There are injective vertex algebra homomorphisms*

$$\pi_L : V^k(\mathfrak{g}) \to \mathcal{D}_G^{ch}(G)\,; \quad x \mapsto x \otimes 1\,,$$
$$j : \mathcal{O}(J_\infty(G)) \to \mathcal{D}_G^{ch}(G)\,; \quad f \mapsto 1 \otimes f\,.$$

In addition, we have a vertex algebra homomorphism

$$\pi_R : V^{k^*}(\mathfrak{g}) \to \mathcal{D}_G^{ch}(G)$$

for $k^* := -k - 2h^\vee$. In order to describe this map, let us recall some facts about derivations and differential forms on $G$.

Let $\Omega^1(G)$ be the space of global differential forms on $G$ and $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ be the space of derivations on $\mathcal{O}(G)$. Then we have $\mathcal{O}(G)$-module isomorphisms

$$\Omega^1(G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))\,; \quad df \mapsto (x \mapsto x_L f)$$
$$\mathcal{O}(G) \otimes \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))\,; \quad 1 \otimes x \mapsto x_L,$$

where $d$ denotes the de Rham differential.

We define $\Omega$ to be the subspace of $\mathcal{D}_G^{ch}(G)$ spanned by $f\partial g$ with $f, g \in \mathcal{O}(G)$). Because of the following lemma, we can identify $\Omega$ with $\Omega^1(G)$.

**Lemma 6.** *There exists a linear isomorphism*

$$\Gamma : \Omega \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))\,; \quad f\partial g \mapsto (x \mapsto f \cdot (x_L g)).$$

Let $\{x^i\}_{i=1}^n$ be a basis of $\mathfrak{g}$ and $\{\omega^i\}_{i=1}^n$ be the dual basis with respect to the $\mathcal{O}(G)$-bilinear pairing $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \times \Omega^1(G) \to \mathcal{O}(G)$. Since $\{x_L^i\}_{i=1}^n$ forms a basis of a free $\mathcal{O}(G)$-module $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$, there exists some invertible matrix $(f^{i,j})$ over $\mathcal{O}(G)$ such that $x_R^i = \sum_{j=1}^n f^{i,j} x_L^j$ for all $i$.

Now we can define $\pi_R$.

**Proposition 7** ([AG, GMS])**.** (i) *For* $x^i = x^i_{(-1)}|0\rangle \in V^{k^*}(\mathfrak{g})$*, set*

$$\pi_R(x^i) = \sum_{i=1}^n f^{i,j} x + \sum_{j,k=1}^n k^* \cdot (x^j, x^k) f^{i,j} \omega^k.$$

*Then* $\pi_R$ *gives rise to a vertex algebra embedding* $\pi_R : V^{k^*}(\mathfrak{g}) \to \mathcal{D}_G^{ch}(G)$.
(ii) $\mathrm{Im}(\pi_R) \subset \mathrm{Com}(\mathrm{Im}(\pi_L), \mathcal{D}_G^{ch}(G)))$, *where* $\mathrm{Com}(W, V)$ *is the commutant of* $W$ *in* $V$ *for a vertex algebra* $V$ *and its vertex subalgebra* $W$.

**Remark 8.** We have decreasing (resp. increasing) filtration $F^s$ (resp. $F'_s$) of $\mathcal{D}_G^{ch}(G)$. Let $\{a_i\}_{i\in I}$ be a set of strong generators of $\mathcal{D}_G^{ch}(G)$. Then $F^s$ and $F'_s$ is the space spanned by the vectors

$$(a_1)_{(-n_1-1)} \cdots (a_m)_{(-n_m-1)}|0\rangle,\ n_1 + \cdots + n_m \geq s,$$
$$(a_1)_{(-n_1-1)} \cdots (a_m)_{(-n_m-1)}|0\rangle,\ \deg(a_1) + \cdots + \deg(a_m) \leq s,$$

respectively.

We have $\mathrm{gr}^F \mathcal{D}_G^{ch}(G) \cong \mathrm{gr}_{F'} \mathcal{D}_G^{ch}(G)$ ([A2]). Let us denote

$$\mathrm{gr}\mathcal{D}_G^{ch}(G) := \mathrm{gr}^F \mathcal{D}_G^{ch}(G) \cong \mathcal{O}(J_\infty T^*G) \cong \mathcal{O}(J_\infty \mathfrak{g}^*) \otimes \mathcal{O}(J_\infty G).$$

The $\mathfrak{g}[t^{\pm 1}]$-action on $\mathrm{gr}\mathcal{D}_G^{ch}(G)$ is as follows:

$$xt^n \cdot (a \otimes f_{(m)}) = a \otimes (x_L f)_{(n+m)},$$
$$xt^{-n-1} \cdot (a \otimes f_{(m)}) = (xt^{-n-1}) a \otimes f_{(m)},$$

where $x \in \mathfrak{g}$, $n, m \geq 0$, $a \in \mathcal{O}(J_\infty \mathfrak{g}^*) \cong \mathrm{Sym}(t^{-1}\mathfrak{g}[t^{-1}])$ and $f \in \mathcal{O}(G)$.

### 1.7 Chiral universal centralizer

The chiral universal centralizer is obtained by performing the quantized Drinfeld-Sokolov reduction of $\mathcal{D}_G^{ch}(G)$ twice.

There are two commuting actions of $\widehat{\mathfrak{g}}_k$ and $\widehat{\mathfrak{g}}_{k^*}$ on $\mathcal{D}_G^{ch}(G)$, corresponding to the vertex algebra homomorphisms

$$\pi_L : V^k(\mathfrak{g}) \to \mathcal{D}_G^{ch}(G),$$
$$\pi_R : V^{k^*}(\mathfrak{g}) \to \mathcal{D}_G^{ch}(G),$$

defined in the previous section.

Since $\mathcal{D}_G^{ch}(G)$ satisfies the condition in Definition 4, we can perform the quantized Drinfeld-Sokolov reduction of $\mathcal{D}_G^{ch}(G)$ first with respect to the former action, and then the latter action.

The first vertex algebra we obtain is said to be **the equivariant affine $W$-algebra** and denoted by $\mathbf{W}_{G,k}$ ([A1]). That is, $\mathbf{W}_{G,k} := H^0_{DS}(\mathcal{D}_G^{ch}(G))$. By Proposition 7 (ii), we know that $\pi_R$ descends to a vertex algebra homomorphism $\pi'_R : V^{k^*}(\mathfrak{g}) \to \mathbf{W}_{G,k}$, which gives rise to a $\widehat{\mathfrak{g}}_{k^*}$ action on $\mathbf{W}_{G,k}$. Then we can consider the quantized Drinfeld-Sokolov reduction of $H^0_{DS}(\mathcal{D}_G^{ch}(G))$ with respect to this action. Finally, we get a vertex algebra

$$\mathbf{I}_{G,k} := H^0_{DS}(\mathbf{W}_{G,k}),$$

called **the chiral universal centralizer** associated with $G$. The level $k$ in $\mathbf{W}_{G,k}$ and $\mathbf{I}_{G,k}$ may be omitted if it is clear.

The chiral universal centralizer is a strict chiral quantization of the (classical) universal centralizer ([A1]). That is, $X_{\mathbf{I}_G} \cong \mathfrak{Z}_G$ as Poisson varieties, $\tilde{X}_{\mathbf{I}_G} \cong \mathfrak{Z}_G$ as schemes and the surjective Poisson vertex algebra homomorphism $\mathcal{O}(J_\infty \tilde{X}_{\mathbf{I}_G}) \to \mathrm{gr}\mathbf{I}_G$ induced by the identity map on $R_{\mathbf{I}_G}$ is an isomorphism. Here, for a vertex algebra $V$, $R_V$ denotes the Zhu's $C_2$ algebra of $V$ ([ZhuY]), $X_V := \mathrm{Specm}(R_V)$ is the associated variety of $V$ and $\tilde{X}_V := \mathrm{Spec}(R_V)$ is the associated scheme of $V$ ([A1]).

The vertex algebra $\mathbf{I}_{G,k}$ is conformal with central charge $2(\mathrm{rk}(\mathfrak{g}) + 24\langle \rho, \rho^\vee \rangle)$, where $\rho$ (resp. $\rho^\vee$) is the half sum of positive root (resp. positive coroot). Moreover, $\mathbf{I}_{G,k}$ is simple ([AM]) because its associated scheme $\mathfrak{Z}_G$ is smooth ([R]) and reduced. Note that we have the following vertex algebra homomorphisms:

$$\begin{aligned}
&\pi_L \otimes \pi_R : V^k(\mathfrak{g}) \otimes V^{k^*}(\mathfrak{g}) \to \mathcal{D}^{ch}_G(G),\\
&H^0_{DS}(\pi_L) \otimes \pi'_R : \mathcal{W}^k(\mathfrak{g}) \otimes V^{k^*}(\mathfrak{g}) \to \mathbf{W}_{G,k},\\
&H^0_{DS}(\pi_L) \otimes H^0_{DS}(\pi'_R) : \mathcal{W}^k(\mathfrak{g}) \otimes \mathcal{W}^{k^*}(\mathfrak{g}) \to \mathbf{I}_G.
\end{aligned}$$

These maps are conformal when $k$ is non-critical level.

Let the level $k$ be irrational. We have a decomposition of $\mathcal{D}^{ch}_{G,k}(G)$ as a $V^k(\mathfrak{g}) \otimes V^{k^*}(\mathfrak{g})$-module ([AG, ZhuM]):

$$\mathcal{D}^{ch}_{G,k}(G) \cong \bigoplus_{\lambda \in P_+} V^k_\lambda \otimes V^{k^*}_{\lambda^*}.$$

Here, $P_+$ is the set of dominant integral weights of $\mathfrak{g}$, $V^k_\lambda$ denotes the Weyl module in level $k$ and $\lambda^* = -\mathbf{w} \cdot \lambda$, where $\mathbf{w}$ is the longest Weyl group element of $\mathfrak{g}$. It follows that $\mathbf{I}_{G,k}$ has a decomposition as a $\mathcal{W}^k(\mathfrak{g}) \otimes \mathcal{W}^{k^*}(\mathfrak{g})$-module:

$$\mathbf{I}_{G,k} \cong \bigoplus_{\lambda \in P_+} T^k_{\lambda,0} \otimes T^{k^*}_{\lambda^*,0},$$

where $T^k_{\lambda,0} := H^0_{DS}(V^k_\lambda)$ is the irreducible $\mathcal{W}^k(\mathfrak{g})$-module studied in [AF]. For $G = SL_2(\mathbb{C})$, this decomposition appeared in [FS] and they called it the modified regular representation of the Virasoro algebra.

## 2 Main Results

### 2.1 Free field realization of $\mathbf{I}_G$

In this section, we construct a free field realization of $\mathbf{I}_G$ in a similar way to [FS].

We denote $U$ the big cell of the Bruhat decomposition of $G$, corresponding to the longest Weyl group element $\mathbf{w}$. This subset $U \subset G$ is open and we can concretely write $U = N\widetilde{\mathbf{w}}TN$. Here $\widetilde{\mathbf{w}}$ is the uplift of $\mathbf{w}$ to the normalizer of $T$ in $G$ and lies in the connected component of the identity.

The vertex algebra $\mathcal{D}^{ch}_G(U)$ is equipped with the commuting actions of $\widehat{\mathfrak{g}}_k$ and $\widehat{\mathfrak{g}}_{k^*}$, induced by composing the restriction map with $\pi_L$ and $\pi_R$:

$$V^k(\mathfrak{g}) \xrightarrow{\pi_L} \mathcal{D}^{ch}_G(G) \xrightarrow{\text{restriction}} \mathcal{D}^{ch}_G(U),$$

$$V^{k^*}(\mathfrak{g}) \xrightarrow{\pi_R} \mathcal{D}^{ch}_G(G) \xrightarrow{\text{restriction}} \mathcal{D}^{ch}_G(U).$$

As in the previous section, we take the quantized Drinfeld-Sokolov reduction twice; first with respect to the $\widehat{\mathfrak{g}}_k$ action, and then with respect to the $\widehat{\mathfrak{g}}_{k^*}$ action.

By performing the first reduction, we obtain the following vertex algebra homomorphisms:

$$\mathcal{W}^k(\mathfrak{g}) \otimes V^{k^*}(\mathfrak{g}) \to \mathbf{W}_{G,k} \to H^0_{DS}(\mathcal{D}^{ch}_G(U)).$$

Then we perform the second reduction with respect to the homomorphisms

$$V^{k^*}(\mathfrak{g}) \to \mathbf{W}_{G,k} \to H^0_{DS}(\mathcal{D}^{ch}_G(U)),$$

and get the following vertex algebra homomorphisms:

$$\mathcal{W}^k(\mathfrak{g}) \otimes \mathcal{W}^{k^*}(\mathfrak{g}) \to \mathbf{I}_G \to H^0_{DS}(H^0_{DS}(\mathcal{D}^{ch}_G(U))).$$

Now we state the following theorem.

**Theorem 9.** *There exists a vertex algebra embedding*

$$\nu : \mathbf{I}_{G,k} \to \mathcal{D}^{ch}_T(T).$$

*Moreover, the corresponding map between associated varieties coincides with $\varphi$ in Prososition 2.*

To prove this theorem, we need the following lemmas.

**Lemma 10.** (i) *Let $G$ be a unipotent algebraic Lie group.*
*The action of $t^{-1}\mathfrak{g}[t^{-1}]$ on $\mathcal{D}^{ch}_G(G)$ is free and the action of $\mathfrak{g}[t]$ on $\mathcal{D}^{ch}_G(G)$ is cofree. Here, these actions come from the vertex algebra homomorphism $V^k(\mathfrak{g}) \to \mathcal{D}^{ch}_G(G)$.*
(ii) $H^0_{DS}(\mathcal{D}^{ch}_N(N)) \cong \mathbb{C}$.

*Proof* (Lemma 10). (i) Let $M := \mathcal{D}^{ch}_G(G)$. It suffices to show that $H_i(t^{-1}\mathfrak{g}[t^{-1}], M) = 0$ and $H^i(\mathfrak{g}[t], M) = 0$ if $i \neq 0$. Set $H_\bullet(N_1) := H_\bullet(t^{-1}\mathfrak{g}[t^{-1}], N_1)$ and $H^\bullet(N_2) := H^\bullet(\mathfrak{g}[t], N_2)$ for a $t^{-1}\mathfrak{g}[t^{-1}]$-module $N_1$ and a $\mathfrak{g}[t]$-module $N_2$.

We have the $J_\infty\mathfrak{g}^*$-action and $J_\infty\mathfrak{g}$-action on $\mathcal{O}(J_\infty T^*G)$ (cf. Remark 8). The former action is free and the latter is cofree. Indeed, the freeness is clear. Since $G$ is unipotent, $J_\infty G$ is so. Hence $J_\infty\mathfrak{g}$-action on $\mathcal{O}(J_\infty T^*G)$ is cofree.

Therefore, $H_i(\mathrm{gr}M) = 0$ and $H^i(\mathrm{gr}M) = 0$ for $i \neq 0$, and $H_0(\mathrm{gr}M) \cong \mathcal{O}(J_\infty\mathfrak{g}^*)$ and $H^0(\mathrm{gr}M) \cong \mathcal{O}(J_\infty G)$. We can get the homology (resp. cohomology) spectral sequence $E^r$ (resp. $E_r$) such that $E^1_\bullet \cong H_\bullet(\mathrm{gr}M)$ and $E^\infty_\bullet = \mathrm{gr}H_\bullet(M)$ (resp. $E_1^\bullet \cong H^\bullet(\mathrm{gr}M)$ and $E_\infty^\bullet = \mathrm{gr}H^\bullet(M)$).

Since $H_i(\mathrm{gr}M) = 0$ if $i \neq 0$ and $H_0(\mathrm{gr}M) \cong \mathcal{O}(J_\infty\mathfrak{g}^*)$, we get $E^1 = E^\infty$. Therefore, $\mathrm{gr}H_\bullet(M) \cong H_\bullet(\mathrm{gr}M)$ and we get $H_i(M) = 0$ if $i \neq 0$. Similarly, $H^i(M) = 0$ if $i \neq 0$. □

(ii) Let $M := \mathcal{D}^{ch}_N(N)$. Let us consider $C^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], M \otimes \mathbb{C}_\chi)$, which is the complex that provides $H^\bullet_{DS}(M) = H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], M \otimes \mathbb{C}_\chi)$, as a double complex with derivations coming from the $\mathfrak{n}[t^{\pm 1}]$-actions on $M$ and $\mathbb{C}_\chi$. Then we find there exists a convergent cohomology spectral sequence $E'_r$ such that $E'^\bullet_1 = H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], M)$ and $E'^\bullet_\infty = H^\bullet_{DS}(M)$.

By Lemma 10 (i) and [V],

$$H^{\frac{\infty}{2}+i}(\mathfrak{n}[t^{\pm 1}], M) \cong \begin{cases} \mathrm{Im}(M^{\mathfrak{n}[t]} \to M \to M/t^{-1}\mathfrak{n}[t^{-1}]M) & (\text{if } i = 0) \\ 0 & (\text{if } i \neq 0). \end{cases}$$

Therefore, $E'_r$ collapses at $r = 1$ and $E'_1 = E'_\infty$, so

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t^{\pm 1}], M) \cong H^\bullet_{DS}(M).$$

We will show that $\mathrm{Im}(M^{\mathfrak{n}[t]} \to M \to M/t^{-1}\mathfrak{n}[t^{-1}]M) = \mathbb{C}|0\rangle$. It suffices to verify that $M^{\mathfrak{n}[t]} = \mathbb{C}|0\rangle \oplus (t^{-1}\mathfrak{n}[t^{-1}]M \cap M^{\mathfrak{n}[t]})$.

Clearly, $M^{\mathfrak{n}[t]} \supset \mathbb{C}|0\rangle \oplus (t^{-1}\mathfrak{n}[t^{-1}]M \cap M^{\mathfrak{n}[t]})$. Assume that $M^{\mathfrak{n}[t]} \supsetneq \mathbb{C}|0\rangle \oplus (t^{-1}\mathfrak{n}[t^{-1}]M \cap M^{\mathfrak{n}[t]})$. Then there exists a nonzero element $a \in M^{\mathfrak{n}[t]}$ such that

$$a = 1 \otimes \sum (f_1)_{(-n_1-1)} \cdots (f_m)_{(-n_m-1)}|0\rangle$$

for non-constant functions $f_1, \ldots, f_m \in \mathcal{O}(N)$ and $n_1, \ldots, n_m \geq 0$. Since $a$ is nonzero, there is an element $xt^n \in \mathfrak{n}[t]$ such that $xt^n \cdot a \neq 0$. This contradicts the assumption that $a \in M^{\mathfrak{n}[t]}$. Therefore, $M^{\mathfrak{n}[t]} = \mathbb{C}|0\rangle \oplus (t^{-1}\mathfrak{n}[t^{-1}]M \cap M^{\mathfrak{n}[t]})$.

Consequently, $H^0_{DS}(\mathcal{D}^{ch}_N(N)) \cong \mathbb{C}$. □

*Proof* (Theorem 9). We have $\mathcal{D}^{ch}_G(U) \cong \mathcal{D}^{ch}_N(N) \otimes \mathcal{D}^{ch}_T(T) \otimes \mathcal{D}^{ch}_N(N)$ as a vertex operator algebra. The action of $\mathfrak{n}[t^{\pm 1}]$ which comes from $\pi_L$ concerns with only the latter $\mathcal{D}^{ch}_N(N)$. Therefore, we can show

$$H^0_{DS}(\mathcal{D}^{ch}_G(U)) \cong \mathcal{D}^{ch}_N(N) \otimes \mathcal{D}^{ch}_T(T),$$

in the same way as Lemma 10 (ii) since $N$ is unipotent. Likewise, the action corresponding to $\pi_R$ (and $\pi'_R$) concerns with only the former $\mathcal{D}^{ch}_G(N)$, so

$$H^0_{DS}(H^0_{DS}(\mathcal{D}^{ch}_G(U))) \cong \mathcal{D}^{ch}_T(T).$$

By composing with $\mathbf{I}_G \to H^0_{DS}(H^0_{DS}(\mathcal{D}^{ch}_G(U)))$, we obtain the map

$$\nu : \mathbf{I}_G \to \mathcal{D}^{ch}_T(T).$$

The injectivity of $\nu$ follows from the simplicity of $\mathbf{I}_G$.

To complete the proof, we need to show that $\nu$ descends to $\varphi$. This follows from the commutativity of the Kostant reduction and the quantized Drinfeld-Sokolov reduction ([A4]). That is,

$$\begin{aligned} X_{\mathbf{W}_{G,k}} &\cong X_{\mathcal{D}^{ch}_G(G)} \times_{\mathfrak{g}^*} S_{f_0} \\ &\cong T^*G \times_{\mathfrak{g}^*} S_{f_0} \\ &\cong G \times S_{f_0}, \end{aligned}$$

and

$$\begin{aligned} X_{\mathbf{I}_G} &\cong X_{\mathbf{W}_{G,k}} \times_{\mathfrak{g}^*} S_{f_0} \\ &\cong (G \times S_{f_0}) \times_{\mathfrak{g}^*} S_{f_0} \\ &\cong \mathfrak{Z}_G. \end{aligned}$$

Similarly, we have $X_{\mathcal{D}^{ch}_G(U)} \cong \mathfrak{Z}_G$ (c.f. Remark 3). The restriction map $\mathcal{D}^{ch}_G(G) \to \mathcal{D}^{ch}_G(U)$ descends to the inclusion map $T^*U \to T^*G$, so $\nu : \mathbf{I}_G \to \mathcal{D}^{ch}_T(T)$ descends to $\varphi : \mathrm{KT}_G \cong \mathfrak{Z}_U \to \mathfrak{Z}_G$. □

### 2.2 An example : $SL_2(\mathbb{C})$-case

In this section we describe the free field realization $\nu$ concretely for $G = SL_2(\mathbb{C})$. In this case, the decomposition $U = N\mathbf{w}TN$ is described as

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{C}, \xi \in \mathbb{C}^\times \right\}.$$

Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be a basis of $\mathfrak{g}$. Take elements $a, b, c, d$ in $\mathcal{O}(G)$, whose values on $g \in G$ are the (1, 1)-entry, the (1, 2)-entry, the (2, 1)-entry and the (2, 2)-entry of $g$, respectively. Then

$$\mathcal{O}(G) = \mathbb{C}[a, b, c, d]/(ad - bc - 1).$$

We can find

$$\begin{aligned} e_R &= -d^2 e_L - cd h_L + c^2 f_L, \\ h_R &= -2bd e_L - (ad + bc) h_L + 2ac f_L, \\ f_R &= b^2 e_L + ab h_L - a^2 f_L, \\ \omega_e &= d \cdot (db) - b \cdot (dd), \\ \omega_h &= \frac{1}{2}(d \cdot (da) + c \cdot (db) - b \cdot (dc) - a \cdot (dd)), \\ \omega_f &= -c \cdot (da) + a \cdot (dc). \end{aligned}$$

Therefore, $\mathcal{D}^{ch}_G(G) \cong U(\widehat{\mathfrak{g}}_k) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathcal{O}(J_\infty G)$ is strongly generated by $e = e_{(-1)} \otimes 1 |0\rangle, h = h_{(-1)} \otimes 1 |0\rangle, f = f_{(-1)} \otimes 1 |0\rangle$ and $a = 1 \otimes a |0\rangle, b = 1 \otimes b |0\rangle, c = 1 \otimes c |0\rangle, d = 1 \otimes d |0\rangle$. More concretely, let $V$ be the vertex algebra generated by fields $e(z), h(z), f(z), a(z), b(z), c(z), d(z)$ with the following nontrivial OPEs:

$$
\begin{aligned}
&e(z)h(w) \sim -\frac{2e(w)}{z-w}, \quad e(z)f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2},\\
&h(z)h(w) \sim \frac{2k}{(z-w)^2}, \quad h(z)f(w) \sim -\frac{2f(w)}{z-w},\\
&e(z)b(w) \sim \frac{a(w)}{z-w}, \quad e(z)d(w) \sim \frac{c(w)}{z-w},\\
&h(z)a(w) \sim \frac{a(w)}{z-w}, \quad h(z)b(w) \sim -\frac{b(w)}{z-w},\\
&h(z)c(w) \sim \frac{c(w)}{z-w}, \quad h(z)d(w) \sim -\frac{d(w)}{z-w},\\
&f(z)a(w) \sim \frac{b(w)}{z-w}, \quad f(z)c(w) \sim \frac{d(w)}{z-w}.
\end{aligned}
$$

Finally, we can obtain $\mathcal{D}^{ch}_G(G)$ as the quotient of $V$ by the submodule generated by the field

$$: a(z)d(z) : - : b(z)c(z) : -1.$$

Next, we consider $\mathcal{D}^{ch}_G(U) \cong \mathcal{D}^{ch}_N(N) \otimes \mathcal{D}^{ch}_T(T) \otimes \mathcal{D}^{ch}_N(N)$.

It is easy to see see that $\mathcal{D}^{ch}_N(N)$ is isomorphic to the $\beta\gamma$ system $\langle \beta, \gamma \rangle$, which is the vertex algebra generated by two fields $\beta(z) = \sum_{n\in\mathbb{Z}} \beta_{(n)} z^{-n-1}, \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_{(n)} z^{-n}$ and the only nontrivial OPE

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}.$$

As for $\mathcal{D}^{ch}_T(T)$, this is generated by $\widetilde{h}(z), \lambda_+(z)$ and $\lambda_-(z)$ with the following nontrivial OPE and relation:

$$\widetilde{h}(z)\widetilde{h}(w) \sim \frac{2k}{(z-w)^2}, \quad \widetilde{h}(z)\lambda_\pm(w) \sim \frac{\pm\lambda_\pm(w)}{z-w},$$

$$: \lambda_+(z)\lambda_-(z) : -1 = 0.$$

For simplicity, we choose $\eta(z) := \widetilde{h}(z) - 2 : \lambda_-(z)\partial\lambda_+(z) :$ instead of $\widetilde{h}(z)$ for the generating field of $\mathcal{D}^{ch}_T(T)$. That is, $\mathcal{D}^{ch}_T(T)$ is isomorphic to $\langle \eta, \lambda_\pm \rangle$, the vertex algebra generated by $\eta(z), \lambda_+(z)$ and $\lambda_-(z)$ with the following nontrivial OPE and relation:

$$\eta(z)\eta(w) \sim \frac{2(k+2)}{(z-w)^2}, \quad \eta(z)\lambda_\pm(w) \sim \frac{\pm\lambda_\pm(w)}{z-w},$$

$$: \lambda_+(z)\lambda_-(z) : -1 = 0.$$

Hence, we can write $\mathcal{D}^{ch}_G(U) \cong \langle \beta_x, \gamma_x \rangle \otimes \langle \eta, \lambda_\pm \rangle \otimes \langle \beta_y, \gamma_y \rangle$, where $\langle \beta_x, \gamma_x \rangle, \langle \beta_y, \gamma_y \rangle$ are the $\beta\gamma$ systems.

The restriction map $r : \mathcal{D}^{ch}_G(G) \to \mathcal{D}^{ch}_G(U)$ is as follows:

$$\begin{aligned}
a(z) &\mapsto\ :\lambda_+(z)\gamma_x(z):\,,\\
b(z) &\mapsto\ :\lambda_+(z)\gamma_x(z)\gamma_y(z): -\lambda_-(z)\,,\\
c(z) &\mapsto\ \lambda_+(z)\,,\\
d(z) &\mapsto\ :\lambda_+(z)\gamma_y(z):\,,\\
e(z) &\mapsto\ \beta_y(z)\,,\\
h(z) &\mapsto\ \eta(z) - 2:\gamma_y(z)\beta_y(z):\,,\\
f(z) &\mapsto\ -:\lambda_-^2(z)\beta_x(z): - :\gamma_y^2(z)\beta_y(z): + :\gamma_y(z)\eta(z): + k\partial\gamma_y(z)\,.
\end{aligned}$$

We will take the quantized Drinfeld-Sokolov reduction with respect to the $\widehat{\mathfrak{g}}_k$-action. Then we get $\mathbf{W}_G := H^0_{DS}(\mathcal{D}^{ch}_G)$ as the quotient of $\widetilde{\mathbf{W}}_G$. $\widetilde{\mathbf{W}}_G$ is a vertex algebra with generating fields $\widetilde{a}(z), \widetilde{b}(z), \widetilde{c}(z), \widetilde{d}(z), \widetilde{f}(z)$ and nontrivial OPEs

$$\begin{aligned}
&\widetilde{a}(z)\widetilde{b}(w) \sim \frac{:\widetilde{a}^2(w):}{2(z-w)}, \quad \widetilde{a}(z)\widetilde{d}(w) \sim \frac{:\widetilde{a}(w)\widetilde{c}(w):}{2(z-w)},\\
&\widetilde{b}(z)\widetilde{c}(w) \sim -\frac{:\widetilde{a}(w)\widetilde{c}(w):}{2(z-w)}, \quad \widetilde{c}(z)\widetilde{d}(w) \sim \frac{:\widetilde{c}^2(w):}{2(z-w)},\\
&\widetilde{b}(z)\widetilde{b}(w) \sim \frac{2k+3}{4}\left\{\frac{:\widetilde{a}^2(w):}{(z-w)^2} + \frac{:(\partial\widetilde{a})(w)\widetilde{a}(w):}{z-w}\right\},\\
&\widetilde{d}(z)\widetilde{d}(w) \sim \frac{2k+3}{4}\left\{\frac{:\widetilde{c}^2(w):}{(z-w)^2} + \frac{:\partial(\widetilde{c})(w)\widetilde{c}(w):}{z-w}\right\},\\
&\widetilde{b}(z)\widetilde{d}(w) \sim \frac{2k+3}{4}\cdot\frac{:\widetilde{a}(w)\widetilde{c}(w):}{(z-w)^2} + \frac{2+(2k+3):(\partial\widetilde{a})(w)\widetilde{c}(w):}{4(z-w)},\\
&\widetilde{f}(z)\widetilde{a}(w) \sim \frac{(2k+1)\widetilde{a}(w)}{4(z-w)^2} + \frac{\widetilde{b}(w)}{z-w}, \quad \widetilde{f}(z)\widetilde{c}(w) \sim \frac{(2k+1)\widetilde{c}(w)}{4(z-w)^2} + \frac{\widetilde{d}(w)}{z-w},\\
&\widetilde{f}(z)\widetilde{b}(w) \sim -\frac{(k+2)(2k+1)\widetilde{a}(w)}{2(z-w)^3} - \frac{(2k+7)\widetilde{b}(w)}{4(z-w)^2} - \frac{:\widetilde{f}(w)\widetilde{a}(w):}{z-w},\\
&\widetilde{f}(z)\widetilde{d}(w) \sim -\frac{(k+2)(2k+1)\widetilde{c}(w)}{2(z-w)^3} - \frac{(2k+7)\widetilde{d}(w)}{4(z-w)^2} - \frac{:\widetilde{f}(w)\widetilde{c}(w):}{z-w},\\
&\widetilde{f}(z)\widetilde{f}(w) \sim -\frac{(k+2)(2k+1)(3k+4)}{2(z-w)^4} - \frac{2(k+2)\widetilde{f}(w)}{(z-w)^2} - \frac{(k+2)(\partial\widetilde{f})(w)}{z-w}.
\end{aligned}$$

The equivariant affine $W$-algebra $\mathbf{W}_G$ is the quotient of $\widetilde{\mathbf{W}}_G$ by the submodule generated by the field

$$:\widetilde{a}(z)\widetilde{d}(z): - :\widetilde{b}(z)\widetilde{c}(z): - :(\partial\widetilde{a})(z)\widetilde{c}(z): -1.$$

The homomorphism $H^0_{DS}(r) : \mathbf{W}_G \to \langle\beta_x, \gamma_x\rangle \otimes \langle\eta, \lambda_\pm\rangle$ is

$$\begin{aligned}
\widetilde{a}(z) &\mapsto\ :\lambda_+(z)\gamma_x(z):\,,\\
\widetilde{b}(z) &\mapsto\ -\frac{1}{2}:\gamma_x(z)\eta(z)\lambda_+(z): -\lambda_-(z)\,,\\
\widetilde{c}(z) &\mapsto\ \lambda_+(z)\,,\\
\widetilde{d}(z) &\mapsto\ -\frac{1}{2}:\eta(z)\lambda_+(z):\,,\\
\widetilde{f}(z) &\mapsto\ -\frac{1}{4}:\eta^2(z): - :\beta_x(z)\lambda_-^2(z): - \frac{k+1}{2}(\partial\eta)(z)\,.
\end{aligned}$$

We will take the quantized Drinfeld-Sokolov reduction with respect to the $\widehat{\mathfrak{g}}_k$-action coming from the vertex algebra homomorphism $V^{k^*}(\mathfrak{g}) \to \mathcal{D}^{ch}_G(G) \to \mathbf{W}_G$.

We get $\mathbf{I}_G$ as the quotient of $\widetilde{\mathbf{I}}_G$, where $\widetilde{\mathbf{I}}_G$ is the vertex algebra generated by $C(z), D(z), F(z)$ subjected to the following nontrivial OPEs:

$$\begin{aligned}
C(z)D(w) &\sim \frac{:C^2(w):}{2(z-w)},\\
D(z)D(w) &\sim \frac{2k+3}{4}\left\{\frac{:C^2(w):}{(z-w)^2} + \frac{:(\partial C)(w)C(w):}{z-w}\right\},\\
F(z)C(w) &\sim \frac{(2k+1)C(w)}{4(z-w)^2} + \frac{D(w)}{z-w},\\
F(z)D(w) &\sim -\frac{(k+2)(2k+1)C(w)}{(z-w)^3} - \frac{(2k+7)D(w)}{4(z-w)^2} - \frac{:F(w)C(w):}{z-w},\\
F(z)F(w) &\sim -\frac{(k+2)(2k+1)(3k+4)}{2(z-w)^4} - \frac{2(k+2)F(w)}{(z-w)^2} - \frac{(k+2)(\partial F)(w)}{z-w}.
\end{aligned}$$

The chiral universal centralizer $\mathbf{I}_G$ is the quotient of $\widetilde{\mathbf{I}}_G$ by the submodule generated by the field

$$\begin{aligned}
:F(z)C^2(z): + :D^2(z): &- \frac{2k+7}{2}(:C(z)(\partial D)(z): - :(\partial C)(z)D(z):)\\
&+ \frac{2k+7}{4}:(\partial C)^2(z): - \frac{2k+3}{8}:(\partial^2 C)(z)C(z): + 1.
\end{aligned}$$

The homomorphism $\nu = H^0_{DS}(H^0_{DS}(r)) : \mathbf{I}_G \to \langle \eta, \lambda_\pm \rangle$ is as follows:

$$\begin{aligned}
C(z) &\mapsto \lambda_+(z),\\
D(z) &\mapsto -\frac{1}{2}:\eta(z)\lambda_+(z):,\\
F(z) &\mapsto -\frac{1}{4}:\eta^2(z): - :\lambda_-^2(z): - \frac{k+1}{2}(\partial\eta)(z).
\end{aligned}$$

When $k=-2$, this coincides with the free field reaization in [BN] by

$$C \longleftrightarrow X, \quad D \longleftrightarrow -Y, \quad F \longleftrightarrow -S,$$

$$\lambda_\pm \longleftrightarrow \gamma^{\mp\frac{1}{2}}, \quad \eta \longleftrightarrow -2b.$$

The conformal vector

$$\begin{aligned}
:\eta(z)(\partial\lambda_+)(z)\lambda_-(z): &+ :\beta_x(z)(\partial\gamma_x)(z): + :\beta_y(z)(\partial\gamma_y)(z):\\
&+ (k+1):(\partial\lambda_+)(z)(\partial\lambda_-)(z): - :(\partial^2\lambda_+)(z)\lambda_-(z):
\end{aligned}$$

in $\mathcal{D}^{ch}_G(U)$ descends to a conformal vector

$$\begin{aligned}
:\eta(z)(\partial\lambda_+)(z)\lambda_-(z): &- :(\partial\lambda_+)(z)(\partial\lambda_-)(z):\\
&- (k+3):(\partial^2\lambda_+)(z)\lambda_-(z): + (\partial\eta)(z)
\end{aligned}$$

in $\mathbf{I}_{G,k}$ with central charge 26, by performing the quantized Drinfeld-Sokolov reduction twice.

*Acknowledgements.* The author is very grateful to his advisor Professor Tomoyuki Arakawa and Ana Kontrec for valuable discussions and useful comments. He also appreciates the organizers deeply for providing an opportunity for giving a talk.

## References


[A1] Tomoyuki Arakawa. Chiral algebras of class S and Moore-Tachikawa symplectic varieties. arXiv:1811.01577.

[A2] Tomoyuki Arakawa. A remark on the $C_2$-cofiniteness condition on vertex algebras. Math. Z.,270(1-2):559-575, 2012.

[A3] Tomoyuki Arakawa. Associated Varieties and Higgs Branches (A Survey), Contemp. Math. 711, 37-44, 2018.

[A4] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and $C_2$-cofiniteness of W-algebras. Int. Math. Res. Not., 2015:11605–11666, 2015.

[AF] Tomoyuki Arakawa and Edward Frenkel. Quantum Langlands duality of representations of $\mathcal{W}$-algebras. Compos. Math.155, no.12, 2235–2262, 2019

[AG] Sergey Arkhipov and Dennis Gaitsgory. Differential operators on the loop group via chiral algebras. Int. Math. Res. Not., (4):165-210, 2002.

[AM] Tomoyuki Arakawa and Anne Moreau. Arc spaces and chiral symplectic cores. Publ. Res. Inst. Math. Sci. 57 , no. 3-4, 795-829, 2021.

[B] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Nat. Acad. Sci. U.S.A., 83(10):3068-3071, 1986.

[BD] Alexander Beilinson and Vladimir Drinfeld. Chiral algebras, volume 51 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[BLL+] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli and Balt C. van Rees. Infinite chiral symmetry in four dimensions. Comm. Math. Phys., 336(3):1359-1433, 2015.

[BFM] Roman Bezrukavnikov, Michael Finkelberg, and Ivan Mirković. Equivariant (K-)homology of affine Grassmannian and Toda lattice, Compositio Mathematica 141 no. 3, 746–768, 2005.

[BMR] Christopher Beem, Carlo Meneghelli and Leonardo Rastelli. Free Field Realizations from the Higgs Branch. arxiv:1903.07624.

[BN] Christopher Beem and Sujay Nair. Free field realisation and the chiral universal centraliser. Annales Henri Poincaré, 2023

[BPRvR] Christopher Beem, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Chiral algebras of class S. J. High Energy Phys., (5):020, front matter+67, 2015.

[BR] Christopher Beem and Leonardo Rastelli. Vertex operator algebras, Higgs branches, and modular differential equations. JHEP 08 114, front matter+70, 2018.

[C] Peter Crooks. Kostant-Toda lattices and the universal centralizer. Journal of Geometry and Physics 150 103595, 2020.

[F] Boris Feĭgin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. Uspekhi Mat. Nauk, 39(2(236)):195-196, 1984.

[FBZ] Edward Frenkel and David Ben-Zvi. Vertex algebras and algebraic curves. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, second edition, 2004.

[FF] Boris Feigin and Edward Frenkel. Quantization of the Drinfel'd-Sokolov reduction. Phys. Lett. B, 246(1-2):75-81, 1990.

[FS] Igor B. Frenkel and Konstantin Styrkas. Modified regular representations of affine and Virasoro algebras, VOA structure and semi-infinite cohomology. Advances in Mathematics 206 no. 1, 57-111, 2006.

[GMS] Vassily Gorbounov, Fyodor Malikov, and Vadim Schechtman. On chiral differential operators over homogeneous spaces. Int. J. Math. Math. Sci., 26(2):83-106, 2001.

[Kac] Victor Kac. Vertex algebras for beginners. University Lecture Series, 10. American Mathematical Society, Providence, RI, second edition, 1998.

[Kos1] Bertram Kostant. On Whittaker vectors and representation theory. Invent. Math. 48(2), 101-184 , 1978.

[Kos2] Bertram Kostant. Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight $\rho$. Selecta Math. (N.S.) 2 , no. 1, 43–9, 1996.

[Kos3] Bertram Kostant. The solution to a generalized Toda lattice and representation theory. Adv. in Math., 34(3):195-338. 1979.

[KRW] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. Comm. Math. Phys., 241(2-3):307-342, 2003.

[MSV] Fyodor Malikov, Vadim Schechtman, and Arkady Vaintrob. Chiral de Rham complex. Comm. Math. Phys., 204(2):439-473, 1999.

[MT] Gregory W. Moore and Yuji Tachikawa. On 2d TQFTs whose values are holomorphic symplectic varieties. In String-Math 2011, volume 85 of Proc. Sympos. Pure Math. pages 191-207. Amer. Math. Soc., Providence, RI, 2012.

[MY] Robert McRae and Jinwei Yang. An $\mathfrak{sl}_2$-type tensor category for the Virasoro algebra at central charge 25 and applications

[R] Simon Riche. Kostant section, universal centralizer, and a modular derived Satake equivalence. Math. Z. 286 , no. 1-2, 223-261, 2017.

[V] Alexander A. Voronov. Semi-infinite homological algebra. Invent. Math., 113(1):103-146, 1993.

[ZhuM] Minxian Zhu. Vertex operator algebras associated to modified regular representations of affine Lie algebras, Adv. Math. 219, 1513-1547, 2008.

[ZhuY] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc., 9(1):237-302, 1996.