Delay Two-Sector Growth Model with Constant Technological Coefficients*

Akio Matsumoto† Ferenc Szidarovszky‡

Abstract

A Leontief two-sector growth model is examined with production and depreciation delays, where the growth rate of labor is exogenously given. After the equilibrium states are characterized, the two-delay dynamic system for labor is formulated. The dynamic analysis is given in three cases: the no delay, one delay and two delays are examined in detail. The presence of a positive eigenvalue makes the systems unstable in the classical sence. The dynamic system is called *stable* if the growth rate of labor as time variable converges to the exogenously given growth rate. Conditions are given for the stability of the systems in this sense. The analysis is based on complete characterizations of the locations of all eigenvalues. The theoretical results are demonstrated with numerical studies.

Keywords: Two-sector growth model, Leontief production technology, Production delay, Depreciation delay, Blanced growth

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*The first author highly acknowledges the financial supports from the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 20K01566). The usual disclaimers apply.

†Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan, akiom@tamacc.chuo-u.ac.jp.

‡Department of Mathematics, Corvinus University, Budapest, Fövám tér 8, 1093, Hungary, szidarka@gmail.com.

1 Introduction

In the classical literature of economic growth, Keynes's short-run analysis is extended into the long-run period by Harrod (1939) and Domar (1946). It is suggested that the key factors for long-run economic growth are the saving rate and the productivity of capital investment. Their model, called the Harrod-Domar model, could give rise to, at best, a knife-edge equilibrium growth only when the warranted growth happens to coincide with the natural rate of growth. Any small derivation from the path is amplified and results in a further deviation, otherwise. This knife-edge instability is due to the Leontief production function in which production has fixed proportions of capital and labor. It is well-known that Solow (1956) and Swan (1956) overcome the instability of Harrod-Domar model's solution by replacing the Leontief production functions with the neoclassical production functions and guarantee the existence and stability of the golden age path. The Solow-Swan model or the neoclassical growth model becomes a prototype model to analyze economic growth and is applied to study multisector growth, nonlinear population growth, cross-country income differences, to name only a few.

However, in real economies business cycles are often observed and very limited research was conducted to explain such phenomenon since in the basic model having the neoclassical properties, the steady state was always locally asymptotically stable. Therefore the basic structure had to be modified. Among others, Day (1982) introduced two opposite nonlinear effects in the increasing capital stock into ála neoclassical growth model. One is the essential source of economic growth and the other is the source of environmental distortion. In the discrete time framework it was shown that persistent irregular fluctuations and even chaos can be observed when the nonlinearities become strong enough. Following the spirit of Kalecki (1935) suggesting that production delay might be a source of economic fluctuations, Matsumoto and Szidarovszky (2011) constructed a continuous time version of Day's discrete time model with production delay and numerically verified the possibility of chaotic behavior through period-doubling cascade. This finding indicates that the delay nonlinear model of a dynamic economy may explain various dynamic behavior of economic variables. In the recent literature, Guerrini et al. (2019a, 2019b) have already confirmed that the neoclassical model with two distinct delays could generate a wide variety of dynamics ranging from monotonic convergence to chaotic dynamics. Further, Matsumoto and Szidarovszky (2020) demonstrate that the two delays can be a source of complex behavior of the extended Solow-Swan model augmented with human capital.

It is also well known that there is another direction for developing the Harrod-Domar model. Shinkai (1960) extends the one-sector Harrod-Domar model to a two-sector model that consists of a capital-good sector and a consumption-good sector. It is shown that the balanced growth can be stable when the consumption-good sector is more capital-intensive. Further, Furuno (1965) introduces production delay in the capital good sector and shows that the destabilizing effect of the production delay shrinks the stability region. In the recent literature, a Leontief two-sector growth model is thrown into the limelight. Nishimura and Yano (1995) demonstrate the possibility of ergodically chaotic optimal accumulation. Deng et al. (2019) present a complete characterization of optimal policy. However, these considerations are confined to a discrete-time framework in which complex dynamics can emerge if some strong nonlinearities are involved. In the existing literature, it is not revealed whether persistent fluctuations of capital accumulation may appear in a continuous time Leontief growth model. The main purpose of this study is to demonstrate to what extent the delays in the production process and depreciation can contribute to the birth of oscillatory behavior in otherwise stable system.

The paper develops as follows. In Section 2, the two-sector growth model with Leontief production functions is introduced, equilibrium conditions are formulated. In Section 3, the stability conditions of the no-delay and the single-delay cases are discussed. Section 4 contains the two-delay cases, with equal and different delays. And finally, Section 5 offers concluding remarks and further research directions.

2 Two-Sector Model

2.1 Model Construction

There are two sectors, investment sector producing capital goods Y_1 and consumption sector producing consumption goods Y_2 and there are two kinds of agents, workers and capitalists, in each sector. Y_i $(i = 1, 2)$ is produced by the Leontief production function having two inputs, capital goods and labor denoted by K_i and L_i, used in technologically pre-determined proportions,

$$Y_i = F_i(K_i, L_i) = \min\left[\frac{1}{a_i}K_i, \frac{1}{b_i}L_i\right] \tag{1}$$

where $1/a_i$ and $1/b_i$ are the capital productivity and the labor productivity of sector i. If no capital and labor is wasted, then

$$\frac{1}{a_i} = \frac{Y_i}{K_i} \text{ and } \frac{1}{b_i} = \frac{Y_i}{L_i}. \tag{2}$$

The capital-labor ratio of sector i is denoted as k_i and considered constant,

$$k_i = \frac{K_i}{L_i} \tag{3}$$

with which a_i can be written as

$$a_i = k_i b_i.$$

Workers only consume while capitalists only save, all savings are reinvested and all labor incomes are expended. The growth rate n of labor $(L = L_1 + L_2)$ is exogenously given,

$$\dot{L}(t) = nL(t). \tag{4}$$

If $k_1 = k_2$, then the two-sector model loses its economic meaning. Hence the following condition is imposed:

Assumption 1. $k_1 \neq k_2$.

An equilibrium is described by the following conditions:

(e_1) $K = K_1 + K_2$: full capacity of capital,
(e_2) $L = L_1 + L_2$: full employment,
(e_3) $P_1 Y_1 = RK$: equity of investment and saving,
(e_4) $P_2 Y_2 = WL$: equity of consumption demand and supply,
(e_5) $P_i Y_i = RK_i + WL_i$: budget constraint for sector i

where K is a total stock of capital, L total supply of labor, R returns to capital, W the wage rate, P_1 price of capital goods and P_2 price of consumption goods that is assumed to be unity (i.e., consumption goods are chosen to be the numeraire). Given $K(t)$ and $L(t)$ at time t, the equilibrium is attained in the following way. The equilibrium allocation of labor is obtained by solving

$$\begin{aligned} k_1 L_1(t) + k_2 L_2(t) &= K(t), \\ L_1(t) + L_2(t) &= L(t). \end{aligned} \tag{5}$$

The solutions are

$$\begin{aligned} L_1^*(t) &= \frac{K(t) - k_2 L(t)}{k_1 - k_2} \\ L_2^*(t) &= \frac{k_1 L(t) - K(t)}{k_1 - k_2} \end{aligned} \tag{6}$$

and from both of which the equilibrium capital stock and the equilibrium output of sector i are determined,

$$K_i^*(t) = k_i L_i^*(t) \text{ and } Y_i^*(t) = \frac{L_i^*(t)}{b_i} \text{ for } i = 1, 2.$$

By (e_4)

$$W^*(t) = \frac{Y_2^*(t)}{L(t)}.$$

By (c_3) and (e_5), solving $RK = RK_1^* + WL_1^*$ for R,

$$R^*(t) = \frac{W^*(t) L_1^*(t)}{K_2^*(t)}$$

which is substituted into (e_3) to obtain

$$P_1^*(t) = \frac{R^*(t) K^*(t)}{Y_1^*(t)}.$$

At time t, the equilibrium conditions (e_1)-(e_5) determine five variables, $L_1^*(t)$, $L_2^*(t)$ (as well as $K_1^*(t)$ and $K_2^*(t)$ through (3)), $W^*(t)$, $R^*(t)$ and $P_1^*(t)$. At the next instantaneous time, the stock of capital increases by investment at time t and so does the labor force by $nL(t)$. With these new values of capital and labor, the new five variables are determined so as to satisfy the equilibrium conditions. Then the economy evolves itself in the same way.

2.2 Dynamic System

Following Furuno (1965), we make the following assumption:

Assumption 2. There is a production delay $\theta_1 > 0$ in investment process,

$$Y_1(t-\theta_1) = \min\left[\frac{K_1(t-\theta_1)}{k_1 b_1}, \frac{L_1(t-\theta_1)}{b_1}\right].$$

In addition to Assumption 1, a delay in depreciation process is also assumed.[1] The capital accumulation is then described by

$$\dot{K}(t) = Y_1(t-\theta_1) - \delta K(t-\theta_2), \tag{7}$$

where $\delta \geq 0$ is the depreciation rate of capital and $\theta_2 \geq 0$ is the depreciation delay. The labor changes according to equation (4). Using the equilibrium conditions (e_1) and (e_2), the dynamic equations (4) and (7) are transformed into

$$\begin{cases} k_1\dot{L}_1(t) + k_2\dot{L}_2(t) = \dfrac{1}{b_1}L_1(t-\theta_1) - \delta\left[k_1 L_1(t-\theta_2) + k_2 L_2(t-\theta_2)\right], \\ \\ \dot{L}_1(t) + \dot{L}_2(t) = nL_1(t) + nL_2(t). \end{cases} \tag{8}$$

We look for exponential solutions,

$$L_1(t) = e^{\lambda t}u_1 \text{ and } L_2(t) = e^{\lambda t}u_2$$

which are substituted into the dynamic equations to obtain the following form,

$$\begin{pmatrix} \lambda k_1 - \dfrac{1}{b_1}e^{-\lambda\theta_1} + \delta k_1 e^{-\lambda\theta_2} & \lambda k_2 + \delta k_2 e^{-\lambda\theta_2} \\ \\ \lambda - n & \lambda - n \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

Then the characteristic equation is obtained,

$$\frac{1}{b_1}(\lambda - n)\left[\lambda(k_1-k_2)b_1 - e^{-\lambda\theta_1} + \delta(k_1-k_2)b_1 e^{-\lambda\theta_2}\right] = 0 \tag{9}$$

that has one positive root,

$$\lambda_1 = n > 0. \tag{10}$$

[1] See Guerrini et al (2019a) for more detailed explanation on the depreciation delay.

Since the characteristic equation has one positive root, the dynamic system is not "stable" in the usual sense that any time trajectories converge to a steady state. In this paper we use "stability" in the following sense. The growth rate of labor is exogenously given as n. We say that the dynamic system is *stable* if the growth rate as time variable converges to n.[2] Hence stability depends on the values of any other roots that solve the equation,

$$\lambda k b_1 - e^{-\lambda\theta_1} + \delta k b_1 e^{-\lambda\theta_2} = 0 \tag{11}$$

where $k = k_1 - k_2$ and $k \neq 0$ by Assumption 1. We identify four cases, depending on the values of the two delays,

$$\begin{aligned} &\text{Case I:} && \theta_1 = \theta_2 = 0, \\ &\text{Case II:} && \theta_1 > 0,\ \theta_2 = 0, \\ &\text{Case III:} && \theta_1 = \theta_2 > 0, \\ &\text{Case IV:} && \theta_1 > 0,\ \theta_2 > 0,\ \theta_1 \neq \theta_2. \end{aligned}$$

Arranging the terms in (8), we have the delay dynamic system of the two-sector model,

$$\begin{aligned} \dot{L}_1(t) &= \frac{1}{k}\left\{-nk_2 \textstyle\sum_{i=1}^2 L_i(t) + \dfrac{1}{b_1} L_1(t-\theta_1) - \delta \textstyle\sum_{i=1}^2 k_i L_i(t-\theta_2)\right\}, \\ \dot{L}_2(t) &= \frac{1}{k}\left\{nk_1 \textstyle\sum_{i=1}^2 L_i(t) - \dfrac{1}{b_1} L_1(t-\theta_1) + \delta \textstyle\sum_{i=1}^2 k_i L_i(t-\theta_2)\right\}. \end{aligned} \tag{12}$$

3 Classic Models Revisited

Although Shinkai (1960) and Furuno (1965) have already examined Cases I and II, we revisit them and complement their results in our way in this section.

3.1 Shinkai Model: $\theta_1 = \theta_2 = 0$.

Case I corresponds to the Shinkai version (1960) of the model. With no delay, equation (11) is reduced to

$$(\lambda + \delta) k b_1 - 1 = 0 \tag{13}$$

and a solution for equation (13) is

$$\lambda_2 = \frac{1 - \delta k b_1}{k b_1}. \tag{14}$$

[2] Consider two functions, $f(t) = ae^{n(t+1)}$ and $g(t) = be^{nt}$. Although the difference between two function does not converge to zero as time goes to infinity, their growh rates converge to the same value (actually, two have the same growth rate). We say that two time trajectories described by these functions are *stable*.

Taking account of normal circumstances in the real economy[3], Shinkai (1960) assumes that the product of the capital coefficient $k_1 b_1$ of the investment section and the growth rate n of labor is less than unity. In Shinkai's model, the depreciation rate δ does not play any crucial role and thus is assumed to be zero. On the other hand, we will consider how the depreciation delay affects the growth of economy, hence we assume not only $\delta > 0$ but the following inequality that extends Shinkai's conjecture:

Assumption 3. $1 - (n+\delta)k_1 b_1 > 0$.

We have $1 - (n+\delta)kb_1 > 0$ under Assumption 3 as $k_1 > k$. The root λ_2 is positive and larger than n if $k > 0$ and negative if $k < 0$,

$$\lambda_2 > n > 0 \text{ if } k > 0 \text{ and } \lambda_2 < 0 \text{ if } k < 0.$$

With two distinct real roots, n and λ_2, the general solution of $L_i(t)$ can be written as

$$L_i^*(t) = A_i e^{nt} + B_i e^{\lambda_2 t} \text{ for } i = 1, 2 \tag{15}$$

where the coefficients A_i and B_i are determined by the parameter specification and the initial values of capital and labor. Except some special cases,[4] the asymptotic behavior is governed by the term $B_i e^{\lambda_2 t}$ if $k > 0$ as the dominant root is λ_2 $(> n)$ and by $A_i e^{nt}$ if $k < 0$ as the dominant root is $\lambda_1 = n$. The results in Case I are summarized as follows, which are the same as those obtained in Shinkai (1960):

Theorem 1 *The growth path of employment is approaching the balanced growth path if the consumption-good industry is more capital-intensive (i.e., $k_1 < k_2$) and moves away from the balanced growth path if the investment-good industry is more capital-intensive (i.e., $k_1 > k_2$) where, due to Assumption 3, k_1 has the upper bound,*

$$k_1 < \frac{1}{(n+\delta)b_1}.$$

3.2 Furuno Model: $\theta_1 > 0$ and $\theta_2 = 0$.

In this section we revisit and further develop the model of Furuno (1965) that introduces the production delay (i.e., $\theta_1 > 0$) in capital accumulation. Our aim

[3] See Section 2 of Sinkai (1960).

[4] The coefficients of equation (15) are

$$A_1 = cL_0, B_1 = \frac{1}{k}\left(K_0 - \frac{k_2}{1 - kb_1(n+\delta)}L_0\right)$$

and

$$A_2 = (1-c)L_0 \text{ and } B_2 = -B_1 \text{ with } c = \frac{(n+\delta)k_2 b_1}{1-(n+\delta)kb_1}$$

where K_0 and L_0 are the initial values. If $B_1 = 0$, then $B_2 = 0$ and $L_1^*(t) = A_1 e^{nt}$, $L_2^*(t) = A_2 e^{nt}$. In addition, if $c = 1 - c$ holds, then $A_1 = A_2 = A$, implying that $L_1^*(t) = L_2^*(t) = Ae^{nt}$. The time trajectories become identical. In both cases, the balanced growth is realized.

of this section is to classify the possible stable and unstable paths depending on the production delay.

For notational simplicity $\theta_1 = \theta$. The characteristic equation (11) is rewritten as

$$F(\lambda) = (\lambda + \delta)kb_1 - e^{-\lambda\theta} = 0. \tag{16}$$

Similarly to Assumption 3, we make the following assumption that is the same as was imposed by Furuno (1965),[5]

Assumption 4. $e^{-n\theta} - (n + \delta)k_1 b_1 > 0$.

Needless to say, Assumption 4 is stronger than Assumption 3 in the sense that its upper bound of k_1 is smaller for $\theta > 0$,

$$\bar{k}_1(\theta) = \frac{e^{-n\theta}}{(n + \delta)b_1} \leq \frac{1}{(n + \delta)b_1} = k_1^m \text{ for } \theta \geq 0.$$

where $\bar{k}_1(0) = k_1^m$. Under Assumption 4, we also have

$$1 - \delta kb_1 > 0. \tag{17}$$

To solve equation (16), we consider two cases according to whether $k > 0$ (i.e., the capital-good sector is capital intensive) or $k < 0$ (i.e., the consumption-good sector is capital intensive).

(I) $k > 0$

Given $\theta > 0$, some properties of $F(\lambda)$ with respect to λ are obtained,

$$F(0) = \delta kb_1 - 1 < 0, \lim_{\lambda\to\infty} F(\lambda) = \lim_{\lambda\to\infty} (\lambda + \delta)kb_1 > 0$$

and

$$\frac{\partial F(\lambda)}{\partial\lambda} = kb_1 + \theta e^{-\lambda\theta} > 0.$$

These inequalities imply the existence of a unique positive solution λ_2 that satisfies

$$F(\lambda_2) = (\lambda_2 + \delta)kb_1 - e^{-\lambda_2\theta} = 0.$$

Further, Assumption 4 and $k_1 > k > 0$ imply

$$F(n) = (n + \delta)kb_1 - e^{-n\theta} < 0.$$

Hence

$$\lambda_2 > n. \tag{18}$$

Let us define the boundary of Assumption 4 in terms of k and θ by

$$\varphi(k, \theta) = e^{-n\theta} - (n + \delta)(k + k_2)b_1 = 0. \tag{19}$$

[5] See his equation (9).

The maximum value k_m of k along the boundary curve satisfies $\varphi(k_m, 0) = 0$ where

$$k_m = \bar{k}_1(0) - k_2$$

and the maximum value $\theta_m$of θ satisfies $\varphi(0, \theta_m) = 0$ where

$$\theta_m = -\frac{1}{n} \log\left[(n+\delta) k_2 b_1\right].$$

Graphically, the boundary curve, $\varphi(k, \theta) = 0$, is illustrated as the downward-sloping solid curve, crossing the horizontal axes at $k = k_m$ and the vertical axis at $\theta = \theta_m$ in Figure 1. Notice that the down-ward sloping dotted curve is the boundary curve in terms of k_1 and θ and is shifted rightward with k_1 from the solid curve and it is defined for $k_2 < k_1 < k_1^m$. Assumption 4 holds in the shaded region with diagonal lines left to the curve and does not hold in the white region right to the curve. Hence, we have the following result.

Lemma 1 *Given Assumption 4 and $k > 0$, the characteristic equation (16) has a positive characteristic root $\lambda_2 > n$ for k and θ such as $\varphi(k, \theta) > 0$.*

Direct consequence of this lemma is that since λ_2 dominates n, the dynamic system (12) with $\theta_1 = \theta > 0$ and $\theta_2 = 0$ can generate only unbalanced growth paths, that is, the system is unstable. Notice that Lemma 1 is essentially the same as the second half of Theorem 1 (i.e., the system is unstable when $k > 0$). However the instability interval of k becomes smaller due to Assumption 4.

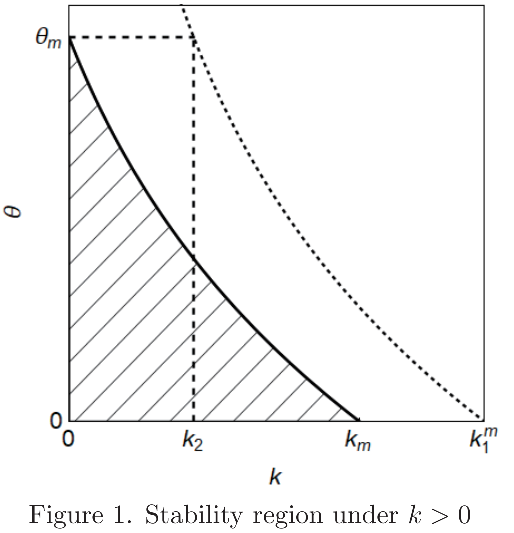

Figure 1. Stability region under $k > 0$

(II) $k < 0$

Let the first term in the right hand side of $F(\lambda)$ be $f(\lambda)$ and the second term $g(\lambda)$. Both are decreasing in λ. When one curve is tangent to the other for $\lambda = \bar{\lambda}$, the following tangency conditions hold:

$$(i)\ f(\bar{\lambda}) = g(\bar{\lambda}) \text{ or } (\bar{\lambda} + \delta)kb_1 = e^{-\bar{\lambda}\theta}\text{: the ordinates are equal}$$

and

$$(ii)\ f'(\bar{\lambda}) = g'(\bar{\lambda}) \text{ or } kb_1 = -\theta e^{-\bar{\lambda}\theta}\text{: the slopes are equal}$$

From (i) and (ii), the abscissa of the tangency point is,

$$\bar{\lambda} = -\frac{1+\delta\theta}{\theta} < 0 \tag{20}$$

that is substituted into equation $F(\lambda)$ to obtain the relation of k and θ that should hold at the tangency point,

$$F(\bar{\lambda}) = -\frac{1}{\theta}\psi(k,\theta) = 0 \tag{21}$$

with

$$\psi(k,\theta) = kb_1 + \theta e^{1+\delta\theta}.$$

Total differentiating $\psi(k,\theta) = 0$ reveals that θ is decreasing in k,

$$\frac{d\theta}{dk} = -\frac{b_1}{(1+\delta\theta)}e^{-(1+\delta\theta)} < 0.$$

If $\psi(k,\theta) < 0$, then the $(\lambda + \delta)kb_1$-line rotates to the right around the horizontal intercept at $\lambda = -\delta$ and becomes steeper.[6] Hence it crosses the $e^{-\lambda\theta}$-curve twice. That is, equation (16) yields two negative roots,

$$\lambda_1 < \bar{\lambda} < \lambda_2 < 0.$$

As t goes to infinity, the terms associated with the negative roots approach zero, regardless of the values of θ. Therefore, the solutions are getting closer to the balanced growth path as time goes on. Hence the first result obtained in case of $k < 0$ is summarized as follows:

Lemma 2 *For k and θ such as $\psi(k,\theta) < 0$, the roots of characteristic equation (16) are negative and thus the time delay does not affect asymptotic behavior of labor and capital.*

On the other hand, if $\psi(k,\theta) > 0$, then the $(\lambda + \delta)kb_1$-line moves counterclockwise and becomes flatter. It then does not intersect the $e^{-\lambda\theta}$-curve and hence equation (16) has infinitely many complex roots. Let $\lambda = \alpha + i\beta$ with $\alpha \geq 0$ and $\beta > 0$ be a complex characteristic root.[7] Substituting it into $F(\lambda)$ presents

$$F(\alpha + i\beta) = (\alpha + i\beta + \delta) - e^{-(\alpha+i\beta)\theta} = 0$$

[6] Remember that $k < 0$ is assumed here.

[7] $\alpha < 0$ does not harm asymptotic stability whereas $\beta > 0$ is assumed only for simplicity as the conjugate root generates exactly the same result.

that is separated into the real and imaginary parts,

$$\begin{aligned} \cos\beta\theta &= (\alpha+\delta)kb_1e^{\alpha\theta}, \\ \sin\beta\theta &= -kb_1\beta e^{\alpha\theta}. \end{aligned} \tag{22}$$

Adding the squares of two equations and solving for β^2 present

$$\beta^2 = \frac{\left[e^{-\alpha\theta} - (\alpha+\delta)kb_1\right]\left[e^{-\alpha\theta} + (\alpha+\delta)kb_1\right]}{(kb_1)^2} \tag{23}$$

where $(kb_1)^2 > 0$ and $e^{-\alpha\theta} - (\alpha+\delta)kb_1 > 0$ since $k < 0$. The sign of $e^{-\alpha\theta} + (\alpha+\delta)kb_1$ is ambiguous in general as the first term is positive and the second is negative. If $e^{-\alpha\theta} + (\alpha+\delta)kb_1 \leq 0$, then $\beta^2 \leq 0$ and thus there is no $\beta > 0$, implying no stability switch occurs. It is already shown that the dynamic system with no delays is stable if $k < 0$ (i.e., see the first half of Theorem 1). Therefore no occurrence of the stability switch means that stability is preserved under the positive delay.

We reveal the parametric condition of k and θ for which the stability switch occurs. If $k_1 \geq -k$ $(= -(k_1 - k_2))$ or $k_1 \geq k_2/2$, then

$$(\alpha+\delta)k_1b_1 \geq (\alpha+\delta)(-k)\,b_1$$

leading to

$$e^{-\alpha\theta} - (\alpha+\delta)k_1b_1 \leq e^{-\alpha\theta} - (\alpha+\delta)(-k)\,b_1. \tag{24}$$

The sign of $e^{-\alpha\theta} - (\alpha+\delta)k_1b_1$ is ambiguous in general as the first term is decreasing in α and the second term is increasing. Assumption 4 implies the existence of a unique and positive $\bar{\alpha} > 0$ such as

$$e^{-\bar{\alpha}\theta} = (\bar{\alpha}+\delta)k_1b_1 \text{ and } \bar{\alpha} > n$$

implying

$$e^{-\alpha\theta} - (\alpha+\delta)k_1b_1 \geq 0 \text{ for } \alpha \leq \bar{\alpha} \tag{25}$$

where the equality holds for $\alpha = \bar{\alpha}$. Hence $k_1 \geq -k$, (24) and (25) result in

$$e^{-\alpha\theta} - (\alpha+\delta)(-k)\,b_1 > 0 \text{ for } 0 < \alpha < \bar{\alpha}. \tag{26}$$

Therefore

$$\beta^2 > 0 \text{ for } 0 < \alpha < \bar{\alpha}. \tag{27}$$

On the other hand, if $k_1 < -k$ or $2k_1 < k_2$, then $(\alpha+\delta)(-k)b_1 > (\alpha+\delta)k_1b_1$. If $e^{-n\theta} + (\alpha+\delta)kb_1 > 0$, then there exists $\tilde{\alpha}$ such as

$$e^{-\tilde{\alpha}\theta} = (\tilde{\alpha}+\delta)(-k)b_1 \text{ and } n < \tilde{\alpha} < \bar{\alpha},$$

otherwise take $\tilde{\alpha} = n$. Then the following relation holds for $\alpha \in (\tilde{\alpha}, \bar{\alpha})$,

$$(\alpha+\delta)(-k)b_1 > e^{-\alpha\theta}$$

which leads to $\beta^2 < 0$. For analytical simplicity, we confine our analysis to the case in which the pair of (k_1, k_2) satisfies the following inequality:

Assumption 5. $k_2 \leq 2k_1$.

(22) implies that $\cos\beta\theta < 0$ and $\sin\beta\theta > 0$ for $(1/2+2m)\pi < \beta\theta < (1+2m)\pi$, for $m = 0, 1, 2, ...$ Then solving the first equation of (22) for θ gives[8]

$$\theta_\alpha(k, m) = \frac{1}{\beta_\alpha}\left\{\cos^{-1}\left[(\alpha+\delta)kb_1e^{\alpha\theta}\right] + 2m\pi\right\} \text{ for } m = 0, 1, 2, ... \tag{28}$$

where

$$\beta_a = \sqrt{\frac{\left[e^{-\alpha\theta} - (\alpha+\delta)kb_1\right]\left[e^{-\alpha\theta} + (\alpha+\delta)kb_1\right]}{(kb_1)^2}}.$$

Notice that equation (28) is an alternative expression of $F(\alpha + i\beta) = 0$. We have already seen from (26) that, for $0 \leq \alpha < \bar{\alpha}$,

$$0 > (\alpha+\delta)kb_1e^{\alpha\theta} > -1$$

under which $\theta_\alpha(k, m)$ is definitely determined. Equation (28) describes the locus of (k, θ) on which $\mathrm{Re}(\lambda) = \alpha$ and $\mathrm{Im}(\lambda) = \beta_a$. The location of the locus depends on the value of m, the shift parameter. Increasing the value of m shifts the locus upward in the (k, θ) plane. We focus on the locus located in the lowest position, that is, we confine attention to the case of $m = 0$ henceforth. Among others, we consider two special cases, $\alpha = 0$ and $\alpha = n$.

When the characteristic root is pure imaginary (i.e., $\alpha = 0$), equation (28) can be written in an implicit form as

$$\phi_0(k, \theta) = \theta - \frac{1}{\beta_0}\cos^{-1}(\delta kb_1) = 0 \tag{29}$$

with

$$\beta_0 = \sqrt{\frac{1 - (\delta kb_1)^2}{(kb_1)^2}}$$

that can be written as $F(i\beta_0) = 0$. When the real part of the characteristic root is equal to the growth rate of labor (i.e., $\alpha = n$), equation (28) is also written in an implicit form as

$$\phi_n(k, \theta) = \theta - \frac{1}{\beta_n}\left\{\cos^{-1}\left[(n+\delta)kb_1e^{n\theta}\right]\right\} = 0 \tag{30}$$

with

$$\beta_n = \sqrt{\frac{\left[e^{-n\theta} - (n+\delta)kb_1\right]\left[e^{-n\theta} + (n+\delta)kb_1\right]}{(kb_1)^2}}$$

that is equivalent to $F(n + i\beta_n) = 0$.

Under Assumption 5 and $k < 0$, the domain of k_1 for which Assumption 4 holds is given by

$$\frac{1}{2}k_2 \leq k_1 < k_2.$$

[8] Solving the second equation yields the same value of θ in the different form.

Returning to (19), the domain of k for $\varphi(k,\theta)$ is changed to

$$-\frac{1}{2}k_2 \leq k < 0. \tag{31}$$

We then summarize the results obtained.

Lemma 3 *Given Assumption 4 (i.e., $\varphi(k,\theta) > 0$), the characteristic equation (16) in case of $k < 0$ has various roots, depending on combinations of k and θ,*

(1) *complex roots with* $\mathrm{Re}(\lambda) < 0$ *for* (k,θ) *such as* $\psi(k,\theta) < \phi_0(k,\theta) < 0$,

(2) *complex roots with* $0 < \mathrm{Re}(\lambda) < n$ *for* (k,θ) *such as* $\phi_n(k,\theta) < 0 < \phi_0(k,\theta)$,

(3) *complex roots with* $\mathrm{Re}(\lambda) > n$ *for* (k,θ) *such as* $\phi_n(k,\theta) > 0$.

Notice the shaded region is the instability region defined by

$$U = \{(k,\theta) \mid k < 0,\ \phi_n(k,\theta) > 0 \text{ and } \varphi(k,\theta) > 0\}.$$

From Lemmas 2 and 3, we have the stability result in Case II:

Theorem 2 *Given Assumptions* 4 *and* 5*, if there is a delay in production process, the equilibrium path is unstable in case of $k > 0$ whereas the equilibrium path could be destabilized for $(k,\theta) \in U$ even in case of $k < 0$.*

Lemmas 2, 3 and Theorem 2 are graphically confirmed in Figure 4 in which $k_2 = 2$ and the following parameter specification is taken,

$$(S) : n = 0.1,\ \delta = 0.05 \text{ and } b_1 = 2.$$

The red and blue curves describes $\phi_0(k,\theta) = 0$ and $\psi(k,\theta) = 0$ and cross the left-side vertical axis at

$$\theta_a \simeq 0.710 \text{ and } \theta_b \simeq 3.359.$$

$\phi_0(k,\theta) < 0$ below the red curve and $\phi_0(k,\theta) > 0$ above. $\psi(k,\theta) < 0$ below the blue curve and $\psi(k,\theta) > 0$ above. The locus of $\varphi(k,\theta) = 0$ corresponds to the boundary of Assumption 4 and is plotted as the downward-sloping green curve that crosses the left-hand vertical axis at $\theta_c \simeq 12.04$ and the right-side vertical axis at $\theta_m \simeq 5.11$. The domain of k is $[-1, 0)$ where $-1 = -k_2/2$. Given Assumption 4, we confine to the region under the green curve in which $\varphi(k,\theta) > 0$. The characteristic equation (16) has two distinct negative roots for (k,θ) below the blue curve (i.e., Lemma 2) and complex roots with $\alpha < 0$ in the region between the blue and the red curves (i.e., Lemma 3(1)). The region above the red curve and below the green curve is divided into two subregions by the black locus of $\phi_n(k,\theta) = 0$. The real part of the complex root is less than n in the subregion left to the black curve (i.e., Lemma 3(2)) and greater than n in the shaded subregion to the right (i.e., Lemma 3(3)). Thus the dynamic system is unstable in the region shaded with diagonal lines (i.e., Theorem 2). The time trajectories starting in the white subregion below the green curve asymptotically approach the balanced growth.[9]

When the value of k is specified, we can obtain the specific values of θ. The vertical dotted line at $k = -1/2$ crosses the red, black and green curve at $\theta_\alpha \simeq 1.61$, $\theta_\beta \simeq 2.24$ and $\theta_\gamma \simeq 7.99$. Hence the characteristic equation (16) has $\mathrm{Re}(\lambda) = 0$ for θ_α, $\mathrm{Re}(\lambda) = n$ for θ_b and Assumption 4 holds for $\theta < \theta_\gamma$. Thus the one-delay dynamic system is unstable for $\theta \in (\theta_\beta, \theta_\gamma)$ and stable for $\theta < \theta_\beta$.

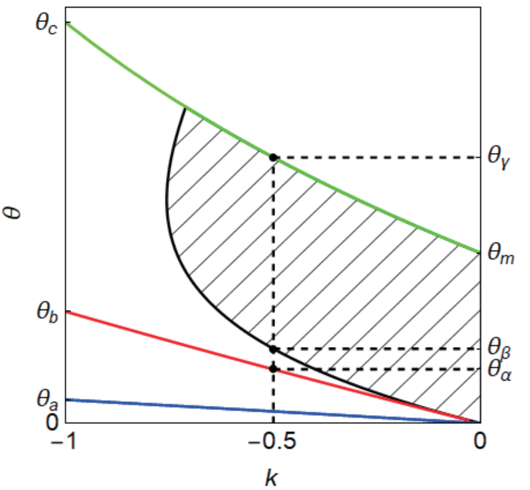

Figure 2. Stability and instability regions

4 Two Delay Models

In Case III of $\theta_1 = \theta_2 = \theta > 0$, equation (11) divided by $1 - \delta k b_1$ is written as

$$\frac{kb_1}{1 - \delta k b_1}\lambda - e^{-\lambda\theta} = 0 \tag{32}$$

This is very similar to equation (16) and thus analytical considerations in Case III might be similar to those done in the previous section. To avoid repetition, we skip this case and jump to Case IV in which $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$. Since Case III is also a special case of Case IV, we will numerically confirm some results that could be obtained in Case III.

[9] It might be possible that initial disturbances make some trajectories to take nagtive values although they are assymptotically stable.

4.1 Analytical Consideration

The characteristic equation (11) with two distinct delays is written as

$$G(\lambda) = kb_1\lambda - e^{-\lambda\theta_1} + \delta kb_1 e^{-\lambda\theta_2} = 0. \tag{33}$$

Since the following inequality holds for $k > 0$ and $\theta_2 > 0$,

$$(n + \delta)k_1 b_1 > (n + \delta e^{-n\theta_2})k_1 b_1,$$

Assumption 4 implies

$$e^{-n\theta_1} - (n + \delta e^{-n\theta_2})k_1 b_1 > 0. \tag{34}$$

As before, we solve equation (33) in two cases, $k > 0$ and $k < 0$.

(I) $k > 0$

Given $\theta_1 > 0$ and $\theta_2 > 0$, we have the followings,

$$G(0) = \delta kb_1 - 1 < 0 \text{ and } \lim_{\lambda\to\infty} G(\lambda) > 0$$

and

$$\frac{dG(\lambda)}{d\lambda} = \left(1 - \delta\theta_2 e^{-\lambda\theta_2}\right) kb_1 + \theta_1 e^{-\lambda\theta_1} > 0$$

where we impose $\delta\theta_2 < 1$ as δ could be very small and θ_2 is thought to be not so large. Hence there is a unique and positive λ_2 solving equation (33),

$$G(\lambda_2) = kb_1\lambda_2 - e^{-\lambda_2\theta_1} + \delta kb_1 e^{-\lambda\theta_2} = 0.$$

Due to (34), $G(n) < 0$ implying

$$\lambda_2 > n.$$

From (34) with given θ_2, we can find a pair of k and θ_1 such as

$$\bar{\varphi}(k, \theta_1, \theta_2) = e^{-n\theta_1} - (n + \delta e^{-n\theta_2})(k + k_2)b_1 = 0 \tag{35}$$

with

$$\frac{\partial\bar{\varphi}(k, \theta_1, \theta_2)}{\partial\theta_2} = (k + k_2)b_1 n\delta e^{-n\theta_2} > 0. \tag{36}$$

The results, which is similar to Lemma 1, are summarized:

Lemma 4 *Given Assumption 4 and $k > 0$, the characteristic equation (33) has a positive characteristic root $\lambda_2 > n$ for k, θ_1 and θ_2 such as $\bar{\varphi}(k, \theta_1, \theta_2) > 0$.*

Graphical representation of Lemma 4 is given in Figure 3 in which three loci of $\bar{\varphi}(\lambda, \theta_1, \theta_2) = 0$ with three different values of θ_2, $\theta_2 = 0$, $\theta_2 = 5$ and

$\theta_2 = 10$, are illustrated as downward-sloping curves where the intercepts with the horizontal axis are

$$k_m^0 \simeq 1.33,\ k_m^5 \simeq 1.84 \text{ and } k_m^{10} \simeq 2.22$$

and the intercepts with the vertical axis are

$$\theta_m^0 \simeq 5.11,\ \theta_m^5 \simeq 6.51 \text{ and } \theta_m^{10} \simeq 7.47.$$

Notice that the locus of $\bar{\varphi}(\lambda, \theta_1, 0) = 0$ is the boundary of the green region in Figure 3 and identical with the downward sloping curve in Figure 1. Due to (36), the curve shifts upward as the value of θ_2 increases. This means that θ_2 is a destabilizing factor as increasing θ_2 enlarges the instability region.

3

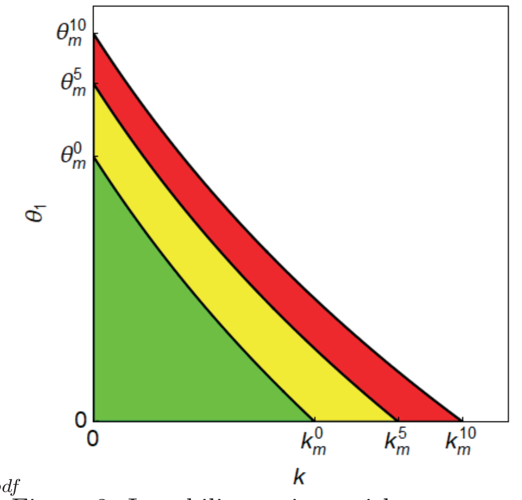

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Figure 3. Instability regions with different values of θ

(II) $k < 0$

To determine the type of the movement controlled by the characteristic equation (33), the non-negative region of the (θ_1, θ_2) plane is divided as seen in Figure 4 in which the parameter specification (S) and $k = -1/2$ are adopted. The dotted three points, the horizontal dotted three lines and the dotted diagonal are ignored for the time being. There the upward-sloping black curve is the boundary of (34). The inequality of (34) holds in the region left to this curve. The real parts of the characteristic roots of (33) are zero on the red-blue curve, negative in region to its left and positive to its right.[10] The stability of the time trajectories is preserved in the region left to the red-blue curve as the real parts of the characteristic equation is negative. We numerically determine pairs of θ_1

[10] The analytical derivation of this red-blue curve is given in the Appendix.

and θ_2 for which the stability is lost and connect these pairs to construct the vertical-like dotted curve. Since one characteristic root is equal to the growth rate of labor, n, the real parts of the other characteristic roots are less than n in the region to its left and greater than n to its right. For θ_1 and θ_2 belonging to the region between the red-blue curve and the dotted curve, the growth rate of any trajectory oscillates and converges to n as $0 < \mathrm{Re}(\lambda) < n$. Lastly for θ_1 and θ_2 in the region between the dotted curve and the black curve, the trajectories oscillate, diverge and take negative values sooner or later, stopping production of output when L_1 becomes zero. Summarizing the result, the constraint (34) holds in the region left to the black curve, the trajectories are stable in the region left to the dashed curve and unstable in the region right.

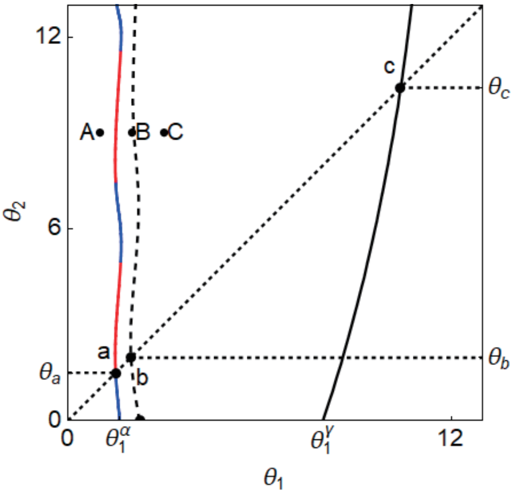


Figure 4. Division of the (θ_1, θ_2) plane

4.2 Numerical Simulations

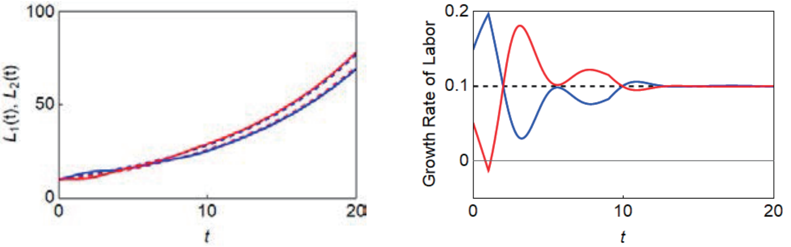
To confirm the analytical results, we simulate the dynamic system (12). The initial values of labor and capital are taken to be $L_0 = 20$ and $K_0 = (k_1+k_2)L_0/2$ and the initial functions are assumed to be constant for $t \leq 0$,

$$\varphi_1(t) = \frac{K_0 - k_2 L_0}{k_1 - k_2} \text{ and } \varphi_2(t) = \frac{k_1 L_0 - K_0}{k_1 - k_2}$$

where $k_1 = 3/2$ and $k_2 = 2$, leading to $k = -1/2$. To see what dynamics is generated before convergence, we first pick up three points denoted as A, B and C in the upper-left area of Figure 4. Their ordinates are the same, $\theta_2 = 9$ but their abscissas are different,

$$\theta_1^A = 1, \ \theta_1^B = 2 \text{ and } \theta_1^C = 3.$$

Notice that point A is located in the left side of the red-blue curve, implying that the real parts of the complex roots or the real roots are negative. Accordingly, the time trajectories of the employment approach the balanced growth path and their growth rates exhibit damping oscillations moving around the balanced growth rate, n. This indicates that equation (33) generates complex roots having negative real parts at point A. The numerical simulations are visualized in Figure 5 in which the variables in the investment sector are illustrated with the blue curve and those of the consumption sector with the red curve. The dotted curves describe the corresponding trajectories generated by the Shinkai model where there are no delays. It is seen in Figures 5(A) and 5(B), the delay trajectories do not converge to the dotted trajectories, however their growth rate quickly converges to n $(= 0.1)$. In Figure 5(B), there are kinks of the trajectories at $t = \theta_1^A$ due to the assumption of the constant initial functions.

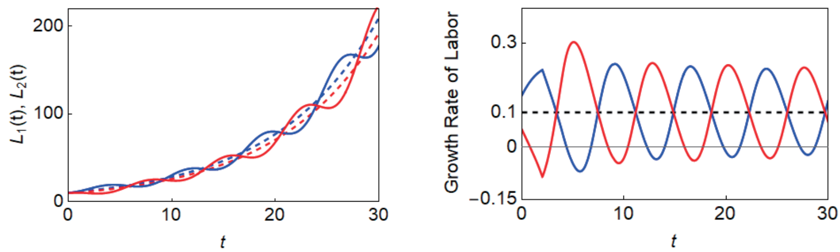


(A) Time trajectories (B) Growth rates

Figure 5. Simulations at point A with $\theta_1 = 1$ and $\theta_2 = 9$

Point B is located in the region left to the red-blue curve and right to the dashed curve.[11] This selection of the point implies two issues; one is that the characteristic equation (33) gives rise to complex roots and the other is that their real parts are positive but less than n. Accordingly, the time trajectories exhibit damping oscillations around the no-delay trajectories as seen in Figure 6(A). It is seen in Figure 6(B) that the growth rates oscillate and approach to n.

[11] The critical value of θ_1 with $\theta_2 = 9$ is $\theta_1 \simeq 2.06$. At point B, $\theta_1^B = 2$, which is strictly less than the critical value and thus point B is located in the left from the dotted curve.

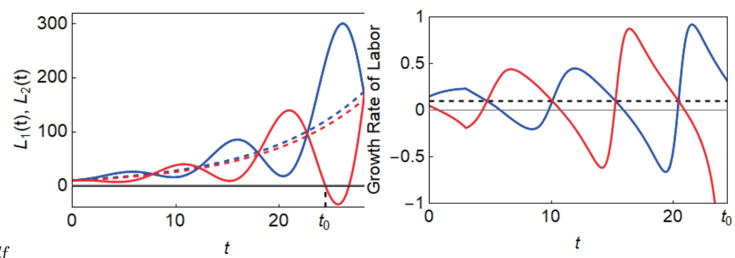


(A) Time trajectories (B) Growth rates

Figure 6. Simulations at point B with with $\theta_1 = 2$ and $\theta_2 = 9$

Point C is in the right side of the dotted curve. The characteristic root is complex and its real part is larger than n. Thus the trajectories exhibit explosive oscillations. As seen in Figure 7(A), the red trajectory takes negative value for $t = t_0(\simeq 24.45)$ and implies that the trajectories alternately take positive and negative values as time goes on. When the red trajectory (i.e., $L_2(t)$) is zero at $t = t_0$, the output of consumption goods ceases and the capital will be over-accumulated, losing economic meaning of the two sector model. In Figure 7(B), the growth rate of $L_2(t)$ (that is, the red curve) goes to negative infinity as time comes closer to t_0.

7A



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(A) Time trajectories (B) Growth rates

Figure 7. Simulations point C with with $\theta_1 = 3$ and $\theta_2 = 9$

This two delay model includes the previous models as its special cases. The Shinkai model corresponds to the origin in which the two delay model is stable in its small neighborhood. Along the horizontal axis, $\theta_1 > 0$ and $\theta_2 = 0$ for which the premise of Furuno model is satisfied. The red-blue, the dotted and black curves cross the horizontal axes, respectively, at

$$\theta_1^\alpha \simeq 1.62,\ \theta_1^\beta \simeq 2.24 \text{ and } \theta_1^\gamma \simeq 7.99$$

where θ_1^β is not labelled in Figure 4 in order to avoid notational congestion. Returning to Figure 2, it can be seen that the vertical dotted line standing at $k = -1/2$ crosses the red curve at θ_α at which Furuno model generates $\mathrm{Re}(\lambda) = 0$, the black curve at θ_β where $\mathrm{Re}(\lambda) = n$ and the green curve at θ_γ where

$$\theta_1^\alpha = \theta_\alpha,\ \theta_1^\beta = \theta_\beta \text{ and } \theta_1^\gamma = \theta_\gamma.$$

As is clear from Figure 2, stable region between the red and black curves becomes larger as the absolute value of k gets larger. In the same way, it can be mentioned that the vertically long stable region between the red-blue and the dotted curves in Figure 4 could become larger if the absolute value of k becomes larger. Along the dashed diagonal, $\theta_1 = \theta_2$ holds that are the premise of Case III. In Figure 4, the diagonal crosses the red-blue, dotted and black curves at points a, b and c where

$$\theta_a \simeq 1.50\theta_b \simeq 1.97 \text{ and } \theta_c \simeq 10.41.$$

Referring to Figure 2, $\theta_a < \theta_1^\alpha$, $\theta_b < \theta_1^\beta$ and $\theta_c > \theta_1^\gamma$ lead us to conclude that the $\mathrm{Re}(\lambda) = 0$ curve shifts downward and the $\mathrm{Re}(\lambda) = n$ curve shifts outward, implying extension of the unstable region in Case III.[12]

5 Concluding Remarks

This paper examined a two-sector growth model with an investment sector producing capital goods and a consumption sector producing consumption goods. Two types of agents, workers and capitalists were assumed in each sector. It was also assumed that a special Leontief production function had two inputs, capital goods and labor, which were used in technologically pre-determined proportions. First the equilibrium was characterized and the two-delay dynamic system of labor was derived. The asymptotical behavior of the delay system was characterized by analyzing the locations of the eigenvalues. Because of the presence of a positive eigenvalue, the classical definition of stability had to be modified accordingly. The study was conducted in three cases. Systems with no delay, one delay and two delays were examined in different subsections in which some results with equal delays are numerically examined. The main result of this paper was to derive conditions under which the growth rate of labor as time variable converges to the exogenously given constant growth rate. This study can be extended in several ways. More general production functions can be introduced. It can be also assumed that the exogenously given growth rate of labor is time dependent. Experimental studies could illustrate further details of the long-term behavior of the time trajectories especially in the unstable cases.

[12] Applying the same precedure used in Case II to Case III reveals that this conclusion is true.

Appendix

In this Appendix, we analytically derive the red-blue curve in Figure 4, applying the method developed by Matsumoto and Szidarovszky (2019) that is based on Gu et al. (2008).

Given two distinct positive delays, the characteristic equation (33) is written as

$$P_0(\lambda) + P_1(\lambda)e^{-\lambda\theta_1} + P_2(\lambda)e^{-\lambda\theta_2} = 0$$

where

$$P_0(\lambda) = kb_1\lambda,\ P_1(\lambda) = -1 \text{ and } P_2(\lambda) = \delta kb_1.$$

It can be checked that the following conditions hold:

(i) $\deg[P_0(\lambda)] = 1 > \max\{\deg[P_1(\lambda)], \deg[P_2(\lambda)]\} = 0$;

(ii) $P_0(0) + P_1(0) + P_2(0) = -1 + \delta kb_1 \neq 0$ if $\delta kb_1 \neq 1$ that holds under Assumption 3;

(iii) polynomials $P_0(\lambda), P_1(\lambda)$ and $P_2(\lambda)$ have no common root;

(iv) $\lim \left\{ \left| \dfrac{P_1(\lambda)}{P_0(\lambda)} \right| + \left| \dfrac{P_2(\lambda)}{P_0(\lambda)} \right| \right\} = 0 < 1.$

The condition (ii) eliminates that $\lambda = 0$ is a root of (33) and if it is violated, then the characteristic function is always unstable. The condition (iii) is to ensure equation (33) has the lowest degree. The condition (iv) is to exclude large oscillations. Dividing equation (33) by $P_0(\lambda)$ to obtain

$$1 + a_1(\lambda)e^{-\lambda\theta_1} + a_2(\lambda)e^{-\lambda\theta_2} = 0$$

where

$$a_1(\lambda) = -\frac{1}{kb_1\lambda} \text{ and } a_2(\lambda) = \frac{\delta}{\lambda}.$$

Now suppose that $\lambda = i\omega$ with $\omega > 0$ for the time being. Substituting this solution into $a_1(\lambda)$ and $a_2(\lambda)$ presents

$$a_1(i\omega) = \frac{i}{kb_1\omega} \text{ and } a_2(i\omega) = -i\frac{\delta}{\omega}.$$

and

$$|a_1(i\omega)| = \frac{1}{|k|\, b_1\omega} \text{ and } |a_2(i\omega)| = \frac{\delta}{\omega}$$

The conditions that 1, $a_1(i\omega)$ and $a_2(i\omega)$ can form a triangle are

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1$$

and

$$-1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1.$$

Substituting $|a_1(i\omega)|$ and $|a_2(i\omega)|$ render these inequalities to

$$\omega \leq \frac{1+\delta\,|k|\,b_1}{|k|\,b_1},$$

and

$$\omega \geq \frac{\delta\,|k|\,b_1 - 1}{|k|\,b_1} \text{ and } \omega \geq \frac{1-\delta\,|k|\,b_1}{|k|\,b_1}.$$

In summary,

$$\left|\frac{1-\delta\,|k|\,b_1}{|k|\,b_1}\right| \leq \omega \leq \frac{1+\delta\,|k|\,b_1}{|k|\,b_1}$$

that is clearly nonempty. The angles of the triangle can be obtained by law of cosine,

$$\alpha_1 = \cos^{-1}\left[\frac{(kb_1\omega)^2 + 1 - (\delta k b_1)^2}{2\,|k|\,b_1\omega}\right]$$

and

$$\alpha_2 = \cos^{-1}\left[\frac{-1 + (\delta k b_1)^2 + (kb_1\omega)^2}{2\delta\omega\,|k|^2\,b_1^2}\right].$$

Notice that

$$\arg\left[a_1(i\omega)\right] = \begin{cases} \dfrac{\pi}{2} & \text{if } k > 0, \\ -\dfrac{\pi}{2} & \text{if } k < 0. \end{cases}$$

and

$$\arg\left[a_2(i\omega)\right] = -\frac{\pi}{2}.$$

So the angles are

$$\theta_1^{n\pm}(\omega) = \frac{1}{\omega}\left\{\arg\left(a_1(i\omega)\right) + (2n-1)\pi \pm \alpha_1\right\}$$

and

$$\theta_2^{m\mp}(\omega) = \frac{1}{\omega}\left\{\arg\left(a_2(i\omega)\right) + (2m-1)\pi \mp \alpha_2\right\}.$$

The red-blue curves on which $\lambda = i\omega$ holds are union Ω of the segments,

$$\left\{\theta_1^{n+}(\omega), \theta_2^{m-}(\omega)\right\} \cup \left\{\theta_1^{n-}(\omega), \theta_2^{m+}(\omega)\right\} \text{ for } n, m = 0, 1, 2, ...$$

where

$$\omega \in \left[\left|\frac{1-\delta\,|k|\,b_1}{|k|\,b_1}\right|, \frac{1+\delta\,|k|\,b_1}{|k|\,b_1}\right].$$

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