THE CLEBSCH-GORDAN COEFFICIENTS OF $U_q(\mathfrak{sl}_2)$ AND GRASSMANN GRAPHS

HAU-WEN HUANG

ABSTRACT. In the first section, I will mention a connection between the Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ and Johnson graphs. In the second section, I will develop a q-analog connection between the Clebsch–Gordan coefficients of $U_q(\mathfrak{sl}_2)$ and Grassmann graphs.

1. THE CLEBSCH–GORDAN COEFFICIENTS OF $U(\mathfrak{sl}_2)$ AND JOHNSON GRAPHS

The notation $\mathbb{N}$ denotes the set of nonnegative integers. The notation $\mathbb{C}$ denotes the complex number field. The unadorned tensor product $\otimes$ is meant to be over $\mathbb{C}$. For any set X the notation $\mathbb{C}^X$ stands for the vector space over $\mathbb{C}$ that has a basis X. A vacuous summation is interpreted as 0. A vacuous product is interpreted as 1. An *algebra* is meant to be a unital associative algebra. An *algebra homomorphism* is meant to be a unital algebra homomorphism. For any two elements x, y in an algebra, the bracket $[x, y]$ is defined as

$$[x, y] = xy - yx.$$

The *universal enveloping algebra* $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ generated by E, F, H subject to the relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

The element

$$\Lambda = EF + FE + \frac{H^2}{2}$$

is called the *Casimir element* of $U(\mathfrak{sl}_2)$. For any $n \in \mathbb{N}$ there exists an $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module L_n that has a basis $\{v_i^{(n)}\}_{i=0}^n$ such that

$$\begin{aligned}
Ev_i^{(n)} &= iv_{i-1}^{(n)} \quad (1 \leq i \leq n), \qquad Ev_0^{(n)} = 0,\\
Fv_i^{(n)} &= (n-i)v_{i+1}^{(n)} \quad (0 \leq i \leq n-1), \qquad Fv_n^{(n)} = 0,\\
Hv_i^{(n)} &= (n-2i)v_i^{(n)} \quad (0 \leq i \leq n).
\end{aligned}$$

Every $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module is isomorphic to L_n.

Recall that the *comultiplication* Δ of $U(\mathfrak{sl}_2)$ is an algebra homomorphism $U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$ given by

$$\begin{aligned}
E &\mapsto E \otimes 1 + 1 \otimes E,\\
F &\mapsto F \otimes 1 + 1 \otimes F,\\
H &\mapsto H \otimes 1 + 1 \otimes H.
\end{aligned}$$

The $U(\mathfrak{sl}_2)$-module $L_m \otimes L_n$ has the basis

$$v_i^{(m)} \otimes v_j^{(n)} \qquad (0 \leq i \leq m;\ 0 \leq j \leq n).$$

The Clebsch–Gordan rule states that the $U(\mathfrak{sl}_2)$-module $L_m \otimes L_n$ is isomorphic to

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.$$

Thus the vectors

$$v_i^{(m+n-2p)} \qquad (0 \le p \le \min\{m,n\};\ 0 \le i \le m+n-2p)$$

can be viewed as a basis for $L_m \otimes L_n$. Roughly speaking the *Clebsch–Gordan coefficients* of $U(\mathfrak{sl}_2)$ are the entries of the transition matrix from the first basis to the second basis for $L_m \otimes L_n$.

The *universal Hahn algebra* $\mathcal{H}$ is an algebra over $\mathbb{C}$ generated by A, B, C and the relations assert that

$$[A,B] = C$$

and each of

$$[C,A] + 2A^2 + B,$$
$$[B,C] + 4BA + 2C$$

is central in $\mathcal{H}$. Note that the algebra $\mathcal{H}$ is generated by A and B. The Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ can be expressed in terms of Hahn polynomials. The phenomenon can be explained as follows:

Theorem 1.1 (Theorem 1.5, [5])**.** *There exists a unique algebra homomorphism* $\natural : \mathcal{H} \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$ *that sends*

$$A \mapsto \frac{H \otimes 1 - 1 \otimes H}{4},$$
$$B \mapsto \frac{\Delta(\Lambda)}{2}.$$

By pulling back via $\natural$ every $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module can be considered as an $\mathcal{H}$-module. Let V denote a $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module. For any $\theta \in \mathbb{C}$ we define

$$V(\theta) = \{v \in V \mid \Delta(H)v = \theta v\}.$$

Since $\Delta(H)$ is in the centralizer of $\natural(\mathcal{H})$ in $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$ the space $V(\theta)$ is an $\mathcal{H}$-submodule of V.

Let Ω stand for a finite set with size D. Let 2^Ω denote the power set of Ω. The notation $\subset\!\!\!\cdot$ stands for the covering relation of this subset lattice $(2^\Omega, \subseteq)$. For any integer k with $0 \le k \le D$ let

$$\binom{\Omega}{k} = \{\text{all } k\text{-element subsets of } \Omega\}.$$

Recall that the *Johnson graph* $J(D,k)$ is a simple connected graph whose vertex set is $\binom{\Omega}{k}$ and two vertices x, y are adjacent if and only if $x \cap y \subset\!\!\!\cdot\, x$. By [2, Theorem 13.2] there exists a $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ given by

$$Ex = \sum_{y \subset\!\!\!\cdot\, x} y \qquad \text{for all } x \in 2^\Omega,$$
$$Fx = \sum_{x \subset\!\!\!\cdot\, y} y \qquad \text{for all } x \in 2^\Omega,$$

$$Hx = (D - 2|x|)x \qquad \text{for all } x \in 2^\Omega.$$

The action of Λ on the $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ is as follows:

$$\Lambda x = \left(D + \frac{(D-2|x|)^2}{2}\right) x + 2 \sum_{\substack{|y|=|x| \\ x\cap y \subset x}} y \qquad \text{for all } x \in 2^\Omega.$$

Note that the above sum corresponds to a direct sum of the adjacency operators of $J(D,k)$ for all integers k with $0 \leq k \leq D$.

Fix an element $x_0 \in 2^\Omega$. The vector spaces $\mathbb{C}^{2^{\Omega\setminus x_0}}$ and $\mathbb{C}^{2^{x_0}}$ are $U(\mathfrak{sl}_2)$-modules. Hence $\mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ is a $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-module. There exists a unique linear map $\iota(x_0) : \mathbb{C}^{2^\Omega} \to \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ that sends

$$x \quad \mapsto \quad (x \setminus x_0) \otimes (x \cap x_0) \qquad \text{for all } x \in 2^\Omega.$$

Note that $\iota(x_0)$ is a linear isomorphism. For any element $X \in U(\mathfrak{sl}_2)$ the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{C}^{2^\Omega} & \xrightarrow{\ \iota(x_0)\ } & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}} \\
{\scriptstyle X}\big\downarrow & & \big\downarrow{\scriptstyle \Delta(X)} \\
\mathbb{C}^{2^\Omega} & \xrightarrow[\ \iota(x_0)\ ]{} & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}
\end{array}$$

By identifying $\mathbb{C}^{2^\Omega}$ with $\mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ via $\iota(x_0)$, this induces a $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-module structure on $\mathbb{C}^{2^\Omega}$. We denote this module by

$$\mathbb{C}^{2^\Omega}(x_0).$$

By pulling back via $\natural$ the $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}(x_0)$ is an $\mathcal{H}$-module. The action of A on the $\mathcal{H}$-module $\mathbb{C}^{2^\Omega}(x_0)$ is as follows:

$$Ax = \left(\frac{D}{4} - \frac{|x_0 \setminus x| + |x \setminus x_0|}{2}\right) x \qquad \text{for all } x \in 2^\Omega.$$

Applying the above commutative diagram with $X = \Lambda$ yields that the action of B on the $\mathcal{H}$-module $\mathbb{C}^{2^\Omega}(x_0)$ is as follows:

$$Bx = \left(\frac{D}{2} + \frac{(D-2|x|)^2}{4}\right) x + \sum_{\substack{|y|=|x| \\ x\cap y \subset x}} y \qquad \text{for all } x \in 2^\Omega.$$

Applying the above commutative diagram with $X = H$ yields that

$$\mathbb{C}^{\binom{\Omega}{k}} = \mathbb{C}^{2^\Omega}(x_0)(D-2k) \qquad (0 \leq k \leq D).$$

Hence $\mathbb{C}^{\binom{\Omega}{k}}$ is an $\mathcal{H}$-submodule of $\mathbb{C}^{2^\Omega}(x_0)$. We denote this $\mathcal{H}$-module by $\mathbb{C}^{\binom{\Omega}{k}}(x_0)$.

Now we assume that $1 \le k \le D-1$ and set $x_0 \in \binom{\Omega}{k}$. Let $\mathbf{T}(x_0)$ denote the *Terwilliger algebra* of $J(D,k)$ with respect to x_0. Since $J(D,k)$ is a P- and Q-polynomial association scheme, the algebra $\mathbf{T}(x_0)$ is the subalgebra of $\mathrm{End}(\mathbb{C}^{\binom{\Omega}{k}})$ generated by the adjacency operator $\mathbf{A}$ and the dual adjacency operator $\mathbf{A}^*(x_0)$ of $J(D,k)$. Recall that

$$\mathbf{A}^*(x_0)x = (D-1)\left(1 - \frac{D(|x_0 \setminus x| + |x \setminus x_0|)}{2k(D-k)}\right)x \qquad \text{for all } x \in \binom{\Omega}{k}.$$

Therefore the following equations hold on the $\mathcal{H}$-module $\mathbb{C}^{\binom{\Omega}{k}}(x_0)$:

$$\mathbf{A} = B - \frac{D}{2} - \frac{(D-2k)^2}{4},$$
$$\mathbf{A}^*(x_0) = \frac{D(D-1)}{k(D-k)}\left(A - \frac{(D-2k)^2}{4D}\right).$$

We have seen the following connection between the Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ and Johnson graphs:

Theorem 1.2 (Theorem 5.9, [5]). *Let $\mathcal{H} \to \mathrm{End}(\mathbb{C}^{\binom{\Omega}{k}})$ denote the representation corresponding to the $\mathcal{H}$-module $\mathbb{C}^{\binom{\Omega}{k}}(x_0)$. Then the following equality holds:*

$$\mathbf{T}(x_0) = \mathrm{Im}\left(\mathcal{H} \to \mathrm{End}(\mathbb{C}^{\binom{\Omega}{k}})\right).$$

2. The Clebsch–Gordan coefficients of $U_q(\mathfrak{sl}_2)$ and Grassmann graphs

Assume that q is a nonzero complex number which is not a root of 1. For any two elements x, y in an algebra over $\mathbb{C}$, the q-bracket $[x,y]_q$ is defined as

$$[x,y]_q = qxy - q^{-1}yx.$$

The q-analog $[n]_q$ of any integer n is defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

My first step is to develop a q-analog of the commutative diagram in Section 1. The *quantum algebra* $U_q(\mathfrak{sl}_2)$ is an algebra over $\mathbb{C}$ generated by $E, F, K^{\pm 1}$ subject to the relations

$$KK^{-1} = K^{-1}K = 1,$$
$$[E,K]_q = [K,F]_q = 0,$$
$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The element

$$\Lambda = (q - q^{-1})^2 EF + q^{-1}K + qK^{-1}$$

is called the *Casimir element* of $U_q(\mathfrak{sl}_2)$. Recall that a common comultiplication Δ of $U_q(\mathfrak{sl}_2)$ is an algebra homomorphism $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ given by

$$\begin{aligned} E &\mapsto E \otimes 1 + K \otimes E, \\ F &\mapsto F \otimes K^{-1} + 1 \otimes F, \\ K^{\pm 1} &\mapsto K^{\pm 1} \otimes K^{\pm 1}. \end{aligned}$$

Now assume that Ω is a vector space over a finite field $\mathbb{F}$ that has finite dimension D. Set the parameter

$$q = \sqrt{|\mathbb{F}|}.$$

The notation $\mathcal{L}(\Omega)$ stands for the set of all subspaces of Ω. This symbol $\lessdot$ now represents the covering relation of this subspace lattice $(\mathcal{L}(\Omega), \subseteq)$. For any integer k with $0 \leq k \leq D$ let

$$\mathcal{L}_k(\Omega) = \{\text{all } k\text{-dimensional subspaces of } \Omega\}.$$

Recall that the Grassmann graph $J_q(D,k)$ is a simple connected graph whose vertex set is $\mathcal{L}_k(\Omega)$ and two vertices x, x' are adjacent if and only if $x \cap x' \lessdot x$. It is known from [6, Section 33] that there exists a $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^{\mathcal{L}(\Omega)}$ given by

$$\begin{aligned}
Ex &= q^{1-D} \sum_{x' \lessdot x} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
Fx &= \sum_{x \lessdot x'} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
Kx &= q^{D - 2\dim x} x \qquad \text{for all } x \in \mathcal{L}(\Omega).
\end{aligned}$$

Fix an element $x_0 \in \mathcal{L}(\Omega)$. Let $\iota(x_0) : \mathbb{C}^{\mathcal{L}(\Omega)} \to \mathbb{C}^{\mathcal{L}(\Omega/x_0)} \otimes \mathbb{C}^{\mathcal{L}(x_0)}$ denote the linear map that sends

$$x \quad \mapsto \quad (x + x_0)/x_0 \otimes x \cap x_0 \qquad \text{for all } x \in \mathcal{L}(\Omega).$$

Unfortunately, the following diagram is not commutative for any element $X \in U_q(\mathfrak{sl}_2)$:

$$\begin{array}{ccc}
\mathbb{C}^{\mathcal{L}(\Omega)} & \xrightarrow{\ \iota(x_0)\ } & \mathbb{C}^{\mathcal{L}(\Omega/x_0)} \otimes \mathbb{C}^{\mathcal{L}(x_0)} \\
{\scriptstyle X}\big\downarrow & & \big\downarrow{\scriptstyle \Delta(X)} \\
\mathbb{C}^{\mathcal{L}(\Omega)} & \xrightarrow[\ \iota(x_0)\ ]{} & \mathbb{C}^{\mathcal{L}(\Omega/x_0)} \otimes \mathbb{C}^{\mathcal{L}(x_0)}
\end{array}$$

I choose another comultiplication Δ of $U_q(\mathfrak{sl}_2)$ [3, Lemma 1.2] which is an algebra homomorphism $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ given by

$$\begin{aligned}
E &\mapsto E \otimes 1 + K^{-1} \otimes E, \\
F &\mapsto F \otimes K + 1 \otimes F, \\
K^{\pm 1} &\mapsto K^{\pm 1} \otimes K^{\pm 1}.
\end{aligned}$$

I consider a more general setting of the $U_q(\mathfrak{sl}_2)$-module structure on $\mathbb{C}^{\mathcal{L}(\Omega)}$ [3, Proposition 11.2]: Suppose that λ is a nonzero scalar in $\mathbb{C}$. Then there exists a unique $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^{\mathcal{L}(\Omega)}$ such that

$$\begin{aligned}
Ex &= \lambda q^{-D} \sum_{x' \lessdot x} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
Fx &= \lambda^{-1} q \sum_{x \lessdot x'} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
Kx &= q^{D - 2\dim x} x \qquad \text{for all } x \in \mathcal{L}(\Omega).
\end{aligned}$$

We denote the $U_q(\mathfrak{sl}_2)$-module by $\mathbb{C}^{\mathcal{L}(\Omega)}(\lambda)$. The previous $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^{\mathcal{L}(\Omega)}$ is identical to the $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^{\mathcal{L}(\Omega)}(q)$. The action of Λ on the $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^{\mathcal{L}(\Omega)}(\lambda)$ is as follows:

$$\begin{aligned}\Lambda x = {} & (q^{D-2\dim x+1} + q^{2\dim x - D+1} + q^{-1-D} - q^{1-D})x \\ & + q^{1-D}(q-q^{-1})^2 \sum_{\substack{\dim x' = \dim x \\ x\cap x' \lessdot x}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega).\end{aligned}$$

Note that the above sum corresponds to a direct sum of the adjacency operators of $J_q(D,k)$ for all integers k with $0 \leq k \leq D$.

Recall the triple coordinate system for the subspace lattice $(\mathcal{L}(\Omega), \subseteq)$, introduced in Dunkl's 1977 paper [1, Section 4]. Define $\mathcal{L}(\Omega)_{x_0}$ to be the set of all triples (y,z,τ) where

- $y \in \mathcal{L}(\Omega/x_0)$;
- $z \in \mathcal{L}(x_0)$;
- τ is a linear map from y into x_0/z.

For any two triples $(y,z,\tau),(y',z',\tau') \in \mathcal{L}(\Omega)_{x_0}$ we write $(y,z,\tau) \subseteq (y',z',\tau')$ whenever the following conditions hold:

- $y \subseteq y'$.
- $z \subseteq z'$.
- $\tau(u) \subseteq \tau'(u)$ for all $u \in y$.

Note that $(\mathcal{L}(\Omega)_{x_0}, \subseteq)$ is a poset. Fix a subspace x_1 of Ω such that $\Omega = x_0 \oplus x_1$. For any $u \in \Omega$ we write u_0 and u_1 for the unique vectors $u_0 \in x_0$ and $u_1 \in x_1$ such that $u = u_0 + u_1$. For any $x \in \mathcal{L}(\Omega)$ we define the linear map $\tau_{x_0}^{x_1}(x) : x + x_0/x_0 \to x_0/x \cap x_0$ by

$$u + x_0 \quad \mapsto \quad u_0 + (x \cap x_0) \qquad \text{for all } u \in x.$$

The map $\Phi_{x_0}^{x_1} : \mathcal{L}(\Omega) \to \mathcal{L}(\Omega)_{x_0}$ given by

$$x \quad \mapsto \quad (x + x_0/x_0, x \cap x_0, \tau_{x_0}^{x_1}(x)) \qquad \text{for all } x \in \mathcal{L}(\Omega)$$

is an order isomorphism. We may identify the subspace lattice $(\mathcal{L}(\Omega), \subseteq)$ with the triple coordinate system $(\mathcal{L}(\Omega)_{x_0}, \subseteq)$. The following linear maps $L_1(x_0), L_2(x_0), R_1(x_0), R_2(x_0) : \mathbb{C}^{\mathcal{L}(\Omega)} \to \mathbb{C}^{\mathcal{L}(\Omega)}$ were mentioned in [1]:

$$\begin{aligned}
L_1(x_0) &: x \mapsto \sum_{\substack{x' \lessdot x \\ x' \cap x_0 = x \cap x_0}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
L_2(x_0) &: x \mapsto \sum_{\substack{x' \lessdot x \\ x' + x_0/x_0 = x + x_0/x_0}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
R_1(x_0) &: x \mapsto \sum_{\substack{x \lessdot x' \\ x' \cap x_0 = x \cap x_0}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
R_2(x_0) &: x \mapsto \sum_{\substack{x \lessdot x' \\ x' + x_0/x_0 = x + x_0/x_0}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega).
\end{aligned}$$

Define the linear maps $D_1(x_0), D_2(x_0) : \mathbb{C}^{\mathcal{L}(\Omega)} \to \mathbb{C}^{\mathcal{L}(\Omega)}$ as follows:

$$\begin{aligned}
D_1(x_0) &: x \mapsto q^{\dim \Omega/x_0 - 2\dim(x + x_0/x_0)} x \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
D_2(x_0) &: x \mapsto q^{\dim x_0 - 2\dim x \cap x_0} x \qquad \text{for all } x \in \mathcal{L}(\Omega).
\end{aligned}$$

Using the triple coordinate system $(\mathcal{L}(\Omega)_{x_0}, \subseteq)$, it is not difficult to me to verify the following properties: For any nonzero $\lambda, \mu \in \mathbb{C}$ the following diagrams commute:

$$\begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\lambda) \\ @V{q^{\dim x_0 - D} L_1(x_0)}VV @VV{E \otimes 1}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\lambda) \end{CD} \qquad \begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \\ @V{q^{\dim x_0 - D} D_1(x_0) \circ L_2(x_0)}VV @VV{1 \otimes E}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \end{CD}$$

$$\begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\lambda) \\ @V{q^{1-\dim x_0} R_1(x_0) \circ D_2(x_0)^{-1}}VV @VV{F \otimes 1}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\lambda) \end{CD} \qquad \begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \\ @V{q^{1-\dim x_0} R_2(x_0)}VV @VV{1 \otimes F}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \end{CD}$$

$$\begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu) \\ @V{D_1(x_0)}VV @VV{K \otimes 1}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu) \end{CD} \qquad \begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)} @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu) \\ @V{D_2(x_0)}VV @VV{1 \otimes K}V \\ \mathbb{C}^{\mathcal{L}(\Omega)} @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu) \end{CD}$$

Applying the above commutative diagrams, we can conclude that

Theorem 2.1 (Theorem 11.15, [3])**.** *The following diagram commutes for each $X \in U_q(\mathfrak{sl}_2)$:*

$$\begin{CD} \mathbb{C}^{\mathcal{L}(\Omega)}(q^{\dim x_0}) @>\iota(x_0)>> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \\ @V{X}VV @VV{\Delta(X)}V \\ \mathbb{C}^{\mathcal{L}(\Omega)}(q^{\dim x_0}) @>>\iota(x_0)> \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(1) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(q^{\dim x_0}) \end{CD}$$

Although Theorem 2.1 is a q-analog of the commutative diagram in Section 1, the linear map $\iota(x_0)$ is not an isomorphism in the general case.

The *universal q-Hahn algebra* $\mathcal{H}_q$ is an algebra over $\mathbb{C}$ generated by A, B, C and the relations assert that each of

$$\frac{[B,C]_q}{q^2 - q^{-2}} + A, \qquad [C,A]_q, \qquad \frac{[A,B]_q}{q^2 - q^{-2}} + C$$

is central in $\mathcal{H}_q$. With respect to the first comultiplication Δ of $U_q(\mathfrak{sl}_2)$, the algebraic treatment of the Clebsch–Gordan coefficients of $U_q(\mathfrak{sl}_2)$ was given in [4, Theorem 2.9]. With respect to the second comultiplication Δ of $U_q(\mathfrak{sl}_2)$, the result [4, Theorem 2.9] can be modified as follows:

Theorem 2.2 (Theorem 1.4, [3]). *There exists a unique algebra homomorphism* $\natural : \mathcal{H}_q \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ *that sends*

$$\begin{aligned} A &\mapsto 1 \otimes K^{-1}, \\ B &\mapsto \Delta(\Lambda), \\ C &\mapsto K^{-1} \otimes 1 - q(q-q^{-1})^2 E \otimes FK^{-1}. \end{aligned}$$

Instead of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, I consider an algebra $\mathfrak{W}_q$ which is inspired by the triple coordinate system $(\mathcal{L}(\Omega)_{x_0}, \subseteq)$ and the equations established in [7, Section 7].

Definition 2.3 (Definition 2.1, [3]). The algebra $\mathfrak{W}_q$ is an algebra over $\mathbb{C}$ defined by generators and relations. The generators are $E_1, E_2, F_1, F_2, K_1^{\pm 1}, K_2^{\pm 1}, I^{\pm 1}$. The relations are as follows:

$$\begin{gathered} I \text{ is central in } \mathfrak{W}_q, \\ II^{-1} = I^{-1}I = 1, \\ K_1K_1^{-1} = K_1^{-1}K_1 = 1, \\ K_2K_2^{-1} = K_2^{-1}K_2 = 1, \\ [K_1, E_2] = [K_1, F_2] = [K_1, K_2] = [K_2, E_1] = [K_2, F_1] = 0, \\ [E_1, K_1]_q = [K_1, F_1]_q = [E_2, K_2]_q = [K_2, F_2]_q = 0, \\ [E_1, E_2] = [E_1, F_2] = [F_1, E_2] = [F_1, F_2] = 0, \\ [E_1, F_1] = \frac{K_1 - IK_1^{-1}}{q - q^{-1}}, \\ [E_2, F_2] = \frac{IK_2 - K_2^{-1}}{q - q^{-1}}. \end{gathered}$$

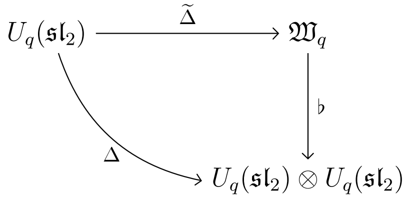
By [3, Theorem 2.2] there exists a unique algebra surjective homomorphism $\flat : \mathfrak{W}_q \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ that sends

$$\begin{aligned} E_1 &\mapsto E \otimes 1, & E_2 &\mapsto 1 \otimes E, \\ F_1 &\mapsto F \otimes 1, & F_2 &\mapsto 1 \otimes F, \\ K_1^{\pm 1} &\mapsto K^{\pm 1} \otimes 1, & K_2^{\pm 1} &\mapsto 1 \otimes K^{\pm 1}, \\ I^{\pm 1} &\mapsto 1 \otimes 1. \end{aligned}$$

Therefore $\mathfrak{W}_q$ is an *algebraic covering* of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$. It can be shown that $\flat$ is not an isomorphism [3, Proposition 2.4]. By [3, Theorem 3.1] there exists a unique algebra homomorphism $\widetilde{\Delta} : U_q(\mathfrak{sl}_2) \to \mathfrak{W}_q$ that sends

$$\begin{aligned} E &\mapsto E_1 + K_1^{-1}E_2, \\ F &\mapsto F_1K_2 + F_2, \\ K^{\pm 1} &\mapsto K_1^{\pm 1}K_2^{\pm 1}. \end{aligned}$$

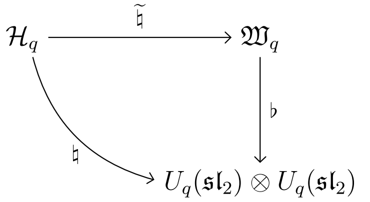
Moreover the following diagram commutes [3, Theorem 3.2]:



Thus $\widetilde{\Delta}$ is a lift of Δ across $\flat$. By [3, Theorem 5.2] there exists a unique algebra homomorphism $\widetilde{\natural} : \mathcal{H}_q \to \mathfrak{W}_q$ that sends

$$\begin{aligned} A &\mapsto K_2^{-1}, \\ B &\mapsto \widetilde{\Delta}(\Lambda), \\ C &\mapsto IK_1^{-1} - q(q-q^{-1})^2 E_1 F_2 K_2^{-1}. \end{aligned}$$

Moreover the following diagram commutes [3, Theorem 5.3]



Thus $\widetilde{\natural}$ is a lift of $\natural$ across $\flat$.

Let $D_3(x_0)$ and $D_4(x_0)$ denote the linear maps $\mathbb{C}^{\mathcal{L}(\Omega)} \to \mathbb{C}^{\mathcal{L}(\Omega)}$ defined as follows:

$$D_3(x_0) \ : \ x \ \mapsto \sum_{\substack{(x+x_0/x_0,\, x\cap x_0,\, \tau)\in\mathcal{L}(\Omega)_{x_0} \\ \operatorname{rk}\left(\tau_{x_0}^{x_1}(x)-\tau\right)=1}} (x+x_0/x_0, x\cap x_0, \tau) \quad \text{for all } x \in \mathcal{L}(\Omega),$$

$$D_4(x_0) \ : \ x \ \mapsto \frac{|x\cup x_0|}{|x\cap x_0|} x \qquad \text{for all } x \in \mathcal{L}(\Omega).$$

It can be shown that the map $D_3(x_0)$ is independent of the choice of x_1. The map $D_3(x_0)$ is a direct sum of the adjacency operators of some bilinear forms graphs. By [3, Lemmas 12.10–12.13] the following equations hold:

- $[D_3(x_0), L_1(x_0)]_q = q^{-1}(1 - q^{\dim x_0} D_2(x_0)) \circ L_1(x_0)$.
- $[D_4(x_0), L_1(x_0)]_q = -(q - q^{-1})(1 - q^{\dim x_0} D_2(x_0)) \circ L_1(x_0)$.
- $[L_2(x_0), D_3(x_0)]_q = q^{-1}(1 - q^{D-\dim x_0} D_1(x_0)^{-1}) \circ L_2(x_0)$.
- $[L_2(x_0), D_4(x_0)]_q = -(q - q^{-1})(1 - q^{D-\dim x_0} D_1(x_0)^{-1}) \circ L_2(x_0)$.
- $[R_1(x_0), D_3(x_0)]_q = q^{-1}(1 - q^{\dim x_0} D_2(x_0)) \circ R_1(x_0)$.
- $[R_1(x_0), D_4(x_0)]_q = -(q - q^{-1})(1 - q^{\dim x_0} D_2(x_0)) \circ R_1(x_0)$.
- $[D_3(x_0), R_2(x_0)]_q = q^{-1}(1 - q^{D-\dim x_0} D_1(x_0)^{-1}) \circ R_2(x_0)$.
- $[D_4(x_0), R_2(x_0)]_q = -(q - q^{-1})(1 - q^{D-\dim x_0} D_1(x_0)^{-1}) \circ R_2(x_0)$.

Thus the map $(q^2-1)D_3(x_0) + D_4(x_0)$ satisfies the following equations [3, Lemma 12.15]

- $[(q^2-1)D_3(x_0) + D_4(x_0), L_1(x_0)]_q = 0$.
- $[L_2(x_0), (q^2-1)D_3(x_0) + D_4(x_0)]_q = 0$.

- $[R_1(x_0), (q^2-1)D_3(x_0) + D_4(x_0)]_q = 0$.
- $[(q^2-1)D_3(x_0) + D_4(x_0), R_2(x_0)]_q = 0$.

In [7] the linear map $(q^2-1)D_3(x_0)+D_4(x_0)$ was mentioned in another way. It can be shown that $(q^2-1)D_3(x_0)+D_4(x_0)$ is invertible [3, Lemma 12.14]. For any nonzero $\lambda, \mu \in \mathbb{C}$ the following diagram commutes [3, Lemma 12.16]:

$$\begin{array}{ccc}
\mathbb{C}^{\mathcal{L}(\Omega)} & \xrightarrow{\iota(x_0)} & \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu) \\
{\scriptstyle (q^2-1)D_3(x_0)+D_4(x_0)}\Big\downarrow & & \Big\downarrow {\scriptstyle q^D K^{-1} \otimes K} \\
\mathbb{C}^{\mathcal{L}(\Omega)} & \xrightarrow[\iota(x_0)]{} & \mathbb{C}^{\mathcal{L}(\Omega/x_0)}(\lambda) \otimes \mathbb{C}^{\mathcal{L}(x_0)}(\mu)
\end{array}$$

Inspired by the aforementioned diagrams, we discover the following result [3, Theorem 13.19]: There exists a unique $\mathfrak{W}_q$-module $\mathbb{C}^{\mathcal{L}(\Omega)}$ given by

$$\begin{aligned}
E_1 &= q^{\dim x_0 - D} L_1(x_0), \\
E_2 &= q^{\dim x_0 - D} D_1(x_0) \circ L_2(x_0), \\
F_1 &= q^{1-\dim x_0} R_1(x_0) \circ D_2(x_0)^{-1}, \\
F_2 &= q^{1-\dim x_0} R_2(x_0), \\
K_1^{\pm 1} &= D_1(x_0)^{\pm 1}, \\
K_2^{\pm 1} &= D_2(x_0)^{\pm 1}, \\
I^{\pm 1} &= q^{\mp D} D_1(x_0)^{\pm 1} \circ D_2(x_0)^{\mp 1} \circ ((q^2-1)D_3(x_0) + D_4(x_0))^{\pm 1}.
\end{aligned}$$

We denote the above $\mathfrak{W}_q$-module by $\mathbb{C}^{\mathcal{L}(\Omega)}(x_0)$. By pulling back via $\widetilde{\natural}$, the $\mathfrak{W}_q$-module $\mathbb{C}^{\mathcal{L}(\Omega)}(x_0)$ is also an $\mathcal{H}_q$-module. The actions of A and B on the $\mathcal{H}_q$-module $\mathbb{C}^{\mathcal{L}(\Omega)}(x_0)$ are as follows:

$$\begin{aligned}
Ax &= q^{2\dim(x \cap x_0) - \dim x_0} x \qquad \text{for all } x \in \mathcal{L}(\Omega), \\
Bx &= (q^{D-2\dim x+1} + q^{2\dim x - D + 1} + q^{-1-D} - q^{1-D})x \\
&\quad + \ q^{1-D}(q-q^{-1})^2 \sum_{\substack{x' \in \mathcal{L}_{\dim x}(\Omega) \\ x \cap x' \subset x}} x' \qquad \text{for all } x \in \mathcal{L}(\Omega).
\end{aligned}$$

Assume that $x_0 \in \mathcal{L}_k(\Omega)$ where k is an integer with $1 \le k \le D-1$. The subspace $\mathbb{C}^{\mathcal{L}_k(\Omega)}$ of $\mathbb{C}^{\mathcal{L}(\Omega)}(x_0)$ is an $\mathcal{H}_q$-submodule of $\mathbb{C}^{\mathcal{L}(\Omega)}(x_0)$. We denote this $\mathcal{H}_q$-module by $\mathbb{C}^{\mathcal{L}_k(\Omega)}(x_0)$. Let

$$\widetilde{\mathbf{T}}(x_0) = \operatorname{Im}\left(\mathcal{H}_q \to \operatorname{End}(\mathbb{C}^{\mathcal{L}_k(\Omega)})\right).$$

Here $\mathcal{H}_q \to \operatorname{End}(\mathbb{C}^{\mathcal{L}_k(\Omega)})$ denotes the representation corresponding to the $\mathcal{H}_q$-module $\mathbb{C}^{\mathcal{L}_k(\Omega)}(x_0)$. Let $J_q(D,k)$ denote the Grassmann graph of $\mathcal{L}_k(\Omega)$. Let $\mathbf{T}(x_0)$ denote the Terwilliger algebra of $J_q(D,k)$ with respect to x_0. Since $J_q(D,k)$ is a P- and Q-polynomial association scheme the algebra $\mathbf{T}(x_0)$ is the subalgebra of $\operatorname{End}(\mathbb{C}^{\mathcal{L}_k(\Omega)})$ generated by the adjacency operator $\mathbf{A}$

and the dual adjacency operator $\mathbf{A}^*(x_0)$ of $J_q(D,k)$. The following equations hold on the $\mathcal{H}_q$-module $\mathbb{C}^{\mathcal{L}_k(\Omega)}(x_0)$:

$$\mathbf{A} = \frac{q^{D-1}B - q^{2D-2k} - q^{2k}}{(q-q^{-1})^2} + \frac{1}{q^2-1},$$

$$\mathbf{A}^*(x_0) = \frac{[D-1]_q}{q-q^{-1}} \left(\frac{q^D [D]_q}{[k]_q [D-k]_q} A - \frac{q^k}{[D-k]_q} - \frac{q^{D-k}}{[k]_q} \right).$$

Therefore $\mathbf{T}(x_0)$ is a subalgebra of $\widetilde{\mathbf{T}}(x_0)$. Please refer to [3, Section 16] for the detailed study of $\mathbf{T}(x_0)$ from the above perspective.

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Hau-Wen Huang, Department of Mathematics, National Central University, Chung-Li 32001 Taiwan

E-mail address: `hauwenh@math.ncu.edu.tw`