On Preservation of Strong Stabilizability Under Sampling

TOMOMICHI HAGIWARA AND MITUHIKO ARAKI

Abstract—**It is shown that a sampled-data system remains strongly stabilizable if the original continuous-time system is so and if the sampling period is sufficiently small. At the same time, it is also shown that a sampled-data system can be strongly stabilizable even if the original continuous-time system is not so, and some sufficient conditions guaranteeing this does not happen are derived.**

I. Introduction

The problem of stabilizing an unstable system by an asymptotically stable compensator is referred to as the strong stabilization problem [1], [2]. Since the recent development of microprocessor techniques has greatly enlarged the opportunity of using discrete-time controllers, it would be significant to study the problem whether strong stabilizability is preserved under sampling. This note deals with this problem for the case of scalar sampled-data control systems which use zero-order holds with sufficiently small sampling periods. It is shown that a sampled-data system remains strongly stabilizable if the original continuous-time system is so. On the other hand, an example of a sampled-data system is given which is strongly stabilizable even though the original continuous-time system is not. Since such possibility exists, sufficient conditions are given for a system not to be strongly stabilizable after sampling.

II. Strong Stabilizability Conditions

Let CS be the stabilizable and detectable scalar continuous-time system described by

$$\frac{dx}{dt}=Ax(t)+bu(t),\ y(t)=cx(t) \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^{n\times 1}$, and $c \in R^{1\times n}$ are real constant matrices. The transfer function of CS is given by $c(sI - A)^{-1}b = N(s)/D(s)$ where

$$N(s)=\det\begin{pmatrix} sI-A & -b \\ c & 0 \end{pmatrix},\ D(s)=\det(sI-A). \tag{2}$$

The real roots of $N(s) = 0$ and $D(s) = 0$ in the closed right-half plane are, respectively, called real unstable zeros (r.u. zeros) and real unstable poles (r.u. poles). Infinity $+\infty$ is also regarded as a r.u. zero since the transfer function of CS is strictly proper. CS is said to be strongly stabilizable if it can be stabilized by an asymptotically stable linear time-invariant continuous-time compensator. It is known [1], [2] that CS is strongly stabilizable if and only if it possesses the parity interlacing property of continuous-time systems:

(***PIP-CS***) the number of r.u. poles counted according to their multiplicities between two r.u. zeros is even.

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The authors are with the Department of Electrical Engineering, Kyoto University, Kyoto, Japan.
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Next, consider the sampled-data system SS described by

$$x(\overline{k+1}T) = \exp(AT)x(kT) + \int_0^T \exp(At)b\,dt \cdot u(kT),$$
$$y(kT) = cx(kT) \quad (3)$$

which is obtained by discretizing CS using the zero-order hold with the sampling period T. The pulse transfer function of SS is given by $N_T(z)/D_T(z)$ where

$$N_T(z) = \det\begin{pmatrix} zI - \exp(AT) & -\int_0^T \exp(At)b\,dt \\ c & 0 \end{pmatrix},$$
$$D_T(z) = \det[zI - \exp(AT)]. \quad (4)$$

The real roots of $N_T(z) = 0$ and $D_T(z) = 0$ outside the open unit disk centered at the origin are, respectively, called r.u. zeros and r.u. poles. Infinity ∞ is also regarded as a r.u. zero since the pulse transfer function of SS is strictly proper.

Let $\lambda_i (i = 1, \cdots, n)$ be the poles of CS and let $\Lambda_i (i = 1, \cdots, n)$ be those of SS. It follows from (2) and (4) that $\{\lambda_i\}$ and $\{\Lambda_i\}$ correspond one-to-one by

$$\Lambda_i = \exp(\lambda_i T) \qquad i = 1, \cdots, n. \quad (5)$$

It is well known that SS is stabilizable and also detectable if and only if the sampling period T satisfies

$$\{\mathrm{Im}(\lambda_i) - \mathrm{Im}(\lambda_j)\} T \neq 2k\pi \text{ whenever } \mathrm{Re}(\lambda_i) = \mathrm{Re}(\lambda_j) \geqq 0 \quad (6)$$

where $k = \pm 1, \pm 2, \cdots$. Since the ordinary stabilizability is necessary for strong stabilizability, we assume throughout this note that the sampling period T satisfies (6). It should be noted that (6) prohibits the imaginary part of a nonreal unstable pole of CS from being a multiple of π/T. This, together with (5), implies the following.

Proposition 1: Any r.u. pole of SS corresponds to a r.u. pole of CS by (5) and vice versa.

Proposition 2: All the r.u. poles of SS are positive.

From Proposition 2 and from the strong stabilizability condition of general discrete-time systems [2], we can conclude that SS is strongly stabilizable (i.e., stabilizable by an asymptotically stable linear time-invariant discrete-time compensator) if and only if it possesses the parity interlacing property of sampled-data systems:

(PIP-SS) the number of positive r.u. poles counted according to their multiplicities between two positive r.u. zeros (including the infinity $+\infty$) is even.

III. Preliminary Study About the Location of Zeros of Sampled-Data Systems

In view of the parity interlacing condition, we have to examine the disposition of the poles and zeros of SS in order to study its strong stabilizability. As noted in the preceding section, the poles of SS are closely related to those of CS, and their exact locations are given by (5). But, as for the zeros of SS, little knowledge is available at present. Åström *et al.* [3] showed that, assuming that the sampling period T is sufficiently small, there are m zeros near $\exp(\gamma T)$ if γ is a zero of CS with the multiplicity m. But this result is not useful for our purpose because both poles of SS and the values $\exp(\gamma T)$ converge to 1 when T becomes small and nothing about their mutual locations are clarified. Hara *et al.* [4] showed that, if the number of real zeros of CS between two real poles (λ_i and λ_j) of CS is even (respectively, odd), the number of real zeros of SS between the corresponding real poles $\exp(\lambda_i T)$ and $\exp(\lambda_j T)$ of SS is also even (respectively, odd) for almost every sampling period T. This result does not contribute to our purpose very much either, because what we need to know is the number of real poles between real zeros and Hara *et al.*'s result cannot give exact knowledge about it in general.[1] So, we make a preliminary study about the location of zeros of SS in this section. The following two lemmas are the keys to clarify the locations of the zeros of SS, where Lemma 2 is a generalization of Lemma 1.

Lemma 1: Let σ be a real number which is not a zero of CS. Then, $N_T(\exp(\sigma T))$ has the same sign as $N(\sigma)$ if the sampling period T satisfies $0 < T < T_\sigma$, where T_σ is a positive number dependent on σ.

Proof: By (4)

$$N_T(\exp(\sigma T)) = \det\begin{pmatrix} \exp(\sigma T)I - \exp(AT) & -\int_0^T \exp(At)b\,dt \\ c & 0 \end{pmatrix}. \quad (7)$$

Since the n rows except the last of the determinant of (7) become 0 when $T = 0$, the following are obtained from the formula about the differential of a determinant:

$$[\partial^k N_T(\exp(\sigma T))/\partial T^k]_{T=0} = 0 \quad (k = 0, \cdots, n-1), \quad (8)$$

$$[\partial^n N_T(\exp(\sigma T))/\partial T^n]_{T=0} = n! \cdot \det\begin{pmatrix} \sigma I - A & -b \\ c & 0 \end{pmatrix} = n! N(\sigma). \quad (9)$$

Since $N(\sigma) \neq 0$, the result of the lemma follows immediately. Q.E.D.

Lemma 2: Let $\mathcal{I}$ be a closed interval of real numbers which does not include zeros of CS. Then, $N_T(\exp(\sigma T))$ has the same sign as $N(\sigma)$ $\forall \sigma \in \mathcal{I}$ if the sampling period T satisfies $0 < T < T_{\mathcal{I}}$ where $T_{\mathcal{I}}$ is a positive number dependent on the interval $\mathcal{I}$.

Proof: Note that $\partial^n N_T(\exp(\sigma T))/\partial T^n$ is continuous with respect to σ and T. By (9) and by the assumption that $\mathcal{I}$ does not include any zeros of CS (i.e., roots of $N(s) = 0$), $[\partial^n N_T(\exp(\sigma T))/\partial T^n]_{n=0}$ has the same sign as $N(\sigma)$ $\forall \sigma \in \mathcal{I}$. Therefore, there exists a positive number $T_{\mathcal{I}}$ such that $\partial^n N_T(\exp(\sigma T))/\partial T^n$ takes the same sign as $N(\sigma)$ $\forall \sigma \in \mathcal{I}$ and $0 \leqq \forall T < T_{\mathcal{I}}$. Since (8) holds true, successive integrations with respect to T lead to the result of the lemma. Q.E.D.

From the above lemmas, we can derive the following propositions about zeros of SS.

Proposition 3: Let γ be a real zero of CS with an odd multiplicity, and let α and β be real numbers such that $\alpha < \gamma < \beta$ and $N(\alpha)N(\beta) < 0$. Then, SS has a real zero in the interval $[\exp(\alpha T), \exp(\beta T)]$ if the sampling period T satisfies $0 < T < \min(T_\alpha, T_\beta)$.

Proof: By Lemma 1, $N_T(\exp(\alpha T))N_T(\exp(\beta T)) < 0$ if $0 < T < \min(T_\alpha, T_\beta)$. The result of the proposition follows readily since $N_T(z)$ is continuous with respect to z. Q.E.D.

Proposition 4: Suppose that the interval $\mathcal{I} = [\alpha, \beta]$ includes no real zeros of CS. Then, SS has no real zeros in the corresponding interval $\exp(\mathcal{I}T) := [\exp(\alpha T), \exp(\beta T)]$ if the sampling period T satisfies $0 < T < T_{\mathcal{I}}$.

Proof: By Lemma 2, $N_T(z)$ has the same sign $\forall z \in \exp(\mathcal{I}T)$ if $0 < T < T_{\mathcal{I}}$. The result of the proposition follows immediately. Q.E.D.

The next proposition is also necessary to deal with the boundary case.

Proposition 5: SS has a zero at $z = 1$ if and only if CS has a zero at $s = 0$.

Proof: Note that the following relation holds true:

$$\begin{pmatrix} I - \exp(AT) & -\int_0^T \exp(At)b\,dt \\ c & 0 \end{pmatrix} = \begin{pmatrix} \int_0^T \exp(At)\,dt & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -A & -b \\ c & 0 \end{pmatrix}. \quad (10)$$

Since $\int_0^T \exp(At)\,dt$ is nonsingular because of the assumption (6), the result of the proposition follows immediately. Q.E.D.

IV. Preservation of the PIP Under Sampling

The main result is as follows.

Theorem 1: If CS possesses (PIP-CS) and if the sampling period T is sufficiently small, then SS possesses (PIP-SS).

[1] Their result can be applied to a certain class of CS's to show that the corresponding SS does not possess the (PIP-SS). But, their result is not helpful if CS does not belong to that class. Details will be stated in the comments following Theorems 2 and 3.

The following notion [4] is used in the proof of this theorem.

Definition 1: Let the r.u. poles of CS repeated according to multiplicities be $\lambda_1, \cdots, \lambda_r$. When CS has no poles and zeros at $s = 0$, define $\mathfrak{X}$ as $\{0, \lambda_1, \cdots, \lambda_r\}$, and when CS has either poles or zeros at $s = 0$, define $\mathfrak{X}$ as $\{\lambda_1, \cdots, \lambda_r\}$. The members of $\mathfrak{X}$ are said to be the extended r.u. poles of CS.

Proof of Theorem 1: Let the extended r.u. poles of CS be $\xi_l \leqq \cdots \leqq \xi_2 \leqq \xi_1$. Note that if $\xi_l \neq 0$ (i.e., if CS has a zero at $s = 0$), then l is even by the assumption that CS possesses the (PIP-CS). Since the interval $\mathcal{I}_i = [\xi_{2i}, \xi_{2i-1}]$ $(2 \leqq 2i \leqq l)$ includes no zeros of CS by the same assumption, it follows from Proposition 4 that SS has no real zeros in the interval $\exp(\mathcal{I}_i T)$ if $0 < T < T_{\mathcal{I}_i}$. This, together with Proposition 1, implies that SS possesses the (PIP-SS) if the sampling period T satisfies $0 < T < \min_i (T_{\mathcal{I}_i})$. Q.E.D.

The above thereom means that SS remains strongly stabilizable if CS is so and if the sampling period T is sufficiently small. But, it seems difficult to find by analysis an interval of T which preserves strong stabilizability.

Next, let us consider the case where CS does not possess the (PIP-CS).

Example: Consider the continuous-time system CS with

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad b=\begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}, \quad c=[1 \ 1 \ 1]. \tag{11}$$

The transfer function is $6(s-3)^2/\{(s-1)(s-2)(s-4)\}$ and, so, CS does not possess the (PIP-CS). Calculation of the Taylor expansion of the discriminant of $N_T(z)$ shows that the corresponding SS does not have real zeros if T is sufficiently small. This implies that SS possesses the (PIP-SS) for sufficiently small T.

The above example shows that there is a case where SS can be strongly stabilizable even if CS is not. This fact motivates us to find sufficient conditions guaranteeing that SS is not strongly stabilizable.

Definition 2: Let distinct positive finite r.u. zeros of CS with odd multiplicities be $\gamma_1, \cdots, \gamma_q$. When CS has no zeros at $s = 0$. Define $\mathcal{O}$ as $\{\gamma_1, \cdots, \gamma_q, +\infty\}$, and when CS has zeros at $s = 0$, define $\mathcal{O}$ as $\{0, \gamma_1, \cdots, \gamma_q, +\infty\}$. The members of $\mathcal{O}$ are said to be the r.u. zeros of CS with odd multiplicities in the generalized sense.

Theorem 2: Suppose that CS has $\omega_i, \omega_j \in \mathcal{O}$ such that an odd number of r.u. poles, counted according to their multiplicities, lie between ω_i and ω_j. Then, SS does not possess the (PIP-SS) if the sampling period T is sufficiently small.

Proof: It follows from the assumption that an odd number of r.u. poles lie between ω and the infinity $+\infty$, where ω is either ω_i or ω_j. If $\omega \neq 0$, choose α and β so that $0 < \alpha < \omega < \beta$, $N(\alpha)N(\beta) < 0$, and $[\alpha, \beta]$ includes no poles of CS. Then, by Proposition 3, there is a r.u. zero of SS in $[\exp(\alpha T), \exp(\beta T)]$ if T is sufficiently small. It follows from Proposition 1 that there are an odd number of r.u. poles of SS between that zero of SS and $+\infty$. This implies that SS does not possess the (PIP-SS). If $\omega = 0$, SS has a zero at $z = 1$ by Proposition 5. By Proposition 1, there are an odd number of r.u. poles of SS between 1 $(= \exp(\omega T))$ and $+\infty$. This implies that SS does not possess the (PIP-SS) for any T. Q.E.D.

From the latter half of the proof, we obtain the next theorem.

Theorem 3: Suppose that CS has zeros at $s = 0$ and has an odd number of real unstable poles. Then, SS does not possess the (PIP-SS) for any sampling period T.

The above two theorems supply sufficient conditions under which SS is not strongly stabilizable. Let $\xi_l \leqq \cdots \leqq \xi_2 \leqq \xi_1$ be the extended r.u. poles of CS. It follows from the result of Hara *et al.* [4] that SS is not strongly stabilizable for almost every sampling period T if there exists an interval $\mathcal{I}_i = [\xi_{2i}, \xi_{2i-1}]$ $(2 \leqq 2i \leqq l)$ which includes an odd number of r.u. zeros. Even when all the intervals $\mathcal{I}_i$ include an even number of r.u. zeros, we can still conclude from Theorem 2 that SS is not strongly stabilizable for sufficiently small T if there exists an interval $\mathcal{I}_i$ which includes a r.u. zero with an odd multiplicity. Therefore, only when all zeros in the intervals $\mathcal{I}_i$ are with even multiplicities, SS can be strongly stabilizable even though CS is not. But, it should be noted that SS is *not always* strongly stabilizable in such cases. Concerning this point, further results can be derived by examining $(n+1)$th and $(n+2)$th derivatives of $N_T(\exp(\sigma T))$ with respect to T. The details are omitted because of the limitation of space.

V. Conclusion

Several results are obtained about strong stabilizability of sampled-data systems. These results are derived under the assumption that the discrete-time compensator does not require any computation time. But, the parallel results hold true even if the compensator is assumed to require the computation time T_c (>0: not necessarily a multiple of the sampling period T). This can be shown by the following steps.

a) let CS-d be the continuous-time system obtained by cascading CS and the pure delay $\exp(-sT_c)$, and let SS-d be the sampled-data system derived from CS-d. Then, the problem reduces to the problem of stabilizing SS-d by an asymptotically stable discrete-time compensator which requires no computation time.

b) Stable poles of SS-d at $z = 0$, which are yielded by the pure delay, have nothing to do with the PIP. Other poles correspond one-to-one to those of CS by (5).

c) The condition (6) is valid also for SS-d.

d) From b) and from c), Propositions 1 and 2 hold true also for SS-d. Therefore, the strong stabilizability condition (PIP-SS) applies also to SS-d.

e) We can derive Propositions 3–5 also for SS-d by slightly modifying the proofs.

Theorems 1 and 3 can be extended to the multivariable case if zeros are replaced by blocking zeros [5], [6]. Details of this as well as the detailed results indicated at the end of the preceding section will be reported as a separate item in the future.

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