Preservation of Reachability and Observability under Sampling with a First-Order Hold

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***Abstract*—This paper gives the necessary and sufficient condition for the reachability of the sampled-data system S_1 obtained by the discretization of a linear time-invariant continuous-time system with a first-order hold. Equivalence of the reachability and controllability of S_1 is also shown. Similar results are given also for observability and reconstructibility. It turns out that S_1 is reachable only if S_0 is reachable, while S_1 is observable if and only if S_4*l0* is observable, where S_0 is the sampled-data system obtained by the discretization with a zero-order hold of the same sampling period.**

I. Introduction

In sampled-data control, hold circuits are used to convert the discrete-time signals from digital compensators into the continuous-time signals to be applied to the continuous-time systems. Hold circuits can be viewed also as filters which attenuate the high-frequency alias spectra generated by sampling continuous-time signals. Typical hold circuits are a zero-order hold and a first-order hold [1], but the former seems to be particularly popular in industrial applications. The primary reason for this is that a zero-order hold can be implemented

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quite easily by using the function of D/A converters while a first-order hold can be implemented only with the aid of some additional analog circuits. Another reason might be that, when viewed as continuous-time filters, the phase lag of a first-order hold is greater than that of a zero-order hold for high-frequency ranges, which seems to be a disadvantage from the point of view of closed-loop stability. However, the latter reason seems to apply mainly in the case when a digital compensator is obtained by a digital redesign method [2] of a continuous-time compensator and closed-loop stability is not necessarily assured theoretically. If we could use a first-order hold in such a way that closed-loop stability can be assured, then it might provide some advantages over a zero-order hold, such as reduction of the intersample ripple of the response.

Based upon the above consideration, the aim of this paper is to provide a basis for the use of a first-order hold in the context of the state-space approach of control system design. For this purpose, we give the necessary and sufficient condition for the reachability of the sampled-data system obtained by the discretization of a linear time-invariant continuous-time system with a first-order hold. In addition, we show the equivalence of the reachability and controllability of this sampled-data system. Furthermore, we give similar results for observability and reconstructibility. (For the standard definitions of these concepts, see [3].)

II. Discretization with a First-Order Hold

We consider the system given by

$$\frac{dx}{dt} = A_c x + B_c u, \quad y = Cx, \tag{1}$$

where $A_c \in \boldsymbol{R}^{n\times n}$, $B_c \in \boldsymbol{R}^{n\times m}$, and $C \in \boldsymbol{R}^{p\times n}$. Suppose a first-order hold is connected to the input. Then, $u(t)$ is given by

$$u(t) = u(kT) + \frac{u(kT) - u(\overline{k-1}T)}{T}(t - kT) \qquad (kT \le t < \overline{k+1}T), \tag{2}$$

where T denotes the sampling period ($u(kT)$ stands for $u(kT+0)$). It should be noted that there is a built-in constraint that the input $u(t)$ ($kT \le t < \overline{k+1}T$) depends not only on $u(kT)$ but also on $u(\overline{k-1}T)$, which shows sharp contrast with a zero-order hold. In particular, $u(t)$ ($0 \le t < T$) depends on $u(-T)$, which has been determined before $t = 0$ and cannot be changed by the compensator at $t = 0$.

The resulting sampled-data system can be described by the equation (see [4], [5])

$$\begin{aligned} x(\overline{k+1}T) &= Ax(kT) + B^+ u(kT) + B^- u(\overline{k-1}T), \\ y(kT) &= Cx(kT) \end{aligned} \tag{3}$$

where,

$$A = \exp(A_c T), \quad B^+ = \int_0^T \left(2 - \frac{t}{T}\right) \exp(A_c t) B_c \, dt,$$

$$B^- = \int_0^T \left(\frac{t}{T} - 1\right) \exp(A_c t) B_c \, dt. \tag{4}$$

We denote the system (3) by S_1, which can be rewritten in the form of the ordinary discrete-time state equation as

$$\begin{bmatrix} x(\overline{k+1}T) \\ u(kT) \end{bmatrix} = \begin{bmatrix} A & B^- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(kT) \\ u(\overline{k-1}T) \end{bmatrix} + \begin{bmatrix} B^+ \\ I_m \end{bmatrix} u(kT), \tag{5}$$

$$y(kT) = [C \;\; 0] \begin{bmatrix} x(kT) \\ u(\overline{k-1}T) \end{bmatrix}. \tag{6}$$

III. Controllability and Reachability of S_1

A. Definitions of Controllability and Reachability and Their Equivalence

In this subsection, we first give the definitions of the controllability and reachability of S_1. In view of the discrete-time state equation (5), let us adopt the following definition.

Definition 1: S_1 is controllable if the pair (A_1, B_1) is controllable, where

$$(A_1, B_1) := \left(\begin{bmatrix} A & B^- \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B^+ \\ I_m \end{bmatrix} \right). \tag{7}$$

Now, let us verify that the above formal definition matches our practical control purposes in spite of the built-in constraint of a first-order hold.

If we regard S_1 simply as an ordinary discrete-time system, then its controllability might be defined as the property that, given any initial condition $x(0)$, there exists a sequence $u(kT)$ ($k = 0, \cdots, {}^{\exists}N-1$) such that $x(NT) = 0$. However, this is not appropriate, because this property does not reflect real purposes of control. Namely, this definition does not always imply the property that there exists a sequence $u(kT)$ ($k = 0, \cdots, N-1, N, \cdots$) such that $x(t) = 0$ (${}^{\forall}t \ge NT$), because of the built-in constraint of a first-order hold as discussed in the previous section. (This can be easily understood if we notice that $u(t) = 0$ ($NT \le t < \overline{N+1}T$) implies $u(\overline{N-1}T) = u(NT) = 0$ from (2).) Therefore, to define controllability, we must require that there exists a sequence $u(kT)$ ($k = 0, \cdots, {}^{\exists}N-2$) such that this together with $u(\overline{N-1}T) = 0$ implies $x(NT) = 0$. Likewise, as discussed in the previous section, $u(t)$ ($0 \le t < T$) is constrained by the unprescribable value $u(-T)$. Despite this constraint, $x(NT)$ is required to be made 0.

From the above consideration, the controllability of S_1 should be defined as the property that, given any initial conditions $x(0)$ and $u(-T)$, there exists a sequence $u(kT)$ ($k = 0, \cdots, {}^{\exists}N-2$) such that this together with $u(\overline{N-1}T) = 0$ implies $x(NT) = 0$. Obviously, this definition is equivalent to the controllability of the discrete-time system (5). Thus, validity of the above definition is assured.

Similarly, we are led to the following definition (see [6] for details).

Definition 2: S_1 is reachable if the pair (A_1, B_1) is reachable.

Now, in spite of the singularity of A_1, we can establish the following result (the straightforward proof [6] is omitted here).

Theorem 1: S_1 is reachable if and only if it is controllable.

B. Condition for Preservation of Reachability

In this subsection, we study the necessary and sufficient condition for the reachability of S_1 in terms of A_c, B_c and T. From Definition 2, it is reachable if and only if

$$\text{rank} \begin{bmatrix} A - zI_n & B & B^+ \\ 0 & I_m - zI_m & I_m \end{bmatrix} = n + m \quad ({}^{\forall}z \in \mathbf{C}) \tag{8}$$

where $B := B^+ + B^- = \int_0^T \exp(A_c t) B_c \, dt$. (Note that (A, B) is nothing but the pair of the sampled-data system obtained by the discretization of (1) with a zero-order hold, which we denote by S_0.) The condition (8) is nothing but the reachability condition for the pair

$$(A_2, B_2) := \left(\begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}, \begin{bmatrix} B^+ \\ I_m \end{bmatrix} \right). \tag{9}$$

This pair can be regarded as the pair obtained by the discretization of the fictitious T-dependent continuous-time pair

$$(A_{2c}, B_{2c}) := \left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B_c \\ I_m/T \end{bmatrix} \right) \tag{10}$$

with a zero-order hold, because $A_2 = \exp(A_{2c}T)$, $B_2 = \int_0^h \exp(A_{2c}t) B_{2c} \, dt$.

Remark 1: (A_{2c}, B_{2c}) is reachable if and only if a) (A_c, B_c) is reachable and b) A_c does not have the eigenvalue $-1/T$.

Since the eigenvalues and left eigenvectors of A_{2c} are not dependent on T, we can apply the necessary and sufficient condition [7] for the reachability of S_0 to the pair (A_{2c}, B_{2c}). Then, the following theorem is obtained (see Appendix for proof).

Theorem 2: S_1 is reachable if and only if the following two conditions are satisfied.

a) S_0 is reachable.

b) A_c does not have the eigenvalue $-1/T$.

Remark 2: Suppose that we define the stabilizability of S_1 by the stabilizability of the pair (A_1, B_1). Then, the necessary and sufficient condition for the stabilizability of S_1 is given by the stabilizability of S_0 (the condition b) can be dropped). Although we can rewrite the reachability/stabilizability of S_0 as the conditions on A_c, B_c, and T (see [7], and Theorem A.1 in Appendix), we did not do this because the importance of the theorem seems to be much clearer in the present form of the statement.

IV. Observability and Reconstructibility of S_1

As in the preceding section, let us consider how to define the observability and reconstructibility of S_1, taking account of practical purposes.

If we regard S_1 simply as an ordinary discrete-time system, then its observability might be defined as the property that its initial state $x(0)$ can be uniquely determined from the input data $u(kT)$ $(k = 0, \cdots, {}^{\exists}N - 1)$ and the output data $y(kT)$ $(k = 0, \cdots, N)$. However, this is not appropriate, because $u(t)$ $(0 \leq t < T)$ cannot be known completely from the knowledge of the above input data, as discussed in Section II, and it is clearly impossible to determine $x(0)$ under this lack of knowledge. Therefore, to define observability, we must assume that $u(t)$ $(0 \leq t < T)$ is also known. This assumption is equivalent to the assumption that $u(-T)$ as well as the above input and output data can be used. Noting that $x(0)$ can be determined uniquely if and only if $[x(0)^T, u(-T)^T]^T$ can be determined uniquely (if we know $u(-T)$), we are led to the following definition.

Definition 3: S_1 is observable if the pair (C_1, A_1) is observable, where

$$C_1 := \begin{bmatrix} C & 0 \\ 0 & I_m \end{bmatrix}. \tag{11}$$

Similarly, we are led to the following definition (see [6] for details).

Definition 4: S_1 is reconstructible if the pair $(\hat{C}_1, A_1)$ is reconstructible, where

$$\hat{C}_1 := [C \quad 0]. \tag{12}$$

$(\hat{C}_1, A_1)$ is reconstructible if and only if (C_1, A_1) is reconstructible. Furthermore, in spite of the singularity of A_1, we can readily show that (C_1, A_1) is reconstructible if and only if it is observable. Thus we obtain the following result.

Theorem 3: S_1 is observable if and only if it is reconstructible.

Now, from Definition 3, S_1 is observable if and only if

$$\text{rank} \begin{bmatrix} A - zI_n & B^- \\ 0 & -zI_m \\ C & 0 \\ 0 & I_m \end{bmatrix} = n + m \quad ({}^{\forall} z \in \mathbf{C}). \tag{13}$$

Since (C, A) is the pair of S_0, we readily obtain the following theorem.

Theorem 4: S_1 is observable if and only if S_0 is observable.

Remark 3: Suppose that we define the detectability of S_1 by the detectability of the pair (C_1, A_1). Then, the necessary and sufficient condition for the detectability of S_1 is given by the detectability of S_0. The condition for observability/detectability of S_0 in terms of C, A_c and T is given by [7] (see also Remark A.1 in Appendix).

V. Conclusion

In this paper, we studied the use of a first-order hold in the context of the state-space approach of control system design. We first studied how to define the controllability and reachability for the sampled-data system S_1 obtained by the discretization of a linear time-invariant continuous-time system with a first-order hold, taking account of the built-in constraint of a first order hold. Next, we showed the equivalence of these two concepts for S_1. Then, we studied the necessary and sufficient condition for the reachability of S_1 in terms of the parameters of the continuous-time system and the sampling period. We also gave similar results for observability and reconstructibility. In particular, it turned out that S_1 is reachable only if S_0 is reachable, while S_1 is observable if and only if S_0 is observable, where S_0 is the sampled-data system for the zero-order hold case. The compensator design problem under the use of a first-order hold is also studied in [6].

Appendix
Proof of Theorem 2

Before proving Theorem 2, we give a more comprehensible statement of the necessary and sufficient condition for the reachability of S_0 derived in [7].

Let $\lambda(A_c)$ denote the set of the eigenvalues of A_c. For each $\lambda_i \in \lambda(A_c)$, we define

$$\Lambda(\lambda_i) := \{\lambda \mid \lambda \in \lambda(A_c), \operatorname{Re}(\lambda) = \operatorname{Re}(\lambda_i), \\ \operatorname{Im}(\lambda) - \operatorname{Im}(\lambda_i) = 2k\pi/T \quad (k = 0, \pm 1, \pm 2, \cdots)\}. \tag{14}$$

Note that $\lambda_i \in \Lambda(\lambda_i)$, and that $\Lambda(\lambda_i) = \Lambda(\lambda_j)$ if $\lambda_j \in \Lambda(\lambda_i)$. Our interest is only in the sets $\Lambda(\lambda_i)$ which have at least two elements. Let Λ_l $(l = 1, \cdots, L)$ be such distinct sets, where we assume that Λ_l $(l = 1, \cdots, L^+ (\leq L))$ are the sets corresponding to the eigenvalues with nonnegative real parts (L and L^+ might be zero). We denote the elements of Λ_l by λ_{lk} $(k = 1, \cdots, K_l)$.

Next, for each $\lambda_i \in \lambda(A_c)$, we define

$$\Gamma(\lambda_i) := \begin{bmatrix} \eta_{i1}^T \\ \vdots \\ \eta_{i\nu_i}^T \end{bmatrix}, \tag{15}$$

where ν_i denotes the geometric multiplicity of the eigenvalue λ_i and η_{ik}^T $(k = 1, \cdots, \nu_i)$ the corresponding linearly independent left eigenvectors. That is to say, all the linearly independent left eigenvectors of A_c corresponding to the eigenvalue λ_i form the rows of $\Gamma(\lambda_i)$. We further define

$$\Gamma(\Lambda_l) := \begin{bmatrix} \Gamma(\lambda_{l1}) \\ \vdots \\ \Gamma(\lambda_{lK_l}) \end{bmatrix} \tag{16}$$

for $l = 1, \cdots, L$. That is to say, all the linearly independent left eigenvectors of A_c corresponding to the eigenvalues in the set Λ_l form the rows of $\Gamma(\Lambda_l)$.

Now, we obtain the following theorem, which is merely a restatement of Theorem 2 of [7].

Theorem A.1: S_0 is reachable (respectively, stabilizable) if and only if the following three conditions hold:

a) (A_c, B_c) is reachable (respectively, stabilizable).

b) A_c does not have the nonzero eigenvalue

$$j2k\pi/T \quad (k = \pm 1, \pm 2, \cdots). \tag{17}$$

c) $\Gamma(\Lambda_l)B_c$ has full row rank for $l = 1, \cdots, L$ (respectively, $l = 1, \cdots, L^+$).

Remark A.1: The conditions for the observability and the detectability of S_0 are given by the dual of the conditions a) and c) of the above theorem. (This is a restatement of Theorem 3 of [7].)

Remark A.2: Note that $\Gamma(\Lambda_l)$ never contains the left eigenvectors for the zero eigenvalue of A_c if the condition b) holds.

Now, we give the proof of Theorem 2.

Proof of Theorem 2: Without loss of generality, we assume that

$$A_c = \begin{bmatrix} \hat{A}_c & 0 \\ 0 & Z \end{bmatrix}, \quad B_c = \begin{bmatrix} \hat{B}_{c1} \\ \hat{B}_{c2} \end{bmatrix}, \tag{18}$$

where $\hat{A}_c$ is nonsingular and all the eigenvalues of Z are zero. Then, applying the similarity transformation by the matrix

$$\begin{bmatrix} I & 0 & -\hat{A}_c^{-1}\hat{B}_{c1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{19}$$

to the pair (10), we obtain the pair

$$\left(\begin{bmatrix} \hat{A}_c & 0 & 0 \\ 0 & Z & \hat{B}_{c2} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} (I + \hat{A}_c^{-1}/T)\hat{B}_{c1} \\ \hat{B}_{c2} \\ I/T \end{bmatrix} \right). \tag{20}$$

Applying Theorem A.1 to the pair (10) for the conditions a) and b), and to the pair (20) for the condition c), and taking Remarks 1 and A.2 into account, we can obtain the following necessary and sufficient condition for the reachability of S_1:

A1) (A_c, B_c) is reachable.

A2) A_c does not have the eigenvalue $-1/T$.

B) A_c does not have the nonzero eigenvalue

$$j2k\pi/T \quad (k = \pm 1, \pm 2, \cdots). \tag{21}$$

C) $\hat{\Gamma}(\hat{\Lambda}_l)(I + \hat{A}_c^{-1}/T)\hat{B}_{c1}$ has full row rank for $l = 1, \cdots, \hat{L}$, where $\hat{\Lambda}_l$, $\hat{\Gamma}(\hat{\Lambda}_l)$ and $\hat{L}$ are defined for $\hat{A}_c$ in a consistent way with the above definitions of Λ_l, $\Gamma(\Lambda_l)$ and L.

Since the rows of $\hat{\Gamma}(\hat{\Lambda}_l)$ are the left eigenvectors of $\hat{A}_c$ by definition, under condition A2) the condition C) is equivalent to

C') $\hat{\Gamma}(\hat{\Lambda}_l)\hat{B}_{c1}$ has full row rank for $l = 1, \cdots, \hat{L}$.

In view of the form of (18), the conditions B) and C') are equivalent to the conditions b) and c) of Theorem A.1. Since the condition A1) is the same as the condition a) of Theorem A.1, and since the condition (A2) is the same as the condition b) of Theorem 2, the proof has become complete. Q.E.D.

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