SYMMETRIC PERFECT 2-COLORINGS ON $J(10,3)$

PAUL TRICOT

Graduate School of Information Sciences, Tohoku University, Japan

1. Introduction

A perfect 2-coloring of a Johnson graph can be associated to one of the non-principal eigenvalue of the graph $\theta_1 > \cdots > \theta_k$. The perfect 2-colorings of the Johnson graphs $J(n,3)$ associated with θ_1 have been characterized by Meyerowitz [7], and those associated with θ_3 by Martin [6]. Those associated with θ_2 have been studied by several mathematicians. Evans, Gavrilyuk, Goryainov and Vorob'ev [1, 3] classified them for n odd and for $n > 10$. Avgustinovich and Mogilnykh also studied these perfect 2-colorings [2, 5], in particular for $n = 6$, 7 and 8.

In [1], Gavrilyuk and Goryainov proved that a perfect 2-coloring of $J(n,3)$ associated with θ_2 and symmetric quotient matrix is possible only when $n \in \{6, 10\}$. In this paper, we survey the known constructions in the case $n = 6$, we give a new construction for the two known perfect 2-colorings in the case $n = 10$, and prove that these are the only possible ones.

2. Preliminaries

The Johnson graph $J(n,k)$ with parameters $n, k \in \mathbb{N}$ has as vertices the subsets of $[n] := \{1, \ldots, n\}$ of size k. The vertices x and y are connected when $|x \cap y| = k - 1$. The distance between the vertices x, y is given by $d(x,y) = k - |x \cap y|$. The graph $J(n,k)$ is regular, with valency $k(n-k)$.

A perfect m-coloring of a regular graph Γ is a partition $P_1, \ldots, P_m$ of the vertices such that there exist fixed numbers $p_{i,j}$ ($i, j \in [m]$) that verify

$$\forall x \in P_i, |\Gamma(x) \cap P_j| = p_{i,j},$$

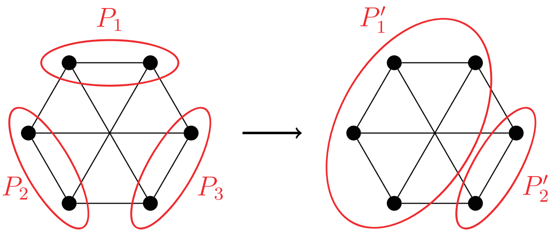
where $\Gamma(x)$ is the neighborhood of x in Γ, which is the set of all its neighbors. The matrix $P = [p_{i,j}]_{1 \le i,j \le m}$ is called the quotient matrix of the coloring. This means that for every vertex $x \in P_i$, x has exactly $p_{i,j}$ neighbors in P_j. We will say that the perfect coloring is symmetric when the quotient matrix is symmetric.

The eigenvalues of $J(n,k)$ are $\theta_i := (k-i)(n-k-i) - i$ for $i \in \{0, \ldots, k\}$.

Lemma 1 (C. Godsil, 1993)**.** *If P is the quotient matrix of a perfect m-coloring of $J(n,k)$, then the eigenvalues of P are among those of $J(n,k)$ including θ_0.*

We say that a perfect 2-coloring is associated with θ_s when the eigenvalues of its quotient matrix are θ_0 and θ_s. We will consider perfect 2-colorings of Johnson graphs associated with θ_2.

A perfect m-coloring can sometimes be merged into a perfect coloring with less parts.



The quotient matrix becomes: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

In general, we have the following:

Lemma 2. *If $P_1, \dots, P_m$ is a perfect m-coloring with quotient matrix $[p_{i,j}]$, and $C_1, \dots, C_l$ is a partition of $[m]$, then*

$$\bigcup_{i \in C_1} P_i, \dots, \bigcup_{i \in C_l} P_i$$

is a perfect l-coloring if and only if each of the submatrices $[p_{x,y}]_{x \in C_i, y \in C_j}$ for $i \in [l]$, $j \in [l-1]$ has constant row sum.

The entry i, j of the quotient matrix of the perfect l-coloring is the row sum of $[p_{x,y}]_{x \in C_i, y \in C_j}$.

3. Symmetric Perfect 2-colorings on $J(6,3)$

In [1], Gavrilyuk and Goryainov proved that a symmetric perfect 2-coloring with eigenvalues θ_0, θ_2 on $J(n,3)$ is possible only when $n \in \{6, 10\}$, and in this case the quotient matrix can only be $\begin{bmatrix} 2n-8 & n-1 \\ n-1 & 2n-8 \end{bmatrix}$. In [5], Avgustinovich and Mogilnykh showed the following construction for the case $n = 6$.

The graph $J(6,3)$ is antipodal of diameter 3, which means that for any vertex v there is a unique vertex at distance 3 from v. Two such vertices are called antipodal vertices. $J(6,3)$ can be partitioned into 10 pairs of antipodal vertices, which forms a perfect 10-coloring with quotient matrix $J - I$ (where J is the all 1 matrix and I the identity matrix). This perfect 10-coloring can be merged into a perfect 2-coloring with quotient matrix $\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$ by taking any two groups of five pairs each.

$J(6,3)$ is small enough that a computer search can be used to list all possible symmetric perfect 2-colorings. It turns out that the only possible ones are those mentioned above, and that they are all isomorphic to one of the two perfect 2-colorings $\{X_1, X_2\}$ and $\{X'_1, X'_2\}$ whose induced subgraphs are represented below.

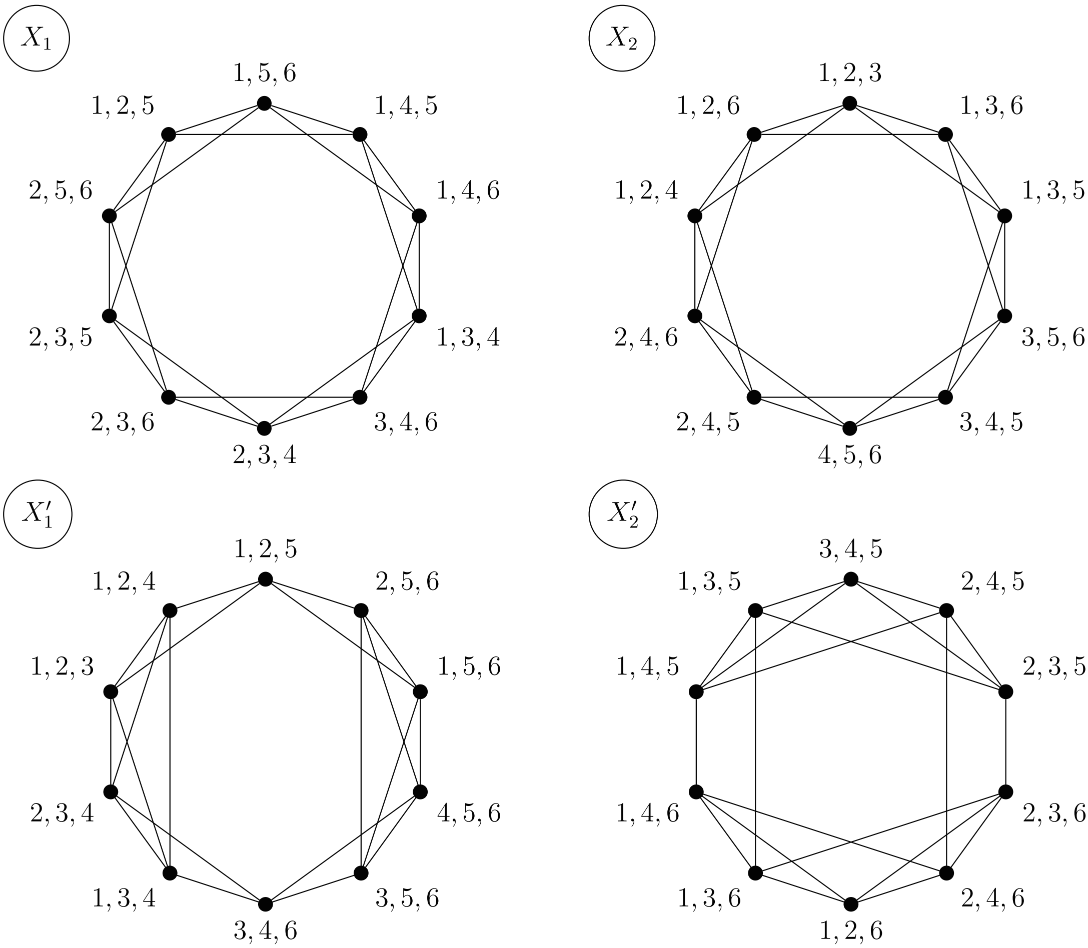


FIGURE 1. Induced subgraphs of symmetric perfect 2-colorings of $J(6,3)$

4. SYMMETRIC PERFECT 2-COLORINGS ON $J(10,3)$

It is mentioned in [1] that for $J(10,3)$ there are only two non-isomorphic perfect 2-colorings with the symmetric quotient matrix $\begin{bmatrix} 12 & 9 \\ 9 & 12 \end{bmatrix}$. But since a formal proof has never been written, we will attempt to do it here by extending the method used in [3] by R.J. Evans, A.L. Gavrilyuk, S. Goryainov and K. Vorob'ev.

There are two known non-isomorphic constructions of perfect 2-colorings of $J(10,3)$ with the above mentioned symmetric quotient matrix. One of the construction was found by Gavrilyuk and Goryainov (but to our knowledge does not appear in any publication), by using the orbit construction method from two 5-cycles. The second construction can be found in [2, Construction 3], using the same method from a complete bipartite graph with parts of size 5, from which we remove a perfect matching.

What follows is a different construction for those two 2-colorings.

Let $\mathcal{G}$ be the cycle graph on 10 vertices, and consider the action of $\mathrm{Aut}(\mathcal{G})$ on $J(10,3)$. The group $\mathrm{Aut}(\mathcal{G})$ is known as the dihedral group of order 20 consisting of 10 rotations (powers of the cycle permutation $(1,2,\dots,10)$) and 10 reflections. For instance it is generated by the two permutations $(1,2,\dots,10)$ and $(2,10)(3,9)(4,8)(5,7)$. Thus the eight orbits of $\mathrm{Aut}(G)$ acting on $J(10,3)$ are:

- $A := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=1, d(b,c)=1, d(a,c)=2\}$,
- $B := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=1, d(b,c)=2, d(a,c)=3\}$,
- $C := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=1, d(b,c)=3, d(a,c)=4\}$,
- $D := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=1, d(b,c)=4, d(a,c)=5\}$,
- $E := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=2, d(b,c)=2, d(a,c)=4\}$,
- $F := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=2, d(b,c)=3, d(a,c)=5\}$,
- $G := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=2, d(b,c)=4, d(a,c)=4\}$,
- $H := \{\{a,b,c\} \in J(10,3) \mid d(a,b)=3, d(b,c)=3, d(a,c)=4\}$.

These orbits corresponds to the 3 "types" of triple of points in the 10-cycle:

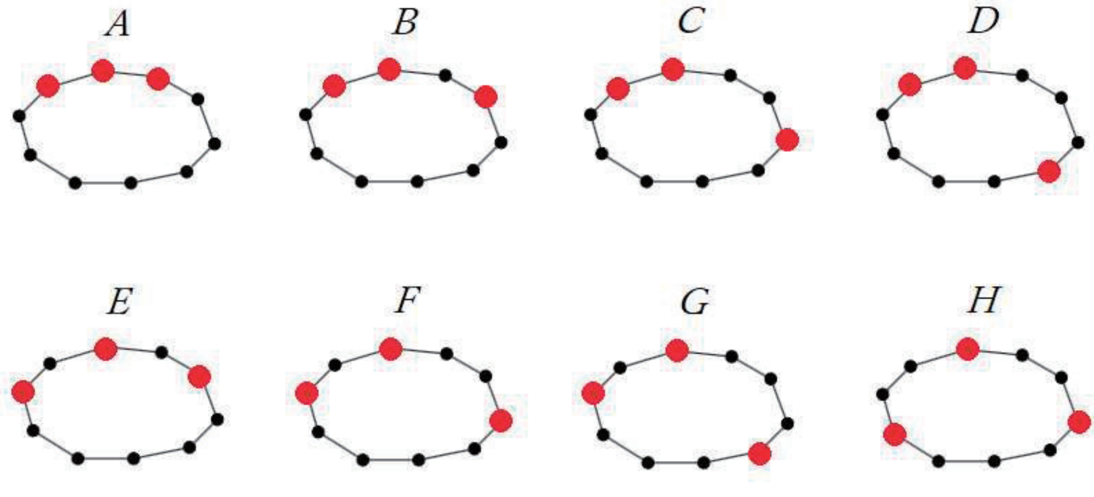


FIGURE 2. Visualisation of the orbits of $\mathrm{Aut}(\mathcal{G})$ acting on $J(10,3)$

This is a perfect 8-coloring of $J(10,3)$ with quotient matrix

$$\begin{bmatrix} 2 & 6 & 4 & 4 & 2 & 2 & 1 & 0 \\ 3 & 3 & 4 & 2 & 2 & 4 & 1 & 2 \\ 2 & 4 & 3 & 4 & 1 & 2 & 2 & 3 \\ 2 & 2 & 4 & 5 & 1 & 4 & 2 & 1 \\ 2 & 4 & 2 & 2 & 2 & 4 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 4 & 2 & 2 & 2 \\ 0 & 4 & 6 & 2 & 1 & 4 & 2 & 2 \end{bmatrix}$$

that can be merged in two ways into perfect 2-colorings. First by $P_1 := A \cup B \cup C \cup H$ and $P_2 := D \cup E \cup F \cup G$. In the quotient matrices below, we can see that the highlighted submatrices have constant row sums, so we can use Lemma 2.

$$\begin{array}{cc} & \begin{array}{cccccccc} A & B & C & D & E & F & G & H \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} A \to \\ B \to \\ C \to \\ D \to \\ E \to \\ F \to \\ G \to \\ H \to \end{array} & \begin{bmatrix} 2 & 6 & 4 & 4 & 2 & 2 & 1 & 0 \\ 3 & 3 & 4 & 2 & 2 & 4 & 1 & 2 \\ 2 & 4 & 3 & 4 & 1 & 2 & 2 & 3 \\ 2 & 2 & 4 & 5 & 1 & 4 & 2 & 1 \\ 2 & 4 & 2 & 2 & 2 & 4 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 4 & 2 & 2 & 2 \\ 0 & 4 & 6 & 2 & 1 & 4 & 2 & 2 \end{bmatrix} \end{array}$$

Secondly, the merging can be done by $P_1' := C \cup D \cup G \cup H$ and $P_2' := A \cup B \cup E \cup F$.

$$\begin{array}{cc} & \begin{array}{cccccccc} A & B & C & D & E & F & G & H \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} A \to \\ B \to \\ C \to \\ D \to \\ E \to \\ F \to \\ G \to \\ H \to \end{array} & \begin{bmatrix} 2 & 6 & 4 & 4 & 2 & 2 & 1 & 0 \\ 3 & 3 & 4 & 2 & 2 & 4 & 1 & 2 \\ 2 & 4 & 3 & 4 & 1 & 2 & 2 & 3 \\ 2 & 2 & 4 & 5 & 1 & 4 & 2 & 1 \\ 2 & 4 & 2 & 2 & 2 & 4 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 4 & 2 & 2 & 2 \\ 0 & 4 & 6 & 2 & 1 & 4 & 2 & 2 \end{bmatrix} \end{array}$$

The two resulting perfect 2-colorings are not isomorphic since it can be computed that the subgraph of $J(n,3)$ induced by P_1 have different eigenvalues than the one induced by P_1' or P_2'.

5. Classification

We now want to prove that $\{P_1, P_2\}$ and $\{P_1', P_2'\}$ are the only perfect 2-colorings (up to isomorphism). We will use the notations and tools of [1] and [3]. In the rest of this section, we consider a perfect coloring of $J(10,3)$ into two parts X_1, X_2 such that the quotient matrix is $\begin{bmatrix} 12 & 9 \\ 9 & 12 \end{bmatrix}$. A simple counting argument shows that $|X_1| = |X_2| = 60$.

We will denote $abc := \{a,b,c\}$, $ab* := \{a,b,x \mid x \in [10] \setminus \{a,b\}\}$, and denote the intersection with X_1 in this way: $\overline{abc} = 1$ if $abc \in X_1$ and 0 otherwise, and for $S \subseteq J(10,3)$, $\overline{S} := |S \cap X_1|$.

The method consists in looking at how the neighborhood of a point can be distributed between the two parts. The subgraph of $J(10,3)$ induced by the neighborhood of a point abc is isomorphic to a 3 by 7 grid, and we can represent its distribution among X_1 and X_2 by the nb-array [3]:

$$\begin{array}{rccccccccl} & \overline{abd} & \overline{abe} & \overline{abf} & \dots & \dots & \dots & \dots & \leftarrow ab\text{-row} \\ N(abc): & \overline{acd} & \overline{ace} & \overline{acf} & \dots & \dots & \dots & \dots & \leftarrow ac\text{-row} \\ & \overline{bcd} & \overline{bce} & \overline{bcf} & \dots & \dots & \dots & \dots & \leftarrow bc\text{-row} \\ & \uparrow & \uparrow & \uparrow & & & & & \\ & d & e & f & \dots & & & & \end{array}$$

The order of the rows and columns is arbitrary.

Most of the proof will rely on the following lemma from [3].

Lemma 3 ([3])**.** *For any five distinct elements $a,b,c,d,e \in [10]$, we have*

$$\overline{ab*} - \overline{ac*} = 3(\overline{abd} + \overline{abe} + \overline{cde} - \overline{acd} - \overline{ace} - \overline{bde}).$$

In particular, since $\overline{ab*} - \overline{abc}$ is the row sum of the row ab in $N(abc)$, we can see that the difference between two row sums in $N(abc)$ is always a multiple of 3. Moreover each row consists of seven 0 and 1, so each row sum is between 0 and 7, and the difference between two row sums is 0, 3 or 6. And since when $\overline{abc} = 1$ the total sum of the nb-array must be 12, we have the following.

Lemma 4. *For $abc \in X_1$, the multiset of row sums of $N(abc)$ is among*

$$\{3,3,6\}, \{4,4,4\}, \{2,5,5\}, \{0,6,6\}, \{1,4,7\}.$$

We also have as a direct consequence of Lemma 3 :

Lemma 5. *For any $a,b,c,d \in [10]$ with $a \neq b$ and $c \neq d$, $\overline{ab*} \equiv \overline{cd*} \pmod 3$.*

Because of this, we can define the type of a part X_1 of a partition to be $k \in \{0,1,2\}$ if for any distinct $a,b \in [10]$, $\overline{ab*} \equiv k \pmod 3$. When $\overline{abc} = 1$, a row sum in $N(abc)$ must be $k-1 \pmod 3$. So we can further restrict the possibilities in Lemma 4.

Lemma 6.
1. *If X_1 is of type 0, for $abc \in X_1$ the row sums of $N(abc)$ is $\{2,5,5\}$.*
2. *If X_1 is of type 1, for $abc \in X_1$ the row sums of $N(abc)$ is $\{3,3,6\}$ or $\{0,6,6\}$.*
3. *If X_1 is of type 2, for $abc \in X_1$ the row sums of $N(abc)$ is $\{4,4,4\}$ or $\{1,4,7\}$.*

We can also use Lemma 3 to eliminate some 2 by 2 patterns that can not appear in $N(abc)$. For a row r of $N(abc)$, denote by $\overline{r}$ the sum of the row r. If we fix the order of rows and columns then nb-arrays can be seen as matrices, so we will use matrices in the next lemma for convenience.

The next lemma is an extension of [3, Lemma 5.4].

Lemma 7. *Let $abc \in J(10,3)$ and r_1, r_2 be two rows of $N(abc)$ such that $\overline{r_1} \geq \overline{r_2}$. Then*

1. $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ *is not a submatrix of* $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$,
2. *If* $\overline{r_1} - \overline{r_2} \geq 3$ *then* $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ *are not submatrices of* $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$,
3. *If* $\overline{r_1} - \overline{r_2} = 6$ *then* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ *are not submatrices of* $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$.

From this lemma, the possible forms of $N(abc)$ can be restricted. For convenience, in the lemmas below let $abc \in X_1$ and $N := N(abc)$.

Lemma 8. 1. *If the row sums of* N *are* $\{2,5,5\}$, *then* N *is*

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{1}$$

$$\text{or} \quad \begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{2}$$

2. *If the row sums of* N *are* $\{3,3,6\}$, *then* N *is*

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{matrix} \tag{3}$$

$$\text{or} \quad \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix} \tag{4}$$

3. *If the row sums of* N *are* $\{0,6,6\}$, *then* N *is*

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{5}$$

$$\text{or} \quad \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{6}$$

4. *If the row sums of N are $\{4,4,4\}$, then N is*

$$\begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix} \tag{7}$$

$$or \quad \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{matrix} \tag{8}$$

$$or \quad \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{matrix} \tag{9}$$

$$or \quad \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{matrix} \tag{10}$$

5. *If the row sums of N are $\{1,4,7\}$, then N is*

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{11}$$

Using Lemma 3, the possible nb-arrays can further be reduced to the following.

Proposition 9. *We have the following.*

1. *If X_1 is of type 0, then for any $abc \in X_1$, $N(abc)$ has the form* (2).
2. *If X_1 is of type 1, then for any $abc \in X_1$, $N(abc)$ has the form* (3) *or* (6).
3. *If X_1 is of type 2, then for any $abc \in X_1$, $N(abc)$ has the form* (10) *or* (11).

These cases occur in the construction depicted at the beginning of the section.

For the first perfect 2-coloring $\{P_1, P_2\}$, P_1 is of type 0 and P_2 is of type 2. When taking $X_1 = P_1$, all of its vertices have an nb-array of the form (2). When taking $X_1 = P_2$, vertices from E and G have an nb-array of the form (10), while vertices from D and F have an nb-array of the form (11).

For the second perfect 2-coloring $\{P_1', P_2'\}$, P_1' and P_2' are both of type 1. When taking $X_1 = P_1'$, vertices from C, D and H have an nb-array of the form (3) and those from G have the form (6). When taking $X_1 = P_2'$, vertices from A, B and F have an nb-array of the form (3) and those from E have the form (6).

Theorem 10. *There is only one perfect* 2*-coloring with a part of type 0, which is also the only perfect* 2*-coloring with a part of type 2, up to isomorphism.*

Proof. Denote t_1 the type of X_1 and t_2 the type of X_2. Since $|ab*\cap X_1|+|ab*\cap X_2| = |ab*| = 8$, we have $t_1+t_2 \equiv 2 \pmod 3$. So if X_1 is of type 0 then X_2 is of type 2, and vice versa. Therefore, it is enough to show that the perfect 2-coloring with a part of type 0 is unique.

Suppose that X_1 is of type 0, and fix $abc \in X_1$. Then from Proposition 9

$$N(abc): \begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \leftarrow ab\text{-row} \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & \leftarrow ac\text{-row} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \leftarrow bc\text{-row} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ d & e & f & g & h & i & j & \end{array}$$

Since the rest of the proof relies on many applications of Lemma 3 and is quite fastidious, we will leave the verification to a computer.

We consider the 120 values $\overline{xyz}$ ($xyz \in J(10,3)$) as variables for multivariate polynomials. The variables having $\{0,1\}$ values translates to $\overline{xyz}^2 - \overline{xyz} = 0$. Lemma 3 and the values fixed in $N(abc)$ also give some multivariate polynomials that must have value 0. The ideal generated by these polynomials can be computed by magma.

We can then check if $\overline{xyz} = \epsilon$ ($\epsilon \in \{0,1\}$) can be deduced by checking if $\overline{xyz} - \epsilon$ belongs to the ideal. Moreover, we can check if $\overline{xyz} = \overline{x'y'z'}$ or $\overline{xyz} \neq \overline{x'y'z'}$ ($xyz, x'y'z' \in J(10,3)$) can be deduced by checking if $\overline{xyz} - \overline{x'y'z'}$ or $\overline{xyz} + \overline{x'y'z'} - 1$ belongs to the ideal.

In this way, 48 of the $\overline{xyz}$ values are deduced to be 1, and 48 of them are 0. The remaining ones are separated into two groups U_1 and U_2 of size 12, with identical value within a group. Since $|X_1| = |X_2| = 60$, there are two possibilities for the coloring (X_1, X_2), either the values in the group U_1 are 0 and those in the group U_2 are 1, or the opposite. But it is computed in the magma code that the transposition (f,g) is an isomorphism between those two possible colorings.

□

Theorem 11. *There is only one perfect* 2*-coloring with a part of type 1 up to isomorphism.*

Proof. Suppose that X_1 is of type 1 and fix $abc \in X_1$. Then from Proposition 9, $N(abc)$ has the form (3) or (6). If $N(abc)$ has the form (6), then the second part of the magma code shows that there exists an element of X_1 which has an nb-array of the form (3). The case (3) is very similar to the proof of Proposition 9 so we will give an almost identical magma code.

□

6. Conclusion

By a proof similar to Theorem 10, we can see that a perfect 2-coloring with a part of type 1 must have the other part also of type 1. Thus, these last two theorems show that there are only two symmetric perfect 2-colorings of $J(10,3)$ associated to θ_2. One of them has parts of type 0 and 2, and the other one has both parts of type 1. Theorem 11 also implies that the two parts of the perfect 2-coloring with parts of type 1 are isomorphic to each other.

7. Acknowledgement

I am grateful to Prof. Akihiro Munemasa for his continuous guidance through this study, and his idea to consider an ideal membership problem for the resolutions by computer.

I am also thankful to Prof. Alexander Gavrilyuk and Prof. Sergey Goryainov for their insight on the known constructions of symmetric perfect 2-colorings of $J(10, 3)$, and for indicating me that the problem resolved in this paper was still open.

This work was supported by JST SPRING (Grant Number JPMJSP2114).

References

[1] Alexander L. Gavrilyuk, Sergey V. Goryainov. On perfect 2-colorings of Johnson graphs $J(v, 3)$. 2012. J. Combin. Designs 21: 232–252, 2013.

[2] S. V. Avgustinovich, I. Yu. Mogil'nykh. Perfect colorings of the Johnson graphs $J(8, 3)$ and $J(8, 4)$ with two Colors. J. Appl. Ind. Math. 5, 19–30 (2011).

[3] R.J. Evans, A.L. Gavrilyuk, S. Goryainov, K. Vorob'ev. Equitable 2-partitions of the Johnson graphs $J(v, 3)$. 2022. arXiv:2206.15341v2 [math.CO].

[4] Konstantin Vorob'ev. Equitable 2-partitions of Johnson graphs with the second eigenvalue. 2020. arXiv:2003.10956v1 [math.CO].

[5] Avgustinovich, S., Mogilnykh, I. (2008). Perfect 2-Colorings of Johnson Graphs $J(6, 3)$ and $J(7, 3)$. In: Barbero, Á. (eds) Coding Theory and Applications. Lecture Notes in Computer Science, vol 5228. Springer, Berlin, Heidelberg.

[6] William J. Martin. Completely regular designs of strength one. Journal of Algebraic Combinatorics, 3(2):170–185, 1994.

[7] Aaron D. Meyerowitz. Cycle-balanced partitions in distance-regular graphs. Discrete Mathematics, 264:149–165, 2003.

[8] S. Morteza Mirafzal. A note on the automorphism groups of Johnson graphs. 2017. arXiv:1702.02568v4 [math.CO].