Radio Labeling on Corona Product

Sarbaini

Abstract

This study aims to determine the labeling $L(3,2,1)$ of the corona product graph of a circle graph C_m. The labeling $L(3,2,1)$ was determined by calculating the lower bound and upper bound based on the maximum degree Δ. The results of the labeling $L(3,2,1)$ of $C_m \odot P_3$ are $2\Delta + 4$ for even m, $2\Delta + 5$ for $m = 3, 7$, and $2\Delta + 6$ for odd m where $m \neq 3, 7$.

1 Introduction

Graph theory finds extensive utility in practical contexts. One of them regulates the technological application of FM frequencies, specifically in radio. To transmit a signal, a frequency must be allocated to each transmitter. The initial challenges regarding frequency assignment arose from the observation that transmitters assigned to identical or nearly similar frequencies were susceptible to interference [5]. In order to mitigate interference, a simple solution involved designating distinct transmitters to non-interfering frequencies, or approaching this objective as closely as possible in light of the constraints. Determining how to select a new frequency and reducing the frequency range utilised to reduce costs is an additional concern.

In 2004, Liu and Shao [6] defined an L(3,2,1) labeling. They get results about $\lambda_{3,2,1}$ in several classes of graphs, with determining boundaries for $\lambda_{3,2,1}$ on irregular graphs, Halin Graphs, and Planar Graphs with a maximum degree Δ. Then in 2005, Clipperton et al. [2] studied $\lambda_{3,2,1}$ on classes of graphs like path graphs, cycle graphs, n-ary tree graphs, and regular caterpillar graphs.

In 2011, Chia et al. [1] studied some general concepts regarding the labeling $L(3,2,1)$ and gave an upper bound of $\lambda_{3,2,1}$ for any graph with a maximum degree Δ. Furthermore, they also described the labeling $L(3,2,1)$ on a tree, rooted tree, and Cartesian product of path graphs and cycle graphs.

2 Definition and Notation

The degree (or valency) of a vertex of a graph G is the number of edges that are incident to the vertex. The degree of a vertex v is denoted by $deg(v)$ and $\Delta(G) = max_{v \in G}\, deg(v)$.

Definition 2.1. *The corona product of G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the $i-th$ vertex of G adjacent to every vertex of the $i-th$ copy of H, where $1 \leq i \leq |V(G)|$.*

3 Radio Labeling

Theorem 3.1. *For any cycle C_n, with $n \geq 3$,*

$$\lambda_{3,2,1}(C_n) = \begin{cases} 6, & \text{if } n = 3, \\ 7, & \text{if } n \text{ is even}, \\ 8, & \text{if } n \text{ is odd and } n \neq 3, 7, \\ 9, & \text{if } n = 7. \end{cases}$$

Theorem 3.2. *For any path P_n, with $n \geq 1$,*

$$\lambda_{3,2,1}(P_n) = \begin{cases} 0, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 5, & \text{if } n = 3, 4, \\ 6, & \text{if } n = 5, 6, 7, \\ 7, & \text{if } n \geq 8. \end{cases}$$

Definition 3.3. *[2] Let $G = (V, E)$ be a graph and f be a mapping $f : V \to \mathbb{N}$. Then f is an $L(3, 2, 1)$-labeling of G if, for all $x, y \in V$,*

$$|f(x) - f(y)| \geq \begin{cases} 3, & \text{if } d(x, y) = 1, \\ 2, & \text{if } d(x, y) = 2, \\ 1, & \text{if } d(x, y) = 3. \end{cases}$$

Definition 3.4. *[2] Let $k \in \mathbb{N} \cup \{0\}$. A $k - L(3, 2, 1)$ labeling is an labeling $L(3, 2, 1)$ such that every label used is not greater than k. The $L(3, 2, 1)$ number on G, denoted as $\lambda_{3,2,1}(G)$, is the smallest number k so that G has the labeling k-$L(3, 2, 1)$.*

Lemma 3.5. *[1] If G' is a subgraph of G, then $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$.*

PROOF. Suppose $\lambda_{3,2,1}(G') > \lambda_{3,2,1}(G)$. Let $\lambda_{3,2,1}(G') = k_1$ and $\lambda_{3,2,1}(G) = k_2$, meaning that k_1 is the smallest number so that G has the labeling k_1-$L(3, 2, 1)$. Let f be a k_1-$L(3, 2, 1)$ labeling on a G. $G' \subset G$, $V(G') \subset V(G)$, $E(G') \subset E(G)$ and f labeling k_1-$L(3, 2, 1)$ on G. Thus, for any vertices $u, v \in V(G') \subset V(G)$, $|f(u) - f(v)| \geq 3$ for any vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 1$, $|f(u)-f(v)| \geq 2$ for any vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 2$ and $|f(u)-f(v)| \geq 1$ for any vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 3$. This means that there is a k_1-$L(3, 2, 1)$ on G'. $\lambda_{3,2,1}(G') = k_2$ and $k_2 > k_1$, meaning that there is a number smaller than k_2 (named k_1) so that G' has labeling k_1-$L(3, 2, 1)$. This contradicts the minimality of $\lambda_{3,2,1}(G')$. Therefore $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$.

Corollary 3.6. *[1] For any graph G with $\Delta(G) = \Delta > 0$ we have $\lambda_{3,2,1}(G) \geq 2\Delta + 1$. If $\lambda_{3,2,1}(G) = 2\Delta + 1$ and f is any labeling 2Δ+1-$L(3, 2, 1)$, then for every $v \in V(G)$ where $d(v) = \Delta$ such that $f(v) \in \{0, 2\Delta + 1\}$.*

Corollary 3.7. *[4] Let G be a graph with $\Delta \geq 1$. If there is $v_1, v_2 \in V(G)$ with $d(v_1, v_2) = 2$ and $d(v_1) = d(v_2) = \Delta$, then $\lambda_{3,2,1}(G) \geq 2\Delta + 2$.*

4 Results

Theorem 4.1. *For m be a positive integer with $m \geq 3$, we have*

$$\lambda_{3,2,1}(C_m \odot P_3) = \begin{cases} 2\Delta + 4, & \text{if } m \text{ even}, \\ 2\Delta + 5, & \text{if } m = 3, 7, \\ 2\Delta + 6, & \text{if } m \text{ odd and } m \neq 3, 7. \end{cases}$$

The proof will be explained in the subsections below.

4.1 The Lower Bounds

Let $V(C_m \odot P_3) = \{v_1, v_2, v_3, \ldots, v_m\} \cup \bigcup_{i=1}^{m} \{v_i^1, v_i^2, v_i^3\}$.

1. Case m even
   It will be shown that $\lambda_{3,2,1}(C_m \odot P_3) \leq 2\Delta + 3$ does not happen.
2. Case $m = 3, 7$
   It will be shown that $\lambda_{3,2,1}(C_m \odot P_3) \leq 2\Delta + 4$ does not happen.
3. Case m odd and $m \neq 3, 7$
   It will be shown that $\lambda_{3,2,1}(C_m \odot P_3) \leq 2\Delta + 5$ does not happen.

4.2 The Upper Bounds

4.2.1 m even

Let $V(C_m \odot P_3) = \{v_1, v_2, v_3, \ldots, v_m\} \cup \bigcup_{i=1}^{m} \{v_i^1, v_i^2, v_i^3\}$.

1. Subcase $m = 4p$ for $p \in \mathbb{Z}_{>0}$
   Claim: If $m = 4p$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 4$.

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \bmod 4, \\ 5, & \text{if } i \equiv 2 \bmod 4, \\ 2, & \text{if } i \equiv 3 \bmod 4, \\ 7, & \text{if } i \equiv 0 \bmod 4. \end{cases}$$

For $i \equiv 1 \pmod 4$

$$f(v_i^j) = \begin{cases} 3, & \text{if } j = 1, \\ 3(j+1), & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod 4$

$$f(v_i^j) = \begin{cases} 8, & \text{if } j = 1, \\ 2^{5-j} + j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 3 \pmod 4$

$$f(v_i^j) = \begin{cases} 14, & \text{if } j = 1, \\ 3(j + 1), & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod 4$

$$f(v_i^j) = \begin{cases} 4, & \text{if } j = 1, \\ 2^{5-j} + j + 3, & \text{if } j = 2, 3. \end{cases}$$

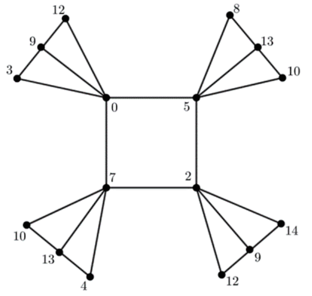

Figure 1: $\lambda_{3,2,1}(C_4 \odot P_3) = 14$

2. Subcase $m = 6p$ for $p \in \mathbb{Z}_{>0}$
   Claim: If $m = 6p$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 4$

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 6, \\ 3, & \text{if } i \equiv 2 \pmod 6, \\ 6, & \text{if } i \equiv 3 \pmod 6, \\ 1, & \text{if } i \equiv 4 \pmod 6, \\ 4, & \text{if } i \equiv 5 \pmod 6, \\ 7, & \text{if } i \equiv 0 \pmod 6. \end{cases}$$

For $i \equiv 1 \pmod 6$

$$f(v_i^j) = \begin{cases} 5, & \text{if } j = 1, \\ 2^{5-j} + j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod 6$ and $i \equiv 4 \pmod 6$

$$f(v_i^j) = \begin{cases} 8, & \text{if } j = 1, \\ 2^{5-j} + j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 3 \pmod 6$ and $i \equiv 5 \pmod 6$

$$f(v_i^j) = \begin{cases} 9, & \text{if } j = 1, \\ 2^{5-j} + j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod 6$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 2^{5-j} + j + 3, & \text{if } j = 2, 3. \end{cases}$$

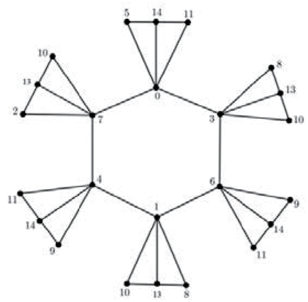

Figure 2: $\lambda_{3,2,1}(C_6 \odot P_3) = 14$

3. Subcase m is other even

   According to **Theorem** 3.1, each labeling $L(3,2,1)$ for C_m even is formed from a combination derived from positive multiplication in the labeling pattern, C_4 and C_6. This case applies also to labeling $L(3,2,1)$ on graphs $C_m \odot P_3$ for m even. Will be shown labeling $L(3,2,1)$ combination results $C_4 \odot P_3$ and $C_6 \odot P_3$ in the Figure 3.

4.2.2 $m = 3, 7$

1. Subcase $m = 3$
   Claim: If $m = 3$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 5$

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3, \\ 5i, & \text{if } i \equiv 2 \pmod 3 \text{ and } i \equiv 0 \pmod 3. \end{cases}$$

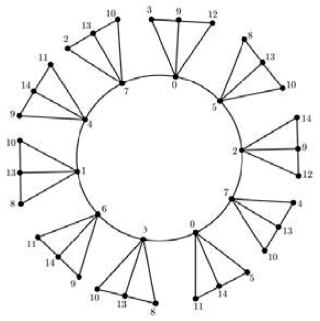

Figure 3: $\lambda_{3,2,1}(C_{10} \odot P_3) = 14$

For $i \equiv 1 \pmod 3$

$$f(v_i^j) = \begin{cases} 3, & \text{if } j = 1, \\ 2^{4-j} + j^{4-j}, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod 3$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 2^{5-j} + 2^{4-j}, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod 3$

$$f(v_i^j) = \begin{cases} 4, & \text{if } j = 1, \\ 2^{5-j} + 2^{4-j}, & \text{if } j = 2, 3. \end{cases}$$

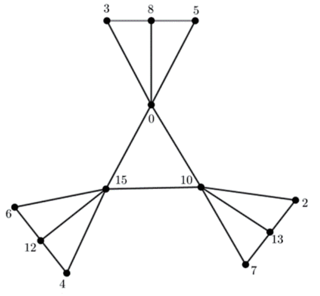

Figure 4: $\lambda_{3,2,1}(C_3 \odot P_3) = 15$

2. Subcase $m = 7$
   Claim: If $m = 7$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 5$.

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 7, \\ 6, & \text{if } i \equiv 2 \pmod 7, \\ 2, & \text{if } i \equiv 3 \pmod 7, \\ 8, & \text{if } i \equiv 4 \pmod 7, \\ 5, & \text{if } i \equiv 5 \pmod 7, \\ 12, & \text{if } i \equiv 6 \pmod 7, \\ 15, & \text{if } i \equiv 0 \pmod 7. \end{cases}$$

For $i \equiv 1 \pmod 7$

$$f(v_i^j) = \begin{cases} 13, & \text{if } j = 1, \\ 2^{5-j} - 1, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod 7$

$$f(v_i^j) = \begin{cases} 9, & \text{if } j = 1, \\ 2^{5-j} + j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 3 \pmod 7$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 2^{5-j} + 2^{4-j} + j + 1, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 4 \pmod 7$

$$f(v_i^j) = \begin{cases} 11, & \text{if } j = 1, \\ -8^{3-j} + 4j, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 5 \pmod 7$

$$f(v_i^j) = \begin{cases} 10, & \text{if } j = 1, \\ -8^{3-j} + 4j + 1, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 6 \pmod 7$

$$f(v_i^j) = \begin{cases} 9, & \text{if } j = 1, \\ 2^j - 1, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod 7$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 2^{5-j} + 2^{4-j} - 2, & \text{if } j = 2, 3. \end{cases}$$

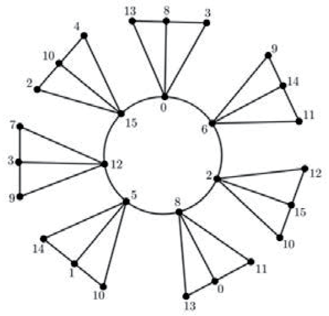

Figure 5: $\lambda_{3,2,1}(C_7 \odot P_3) = 15$

4.2.3 m odd and $m \neq 3, 7$

1. Subcase $m = 5p$ for $\forall p \in \mathbb{Z}_{>0}$
   Claim: If $m = 5p$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 6$

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 5, \\ 5, & \text{if } i \equiv 2 \pmod 5, \\ 13, & \text{if } i \equiv 3 \pmod 5, \\ 10, & \text{if } i \equiv 4 \pmod 5, \\ 7, & \text{if } i \equiv 0 \pmod 5. \end{cases}$$

For $i \equiv 1 \pmod 5$

$$f(v_i^j) = \begin{cases} 3, & \text{if } j = 1, \\ 3j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod 5$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 4j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 3 \pmod 5$

$$f(v_i^j) = \begin{cases} 3, & \text{if } j = 1, \\ 2^{j+1}, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 4 \pmod 5$

$$f(v_i^j) = \begin{cases} 15, & \text{if } j = 1, \\ 3j - 5, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod 5$

$$f(v_i^j) = \begin{cases} 16, & \text{if } j = 1, \\ 2^{j+1} - 2^{5-j} + 2, & \text{if } j = 2, 3. \end{cases}$$

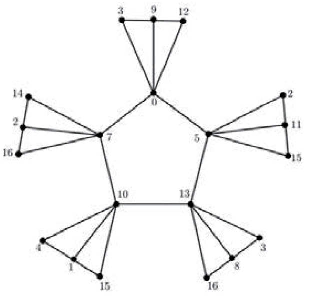

Figure 6: $\lambda_{3,2,1}(C_5 \odot P_3) = 16$

2. Subcase $m = 11$
Claim: If $m = 11$, then $\lambda_{3,2,1}(C_m \odot P_3) = 2\Delta + 6$.

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod{11}, \\ & \text{if } i \equiv 7 \pmod{11}, \\ 3, & \text{if } i \equiv 2 \pmod{11}, \\ 6, & \text{if } i \equiv 8 \pmod{11}, \\ 1, & \text{if } i \equiv 3 \pmod{11}, \\ 4, & \text{if } i \equiv 9 \pmod{11}, \\ 7, & \text{if } i \equiv 4 \pmod{11}, \\ 5, & \text{if } i \equiv 2 \pmod{11}, \\ 8, & \text{if } i \equiv 5 \pmod{11}, \\ 2, & \text{if } i \equiv 6 \pmod{11}, \\ 14, & \text{if } i \equiv 0 \pmod{11}. \end{cases}$$

For $i \equiv 1 \pmod{11}$

$$f(v_i^j) = \begin{cases} 11, & \text{if } j = 1, \\ 2^j + 1, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 2 \pmod{11}$ and $i \equiv 4 \pmod{11}$

$$f(v_{2i}^j) = \begin{cases} 8, & \text{if } j = 1, \\ 2^{5-j} + j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 3 \pmod{11}$ and $i \equiv 5 \pmod{11}$

$$f(v_{2i-1}^{j}) = \begin{cases} 9, & \text{if } j = 1, \\ 2^{5-j} + j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 6 \pmod{11}$

$$f(v_i^j) = \begin{cases} 2, & \text{if } j = 1, \\ 3j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 7 \pmod{11}$

$$f(v_i^j) = \begin{cases} 11, & \text{if } j = 1, \\ 2^{5-j} + 2^{j-1} + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 8 \pmod{11}$

$$f(v_i^j) = \begin{cases} 10, & \text{if } j = 1, \\ 2^{5-j} + 2^{4-j} + 2j, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 9 \pmod{11}$

$$f(v_i^j) = \begin{cases} 14, & \text{if } j = 1, \\ 2^{5-j} + 2j + 3, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 10 \pmod{11}$

$$f(v_i^j) = \begin{cases} 6, & \text{if } j = 1, \\ 4j + 4, & \text{if } j = 2, 3. \end{cases}$$

For $i \equiv 0 \pmod{11}$

$$f(v_i^j) = \begin{cases} 13, & \text{if } j = 1, \\ 3j + 1, & \text{if } j = 2, 3. \end{cases}$$

3. Subcase m is other odd

   According to **Theorem** 3.1, each labeling $L(3,2,1)$ for C_m even is formed from a combination derived from positive multiplication in the labeling pattern, C_5 and C_4. This case applies also to labeling $L(3,2,1)$ on graphs $C_m \odot P_3$ for m odd, $m > 7$ and $m \neq 11$. Will be shown labeling $L(3,2,1)$ combination results $C_4 \odot P_3$ and $C_5 \odot P_3$ in the Figure 8.

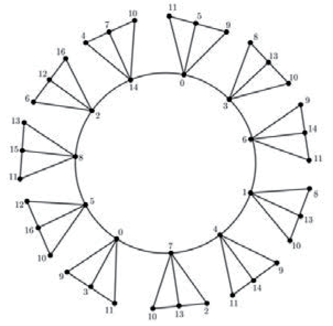

Figure 7: $\lambda_{3,2,1}(C_{11} \odot P_3) = 16$

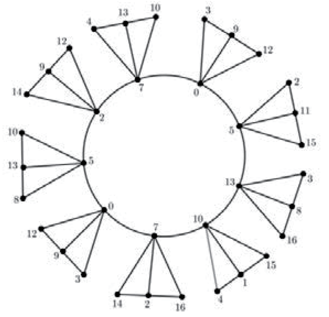

Figure 8: $\lambda_{3,2,1}(C_9 \odot P_3) = 14$

REFERENCES

[1] Chia. Ma-Lian, Kuo. David, Liao. Hong-ya, Yang. Cian-Hui, and Yeh. Roger K: $L(3, 2, 1)$ -labeling of graphs. Taiwanese J. Math.**15**(2011), no.6, 2439–2457.

[2] Clipperton, J., Gehrtz, J., Szaniszlo, Z., and Torkornoo, D: L(3,2,1)-Labeling of Simple Graphs, *VERUM*(2005), Valparaiso University.

[3] Gallian, Joseph A : **A dynamic survey of graph labeling**. Electron. J. Combin.**5**(1998), Dynamic Survey 6, 43 pp.

[4] Ganesha L.P.: *Pelabelan L(3,2,1) pada Beberapa Kelas Graf*, Tesis Program Magister (2018), Institut Teknologi Bandung.

[5] Hale, W.K : Frequency Assignment, Theory and Applications. *Proc. IEEE* (1980), **68**, 1497-1414.

[6] Shao. Zhen-dong and Liu. Jia-zhuang: The $L(3, 2.1)$ **Labeling Problem on Graphs**. Math. Appl. (Wuhan)**17**(2004), no.4, 596–602.