FR-Operator Approach to the H_2 Analysis and Synthesis of Sampled-Data Systems

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Abstract— **Recently, a frequency-domain operator called FR-operator (where FR stands for frequency response) was defined and shown to represent the transfer characteristics of a stable sampled-data system. Using this novel frequency-domain notion and introducing its extended notion called hybrid FR-operator, we define an H_2-norm for sampled-data systems in this paper. Then, sampled-data H_2 control problems are formulated and solved, whereby the usefulness of these frequency-domain notions is demonstrated both in the analysis and synthesis aspects of sampled-data systems. For the case of sampled-data systems with hybrid (i.e., both continuous-time and discrete-time) input and output signals, the H_2-norm defined by a hybrid FR-operator turns out to be slightly different from that defined in previous studies. The source of the discrepancy is also identified.**

I. Introduction

THE study of sampled-data systems taking account of intersample behavior has been attracting much interest for the last several years (see e.g., [1]–[4] and their references). The H_∞ and H_2 control problems are naturally extended to such sampled-data setting, and their solutions have been derived by different approaches, e.g., in [1], [3]–[5] and [6]–[13], respectively. In many of these studies, the lifting technique (see, e.g., [2] and [3]) plays an important role.

Recently, another key technique for the analysis and synthesis of sampled-data systems was developed by the authors and their colleague [14]–[16] based on frequency-domain considerations (see also [17]–[19] for seemingly different but equivalent studies in which the relationship to the lifting technique is clearer). Namely, a frequency-domain operator called FR-operator was defined, where FR stands for frequency response, and it was shown to represent the "steady-state" transfer characteristics (with intersample behavior taken into account) of a stable sampled-data system for sinusoidal inputs of a generalized sense. Furthermore, the sensitivity and complementary sensitivity FR-operators were introduced and were shown to possess the properties corresponding to the sensitivity and complementary sensitivity functions of continuous-time systems. From these facts, it was suggested that FR-operators can be a tool which is powerful enough for the analysis and synthesis of sampled-data systems. The emphasis of the previous papers [14]–[16] was placed on the analysis aspect rather than on the synthesis aspect. The purposes of this paper lie in introducing an extended notion called hybrid FR-operator and in demonstrating that the FR-operator and the hybrid FR-operator can be a tool for the synthesis of sampled-data control systems, not merely in giving a frequency-domain method for the analysis and synthesis of sampled-data systems. In this paper, we study the H_2 problem of sampled-data systems using FR-operators; H_∞ and related problems are studied in [20] and [21].

The contents of this paper are as follows. In the first part, a sampled-data system with only continuous-time input and output signals is studied. In Section II, the notation and basic facts used in this paper are summarized, and the notion of an FR-operator is quickly reviewed. In Section III, the definition of an H_2-norm of the sampled-data system is introduced in terms of an FR-operator, and a sampled-data H_2 control problem is formulated; it is solved in Section IV using only elementary frequency-domain notions such as (conventional) z-transformation, pulse transfer function, and impulse modulation formula. In Section V, the H_2-norm of a given stable sampled-data system is computed by which the H_2-norm defined in terms of the FR-operator is shown equivalent to those definitions employed in the previous studies [7]–[11]. In Section VI, the above arguments are generalized to the case of a sampled-data system with hybrid (i.e., both continuous-time and discrete-time) input and output signals so that the effect of discrete-time signals such as measurement noises and quantization errors can be taken into account. For this purpose, a hybrid FR-operator is introduced, by which an H_2-norm of such a sampled-data system is defined. It will turn out that, for the hybrid case, the definition based on the hybrid FR-operator slightly differs from those in the previous studies [7], [9]–[11]. The source of the discrepancy is also identified. Section VII is the conclusion, where the results of this paper are summarized and some comments are given on its relationship to other related contributions [12], [13], [22].

II. Preliminaries

A. Notation and Facts

The notation and basic mathematical facts used in this paper are summarized in this subsection. Most of the notation is standard: X^T denotes the transpose of X, X^* denotes the complex conjugate transpose of X, and $\tilde{X}$ is defined as

$$\tilde{X}(s) := X(-s)^T \quad \text{(for a continuous-time transfer function matrix } X(s)), \tag{1}$$

$$\tilde{X}(z) := X(z^{-1})^T \quad \text{(for a discrete-time (pulse) transfer function matrix } X(z)). \tag{2}$$

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It is a fact that $X^{\sim}(j\varphi) = X(j\varphi)^*$ for a continuous-time $X(s)$ and $X^{\sim}(e^{j\varphi h}) = X(e^{j\varphi h})^*$ for a discrete-time $X(z)$. For the pulse transfer function matrix

$$X(z) = \begin{bmatrix} X_{11}(z) & X_{12}(z) \\ X_{21}(z) & X_{22}(z) \end{bmatrix} \tag{3}$$

its lower LFT (linear fractional transformation) with respect to $Y(z)$ is defined by

$$\mathcal{F}_l(X(z), Y(z)) := X_{11}(z) + X_{12}(z)(I - Y(z)X_{22}(z))^{-1} \cdot Y(z)X_{21}(z). \tag{4}$$

The H_2-norm of a stable pulse transfer function matrix is denoted by $\|\cdot\|_2$. The L_2-norm of a pulse transfer function matrix is also denoted by $\|\cdot\|_2$.

To facilitate the descriptions in this paper, we introduce the following shorthand notation about the trace of a matrix

$$X \overset{\text{tr}}{=} Y \Leftrightarrow \text{trace}(X) = \text{trace}(Y). \tag{5}$$

With a slight abuse of the notation, the following shorthand notation is also used

$$X \overset{\text{tr}}{=} Y + Z \Leftrightarrow \text{trace}(X) = \text{trace}(Y) + \text{trace}(Z). \tag{6}$$

Note that the sizes of the matrices X, Y, and Z may differ in the above two equations, and thus $Y + Z$ may be meaningless, in particular. In spite of this, the above shorthand notation helps us keep the equations concise and thus saves much space.

The following relation will be used repeatedly

$$XY \overset{\text{tr}}{=} YX. \tag{7}$$

The following fact is well known.

Lemma 1: Let $X(z) = X_0(z) + D$ be a stable rational pulse transfer function matrix, where $X_0(z)$ is strictly proper (i.e., $X_0(\infty) = 0$) and D is a constant matrix. Then

$$\|X(z)\|_2^2 = \|X_0(z)\|_2^2 + \text{trace}(D^T D). \tag{8}$$

B. FR-Operator of a Sampled-Data System

In this paper, we deal with the sampled-data control system shown in Fig. 1, where $P(s)$ is the finite-dimensional linear time-invariant (FDLTI) generalized plant given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] =: \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \tag{9}$$

and $\Psi(z)$ is a proper FDLTI discrete-time controller. $\mathcal{S}$ is the ideal sampler of sampling period h, and $\mathcal{H}$ is assumed to be the zero-order hold, for simplicity, whose transfer function is given by

$$H(s) = (1 - e^{-sh})/s. \tag{10}$$

Solid lines and dashed lines in the block diagram are used to denote continuous-time signals and discrete-time signals, respectively.

Throughout the paper, we assume the following:

A1) The plant is strictly proper (i.e., $D_{22} = 0$).

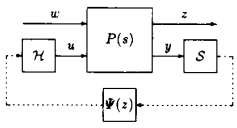

Fig. 1. Sampled-data control system with continuous-time input and output.

A2) (A, B_2) and (C_2, A) are respectively stabilizable and detectable.

In addition, we assume:

A3) $D_{21} = 0$

so that internal stability of the closed-loop system implies its L_2 stability [23]. Furthermore, we also assume

A4) $D_{11} = 0$

so that the H_2-norm introduced in this paper becomes finite. Finally, we assume:

A5) The matrix A has no eigenvalues on the imaginary axis.

This assumption is fairly strong, but here we assume it to avoid some technical difficulties especially in the arguments of Sections IV-B and V. To remove this assumption in a rigorous fashion, we would need some arguments about the pole-zero cancellation in the FR-operator representation, which would be an interesting topic in itself but will not be pursued here.

The purpose of this subsection is to provide a quick review of the notion of the FR-operator (for more details, refer to [14]–[16]). For this purpose, we assume that the sampled-data system is internally stable and define the signal set $\mathcal{X}_\varphi$ as the set of all signals having finite power and consisting of sinusoidal components with equally spaced frequencies φ_m, where

$$\varphi_m := \varphi + m\omega_s \quad (m = 0, \pm 1, \pm 2, \cdots). \tag{11}$$

Here, ω_s $(:= 2\pi/h)$ is the sampling angular frequency, and without loss of generality we assume

$$\varphi \in (-\omega_s/2, \omega_s/2]. \tag{12}$$

More specifically, the signal set $\mathcal{X}_\varphi$ is defined as

$$\mathcal{X}_\varphi := \left\{ x(t) | x(t) = \sum_{m=-\infty}^{\infty} x_m e^{j\varphi_m t}, \sum_{m=-\infty}^{\infty} \|x_m\|^2 < \infty \right\}. \tag{13}$$

We call a member of $\mathcal{X}_\varphi$ an SD-sinusoid of angular frequency φ, where SD stands for "sampled data." An SD-sinusoid is uniquely determined by φ and the bidirectional series (or generalized amplitude/phase) of the complex coefficient vectors x_m, which we denote by an infinite dimensional vector

$$\underline{x} = [\cdots, x_{-2}^T, x_{-1}^T, x_0^T, x_1^T, x_2^T, \cdots]^T \quad (\in l_2). \tag{14}$$

For each φ, we identify $\mathcal{X}_\varphi$ with l_2, and call $\underline{x}$ the (frequency-domain) l_2-expression of the SD-sinusoid $x(t)$. The importance of the signal set $\mathcal{X}_\varphi$ lies in the facts that the sampled-data system of Fig. 1 maps, in the steady state, $w \in \mathcal{X}_\varphi$ to $z \in \mathcal{X}_\varphi$ and that the mapping is bounded. From these facts, we can associate an operator $\mathcal{G}(j\varphi)$ with the sampled-data system of Fig. 1 whose input signals are

restricted to within the signal set $\mathcal{X}_\varphi$. We call this operator the FR-operator of the sampled-data system, where FR stands for "frequency response."

With respect to the standard basis for (14), the FR-operator $\mathcal{G}(j\varphi)$ can be represented by the infinite-dimensional matrix

$$\underline{G}(j\varphi) = \underline{P_{11}}(j\varphi) + \frac{1}{h}\underline{P_{12}H}(j\varphi)\cdot\Lambda(e^{j\varphi h})\cdot\underline{P_{21}}(j\varphi) \quad (15)$$

where $\cdot$ is an ordinary matrix multiplication, $\underline{P_{11}}(j\varphi)$, $\underline{P_{12}H}(j\varphi)$ and $\underline{P_{21}}(j\varphi)$ are respectively given by

$$\underline{P_{11}}(j\varphi) := \begin{bmatrix} \ddots & & & & O \\ & P_{11}(j\varphi_{-1}) & & & \\ & & P_{11}(j\varphi_0) & & \\ & & & P_{11}(j\varphi_1) & \\ O & & & & \ddots \end{bmatrix} \quad (16)$$

$$\underline{P_{12}H}(j\varphi) := \begin{bmatrix} \vdots \\ P_{12}(j\varphi_{-1})H(j\varphi_{-1}) \\ P_{12}(j\varphi_0)H(j\varphi_0) \\ P_{12}(j\varphi_1)H(j\varphi_1) \\ \vdots \end{bmatrix}, \quad (17)$$

$$\underline{P_{21}}(j\varphi) := [\cdots \quad P_{21}(j\varphi_{-1}) \quad P_{21}(j\varphi_0) \quad P_{21}(j\varphi_1) \quad \cdots] \quad (18)$$

and $\Lambda(z)$ is given by

$$\Lambda(z) := (I - \Psi(z)\Pi_{22}(z))^{-1}\Psi(z) \quad (19)$$

with Π_{22} defined as the "discretized version" of P_{22}

$$\Pi_{22}(z) = \left[\begin{array}{c|c} \exp(Ah) & \int_0^h \exp(At)B_2\,dt \\ \hline C_2 & 0 \end{array}\right]. \quad (20)$$

III. Definition of an H_2-Norm of a Sampled-Data System and a Sampled-Data H_2 Optimal Control Problem

In the preceding section, we introduced the FR-operator describing the steady-state transfer characteristics from w to z of the sampled-data system of Fig. 1. Now, we define an H_2-norm of this sampled-data system as follows.

Definition 1: The quantity

$$\|\mathcal{G}\|_2 := \left\{\frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi))\,d\varphi\right\}^{1/2} = h^{-1/2}\cdot\left\{\frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi))\,d\varphi\right\}^{1/2} \quad (21)$$

is the H_2-norm of the FR-operator $\mathcal{G}(j\varphi)$ ($\varphi \in (-\omega_s/2, \omega_s/2]$) associated with the input w and output z of the sampled-data system shown in Fig. 1.

Hereafter, we will simply call the above quantity the H_2-norm of the sampled-data system. It is easy to verify that the above definition is consistent with the definition of the H_2-norm of a continuous-time system in the following sense. Namely, by letting $\Psi(z) \equiv 0$ (and so $\Lambda(z) \equiv 0$) so that the mapping from w to z of the sampled-data system of Fig. 1 reduces to that of the continuous-time system $P_{11}(s)$, we can easily verify that (21) reduces to

$$\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}(P_{11}(j\omega)^*P_{11}(j\omega))\,d\omega\right\}^{1/2} \quad (22)$$

which is nothing but the definition of the H_2-norm of the continuous-time system $P_{11}(s)$ (assuming it is stable). Also, we can show that this H_2-norm gives the steady-state power of the output z for a white noise input w with the unit intensity (i.e., with the identity covariance matrix). Furthermore, Definition 1 will actually turn out to be equivalent to other definitions of an H_2-norm of the sampled-data system of Fig. 1 introduced by previous studies [7], [8], and [10], as will be discussed later. Showing the equivalence is also one of the topics of the present paper.

Now, having given the definition of an H_2-norm, the sampled-data H_2 optimal control problem reads as follows.

Problem 1: Suppose that the generalized plant (9) is given, where Assumptions A1)–A5) are satisfied. For a given sampling period h, find, if one exists, an optimal proper FDLTI discrete-time controller $\Psi(z)$ such that the closed-loop system of Fig. 1 is internally stable and the H_2-norm of the FR-operator from w to z is minimized.

IV. Solution to the Sampled-Data H_2 Optimal Control Problem

In this section, we show that the sampled-data H_2 optimal control problem posed in the preceding section can be reduced to an equivalent discrete-time H_2 control problem, as in the similar studies in [6]–[11]. The basic idea and technique used in our derivation are simple and are basically the same as the ones used to calculate the frequency response gains and solve an H_∞ control problem of a sampled-data system using FR-operator representations [20]. Unfortunately, however, if we simply describe the derivation process for a general setting, the underlying idea does not seem to become transparent enough. Rather, we believe that the heart of the idea could be most easily understood when we confine ourselves to the case of $P_{11}(s) \equiv 0$. For this reason, we first describe the idea for this particular case and then extend it to the general case.

A. The Case of $P_{11}(s) \equiv 0$

Let us first consider the case of $P_{11}(s) \equiv 0$. In this case, from (15) and relation (7), we have

$$\underline{G}(j\varphi)^*\underline{G}(j\varphi) \overset{\text{tr}}{=} \Lambda(e^{j\varphi h})^*\left(\frac{1}{h}\underline{P_{12}H}(j\varphi)^*\underline{P_{12}H}(j\varphi)\right) \cdot \Lambda(e^{j\varphi h})\left(\frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{21}}(j\varphi)^*\right). \quad (23)$$

Here, we have

$$\frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{21}}(j\varphi)^* = \frac{1}{h}\sum_{m=-\infty}^{\infty} P_{21}(j\varphi_m)\tilde{P_{21}}(j\varphi_m) = Z[P_{21}(s)\tilde{P_{21}}(s)]|_{z=\exp(j\varphi h)} \quad (24)$$

by the impulse modulation formula [24], where

$$Z[\cdot] := \mathcal{Z}\mathcal{S}\mathcal{L}^{-1}[\cdot] \tag{25}$$

with $\mathcal{L}^{-1}$ denoting the inverse Laplace transform and $\mathcal{Z}$ denoting the z-transform. As shown in Appendix A, we can do the factorization

$$Z[P_{21}(s)\tilde{P}_{21}(s)] = \Pi_{21}(z)\tilde{\Pi}_{21}(z) \tag{26}$$

with

$$\Pi_{21}(z) := \left[\begin{array}{c|c} \exp(Ah) & W \\ \hline C_2 & 0 \end{array}\right] \tag{27}$$

where W is any matrix such that

$$WW^T = \int_0^h \exp(At)B_1B_1^T\exp(A^Tt)\,dt. \tag{28}$$

From (24) and (26), we have

$$\frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{21}}(j\varphi)^* = \Pi_{21}(e^{j\varphi h})\Pi_{21}(e^{j\varphi h})^*. \tag{29}$$

Similarly, we have

$$\begin{aligned}\frac{1}{h}&\underline{P_{12}H}(j\varphi)^*\underline{P_{12}H}(j\varphi)\\ &= Z[\tilde{H}(s)\tilde{P}_{12}(s)P_{12}(s)H(s)]|_{z=\exp(j\varphi h)}\\ &= (1-z^{-1})^{\sim}\cdot Z[(P_{12}(s)/s)^{\sim}(P_{12}(s)/s)]\\ &\quad\cdot(1-z^{-1})|_{z=\exp(j\varphi h)}.\end{aligned} \tag{30}$$

Here, if we note

$$P_{12}(s)/s = \left[\begin{array}{cc|c} A & B_2 & 0 \\ 0 & 0 & I \\ \hline C_1 & D_{12} & 0 \end{array}\right] \tag{31}$$

then, by a dual argument, we can do the factorization

$$Z[(P_{12}(s)/s)^{\sim}(P_{12}(s)/s)] = \hat{\Pi}_{12}^{\sim}(z)\hat{\Pi}_{12}(z) \tag{32}$$

with

$$\begin{aligned}\hat{\Pi}_{12}(z) &:= \left[\begin{array}{c|c} \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}h\right) & \begin{matrix} 0 \\ I \end{matrix} \\ \hline V_1 \quad V_2 & 0 \end{array}\right]\\ &= \Pi_{12}(z)\cdot(z-1)^{-1}\\ &\left(\Pi_{12}(z) := \left[\begin{array}{c|c} \exp(Ah) & \int_0^h \exp(At)B_2\,dt \\ \hline V_1 & V_2 \end{array}\right]\right)\end{aligned} \tag{33}$$

where V_1 and V_2 are any matrices such that

$$\begin{aligned}[V_1 \quad V_2]^T[V_1 \quad V_2] &= \int_0^h \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}^T t\right)\\ &\quad\cdot[C_1 \quad D_{12}]^T[C_1 \quad D_{12}]\\ &\quad\cdot\exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}t\right)dt.\end{aligned} \tag{34}$$

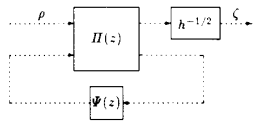

Fig. 2. Discrete-time control system corresponding to Fig. 1.

Now, since $(z-1)^{-1}(1-z^{-1}) = z^{-1}$ and $(z^{-1})^{\sim}z^{-1} = 1$, we obtain from (30), (32), and (33)

$$\frac{1}{h}\underline{P_{12}H}(j\varphi)^*\underline{P_{12}H}(j\varphi) = \Pi_{12}(e^{j\varphi h})^*\Pi_{12}(e^{j\varphi h}). \tag{35}$$

Now, substituting (29) and (35) into (23) and using (7) again, we obtain

$$\begin{aligned}\underline{G}(j\varphi)^*\underline{G}(j\varphi) &\stackrel{\text{tr}}{=} (\Pi_{12}(e^{j\varphi h})\Lambda(e^{j\varphi h})\Pi_{21}(e^{j\varphi h}))^*\\ &\quad\cdot(\Pi_{12}(e^{j\varphi h})\Lambda(e^{j\varphi h})\Pi_{21}(e^{j\varphi h})).\end{aligned} \tag{36}$$

Therefore, from (21), we have

$$||\mathcal{G}||_2 = h^{-1/2}||\Pi_{12}(z)\Lambda(z)\Pi_{21}(z)||_2 \tag{37}$$

where $||\cdot||_2$ in the right-hand side denotes the L_2-norm of a pulse transfer function matrix.

Now let us introduce the discrete-time generalized plant $\Pi(z)$ given by

$$\Pi(z) = \left[\begin{array}{c|cc} \exp(Ah) & W & \int_0^h \exp(At)B_2\,dt \\ \hline V_1 & 0 & V_2 \\ C_2 & 0 & 0 \end{array}\right]. \tag{38}$$

Then, noting that $C_1\exp(At)B_1 \equiv 0$ from the assumption $P_{11}(s) = 0$, it is easy to verify

$$\Pi(z) = \begin{bmatrix} 0 & \Pi_{12}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) \end{bmatrix}. \tag{39}$$

From the above equation and from (19), it follows that

$$\Pi_{12}(z)\Lambda(z)\Pi_{21}(z) = \mathcal{F}_l(\Pi(z), \Psi(z)) \tag{40}$$

where $\mathcal{F}_l$ denotes the lower LFT. This, together with (37), means that the H_2-norm of the sampled-data system of Fig. 1 coincides with that of the discrete-time system of Fig. 2, provided that both are stable. From the fact that Π_{22} is the "discretized version" of P_{22}, it readily follows that the latter discrete-time system is internally stable if and only if the former sampled-data system is. From these arguments, we can conclude the following (temporarily under the assumption $P_{11}(s) \equiv 0$).

Theorem 1: The task of finding an optimal discrete-time controller $\Psi(z)$ for Problem 1 is equivalent to the discrete-time H_2 control problem of finding an optimal controller $\Psi(z)$ such that the closed-loop system of Fig. 2 is internally stable and the H_2-norm of the pulse transfer function matrix from ρ to ζ (i.e., $h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))$) is minimized, where $\Pi(z)$ is given by (38).

B. The General Case

This subsection is devoted to the proof of Theorem 1 in the general case. Although the result has been shown by different approaches [7]–[11], our proof again emphasizes the usefulness of the impulse modulation formula in the frequency-domain study of sampled-data systems.

Proof of Theorem 1: In the general case, from (15) we have

$$\begin{aligned}\underline{G}(j\varphi)^*\underline{G}(j\varphi) \overset{\text{tr}}{=} & \Lambda(e^{j\varphi h})^* \cdot \left(\frac{1}{h}\underline{P_{12}H}(j\varphi)^*\underline{P_{12}H}(j\varphi)\right) \\ & \cdot \Lambda(e^{j\varphi h}) \cdot \left(\frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{21}}(j\varphi)^*\right) \\ & + \Lambda(e^{j\varphi h})^* \cdot \frac{1}{h}\underline{P_{12}H}(j\varphi)^*\underline{P_{11}}(j\varphi)\underline{P_{21}}(j\varphi)^* \\ & + \frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{11}}(j\varphi)^*\underline{P_{12}H}(j\varphi) \\ & \cdot \Lambda(e^{j\varphi h}) + h \cdot \frac{1}{h}\underline{P_{11}}(j\varphi)^*\underline{P_{11}}(j\varphi). \end{aligned} \tag{41}$$

The first term in the right-hand side is nothing but the one we dealt with in the preceding subsection, so we must evaluate the remaining three terms. In the same manner as in the preceding subsection, the last term can be easily evaluated, and we obtain

$$h \cdot \frac{1}{h}\underline{P_{11}}(j\varphi)^*\underline{P_{11}}(j\varphi) \overset{\text{tr}}{=} h\Pi_0(e^{j\varphi h})^*\Pi_0(e^{j\varphi h}) \tag{42}$$

where $\Pi_0(z)$ is given by

$$\Pi_0(z) := \left[\begin{array}{c|c} \exp(Ah) & B_1 \\ \hline V_1 & 0 \end{array}\right] \tag{43}$$

with V_1 same as in (34).

The evaluation of the second and third terms of (41) is a little troublesome, but, basically by using the same technique (i.e., the impulse modulation formula) as before, we can show that ([25])

$$\begin{aligned} & \frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{11}}(j\varphi)^*\underline{P_{12}H}(j\varphi) \cdot \Lambda(e^{j\varphi h}) \\ & \quad = \Pi_{21}(e^{j\varphi h})\hat{\Pi}_{11}^*(e^{j\varphi h})\Pi_{12}(e^{j\varphi h}) \cdot \Lambda(e^{j\varphi h}) \quad (44) \\ & \quad \overset{\text{tr}}{=} \hat{\Pi}_{11}(e^{j\varphi h})^*\Pi_{12}(e^{j\varphi h})\Lambda(e^{j\varphi h})\Pi_{21}(e^{j\varphi h}) \quad (45) \end{aligned}$$

where $\hat{\Pi}_{11}(z)$ is given by

$$\hat{\Pi}_{11}(z) = \left[\begin{array}{c|c} \exp(Ah) & W \\ \hline V_1 & \Delta \end{array}\right]. \tag{46}$$

Here, Δ is any matrix satisfying

$$\begin{aligned} W\Delta^T[V_1 \quad V_2] = & \int_0^h \int_t^h \exp(A\sigma)B_1B_1^T \\ & \cdot \exp(A^T(\sigma - t))C_1^T[C_1 \quad D_{12}] \\ & \cdot \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(h-t)\right) d\sigma\, dt. \end{aligned} \tag{47}$$

The explicit representation for Δ is not necessary in the following argument, but it is important to note that a solution to the above equation always exists because of the form of the equation and the definitions of W and $[V_1 \quad V_2]$. Also note that Δ is independent of the controller $\Psi(z)$. Now, for later use, introduce

$$\Pi_{11}(z) := \left[\begin{array}{c|c} \exp(Ah) & W \\ \hline V_1 & 0 \end{array}\right] \tag{48}$$

so that $\Pi(z)$ of (38) becomes

$$\Pi(z) = \begin{bmatrix} \Pi_{11}(z) & \Pi_{12}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) \end{bmatrix} \tag{49}$$

and

$$\hat{\Pi}_{11}(z) = \Pi_{11}(z) + \Delta. \tag{50}$$

Combining the results summarized in (36), (42), and (45) and rearranging the results, we obtain

$$\begin{aligned} & \underline{G}(j\varphi)^*\underline{G}(j\varphi) \\ & \quad \overset{\text{tr}}{=} (\hat{\Pi}_{11}(e^{j\varphi h}) + \Pi_{12}(e^{j\varphi h})\Lambda(e^{j\varphi h})\Pi_{21}(e^{j\varphi h}))^* \\ & \qquad \cdot (\hat{\Pi}_{11}(e^{j\varphi h}) + \Pi_{12}(e^{j\varphi h})\Lambda(e^{j\varphi h})\Pi_{21}(e^{j\varphi h})) \\ & \qquad + h\Pi_0(e^{j\varphi h})^*\Pi_0(e^{j\varphi h}) - \hat{\Pi}_{11}^*(e^{j\varphi h})\hat{\Pi}_{11}(e^{j\varphi h}). \end{aligned} \tag{51}$$

Therefore, from (21), we obtain

$$\|\mathcal{G}\|_2^2 = h^{-1}\|\hat{\Pi}_{11}(z) + \Pi_{12}(z)\Lambda(z)\Pi_{21}(z)\|_2^2 + \hat{\gamma} \tag{52}$$

where

$$\hat{\gamma} = \|\Pi_0(z)\|_2^2 - h^{-1}\|\hat{\Pi}_{11}(z)\|_2^2 \tag{53}$$

where $\|\cdot\|_2$ in the right-hand sides of (52) and (53) denotes L_2-norm. Since $\hat{\gamma}$ is independent of the controller $\Psi(z)$, minimizing $\|\mathcal{G}\|_2$ is equivalent to minimizing $h^{-1/2}\|\hat{\Pi}_{11}(z) + \Pi_{12}(z)\Lambda(z)\Pi_{21}(z)\|_2$. Now, from (49) and (50), we can easily verify that

$$\begin{aligned} & \hat{\Pi}_{11}(z) + \Pi_{12}(z)\Lambda(z)\Pi_{21}(z) \\ & \quad = \mathcal{F}_l(\Pi(z) + \text{diag}\,[\Delta, 0], \Psi(z)). \end{aligned} \tag{54}$$

Therefore, by the same argument as in the preceding subsection, we can conclude that the task of finding an optimal controller for Problem 1 reduces to that for an equivalent discrete-time H_2 control problem for the system shown in Fig. 2 with $\Pi(z)$ replaced by $\Pi(z) + \text{diag}\,[\Delta, 0]$, including stability constraint. Although Δ here is given as a solution to (47) and is nonzero in general, an optimal controller for the discrete-time H_2 control problem is independent of Δ, because the D_{21}-matrix of the realization of $\Pi(z)$ is zero. Therefore, as long as the task of finding an optimal controller is concerned, we may regard Δ as if it were zero. This implies that Theorem 1 is true even in the general case. Q.E.D.

V. Computation of the H_2-Norm and Its Equivalence to Other Definitions

In the preceding section, we showed that the task of finding an optimal controller for Problem 1 can be reduced to that for an equivalent discrete-time H_2 problem. The optimal H_2-norm for the equivalent discrete-time problem, however, does not give the optimal H_2-norm of the original sampled-data control

system, in general, because of the presence of the term $\hat{\gamma}$ in (52) and the neglected direct-feedthrough term Δ. Motivated by this issue, in this section we clarify the relationship between $||\mathcal{G}||_2$ and $||h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))||_2$. Specifically, we give an explicit representation for

$$\gamma := ||\mathcal{G}||_2^2 - ||h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))||_2^2 \tag{55}$$

in terms of the state-space representation of $P(s)$. Note from the arguments in the preceding section that γ is independent of the choice of the stabilizing controller $\Psi(z)$. Since $||h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))||_2$ can be calculated using a standard state-space formula, the representation of γ will provide us with a method for computing the H_2-norm of a given (not necessarily H_2-optimal) stable sampled-data system. Also, equivalence of the H_2-norm definition (21) to other definitions [7], [8], [10] (for the sampled-data system of Fig. 1) will turn out to follow as a consequence of that representation.

The main result of this section is as follows (the proof is given in Appendix B).

Theorem 2: The H_2-norm $||\mathcal{G}||_2$ of the FR-operator $\mathcal{G}(j\varphi)$ ($\varphi \in (-\omega_s/2, \omega_s/2]$) associated with the input w and output z of the stable sampled-data system of Fig. 1 is given by

$$||\mathcal{G}||_2 = \{||h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))||_2^2 + \gamma\}^{1/2} \tag{56}$$

where $\Pi(z)$ is given by (38), γ is given by

$$\gamma = h^{-1}\,\mathrm{trace}\left(\int_0^h \int_0^\sigma B_1^T \exp(A^T t) C_1^T C_1 \cdot \exp(At) B_1\, dt\, d\sigma\right) \tag{57}$$

and $||\cdot||_2$ in the right-hand side denotes the H_2-norm of a stable pulse transfer function matrix [see Fig. 2 for the system corresponding to $h^{-1/2}\mathcal{F}_l(\Pi(z), \Psi(z))$].

Since the above result is (essentially) the same as the results given in [7]–[11], we can conclude that the H_2-norm definition (21) is equivalent to the ones employed in those previous studies. Namely, we have the following result. (For the definitions of the operator-valued transfer function matrix $\hat{G}(z)$ and its H_{HS}^2-norm, refer to [8].)

Theorem 3: The H_2-norm $||\mathcal{G}||_2$ of the FR-operator $\mathcal{G}(j\varphi)(\varphi \in (-\omega_s/2, \omega_s/2])$ associated with the input w and output z of the stable sampled-data system of Fig. 1 coincides with the H_{HS}^2-norm of the operator-valued transfer function matrix $\hat{G}(z)$ associated with the lifted equivalent (with respect to $L_2[0, h]$) of the mapping from w to z of the sampled-data system.

A result similar to Theorem 3 has been shown for the H_∞-norm in a stronger form [17], which in fact can be stated more explicitly as Theorem 4 below. Actually, Theorem 3 is also a direct consequence of this theorem.

Theorem 4: All the singular values of $\underline{G}(j\varphi)$ and $\hat{G}(e^{j\varphi h})$ coincide for each $\varphi \in (-\omega_s/2, \omega_s/2]$.

Proof: As implied in the arguments of [17] and [19], $\underline{G}(j\varphi)$ and $\hat{G}(e^{j\varphi h})$ are related by

$$\underline{G}(j\varphi)F_\varphi = F_\varphi \hat{G}(e^{j\varphi h}) \tag{58}$$

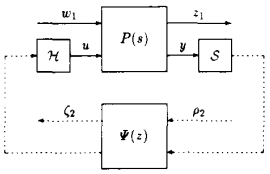

Fig. 3. Sampled-data control system with hybrid input and output.

where

$$F_\varphi : x(t) \in L_2[0, h] \mapsto \underline{x} = [\cdots, x_{-1}^T, x_0^T, x_1^T, \cdots]^T \in l_2 \tag{59}$$

denotes the Fourier-like transformation defined by

$$x_m = \frac{1}{h}\int_0^h e^{-j\varphi_m t} x(t)\, dt. \tag{60}$$

Since $F_\varphi^* F_\varphi$ equals h times the identity operator, the assertion follows readily from (58). Q.E.D.

VI. Extension to the Case of a Sampled-Data System with Hybrid Input and Output Signals

A. Hybrid FR-Operator

In this section, we consider the case where a sampled-data system has hybrid input and output signals (i.e., both continuous-time and discrete-time input and output signals) and extend the results obtained in the above to such a case. By such an extension, the effect of discrete-time signals such as measurement noises and quantization errors can be taken into account. For example, A/D converters may produce some discrete-time noises [7], [9] (or, as in the classical approach to sampled-data LQG (linear quadratic Guassian) problems [26], "sampled continuous-time measurement noises" may be treated as discrete-time measurement noises). Alternatively, quantization errors (including the finite-word-length effect) may be analyzed by approximating them as uniformly distributed noises over the quantization width and further approximating by Gaussian white noises (in terms of the first and second moments).

Let us consider the sampled-data system shown in Fig. 3 with hybrid input and output, where $P(s)$ satisfies Assumptions A1)–A5) as before, and $\Psi(z)$ is a proper FDLTI pulse transfer function matrix given by

$$\Psi(z) = \begin{bmatrix} \Psi_{11}(z) & \Psi_{12}(z) \\ \Psi_{21}(z) & \Psi_{22}(z) \end{bmatrix}. \tag{61}$$

We assume that the closed-loop system is internally stable.

Now, let us introduce the signal set Ξ_φ of all discrete-time sinusoids of angular frequency φ and period h. Specifically, we define

$$\Xi_\varphi := \{\{\xi_k\} | \xi_k = \xi e^{j\varphi k h}\} \tag{62}$$

where ξ is a finite-dimensional complex vector representing the amplitude and phase. Furthermore, we define the hybrid

signal space

$$\tilde{\mathcal{X}}_\varphi := \left\{ \begin{bmatrix} x(t) \\ \{\xi_k\} \end{bmatrix} \middle| x(t) \in \mathcal{X}_\varphi, \quad \{\xi_k\} \in \Xi_\varphi \right\} \tag{63}$$

and call its member a hybrid SD-sinusoid of angular frequency φ. As was the case for an SD-sinusoid, a hybrid SD-sinusoid is uniquely determined by φ and the generalized amplitude/phase

$$\underline{\tilde{x}} = \begin{bmatrix} \underline{x} \\ \xi \end{bmatrix} (\in l_2) \tag{64}$$

where, with a slight abuse of notation, we identified $l_2 \oplus C^{\dim(\xi)}$ with l_2, with its norm defined by $||\underline{\tilde{x}}|| := (||\underline{x}||^2 + ||\xi||^2)^{1/2}$. We call (64) the l_2-expression of a hybrid SD-sinusoid of angular frequency φ. Using the same technique as in [14]–[16], we can readily show that the sampled-data system of Fig. 3 maps, in the steady state, an input signal $[w_1^T, \rho_2^T]^T \in \tilde{\mathcal{X}}_\varphi$ to an output signal $[z_1^T, \zeta_2^T]^T$ within the identical set. Furthermore, the mapping is bounded on l_2. From these facts, we can associate an operator $\tilde{\mathcal{G}}(j\varphi)$ with the hybrid input/output sampled-data system of Fig. 3 whose input signals are restricted to within the signal set $\tilde{\mathcal{X}}_\varphi$. We call this operator the hybrid FR-operator of the sampled-data system shown in Fig. 3. It can be regarded as a mapping on l_2, and, with respect to the standard basis for the l_2-expression (64), its matrix representation is given by

$$\underline{\tilde{G}}(j\varphi) = \begin{bmatrix} \underline{G}(j\varphi) & h^{-1}\underline{P_{12}H}(j\varphi)\Lambda_{21}(e^{j\varphi h}) \\ \Lambda_{12}(e^{j\varphi h})\underline{P_{21}}(j\varphi) & \Lambda_{11}(e^{j\varphi h}) \end{bmatrix} \tag{65}$$

where (see (66) at the bottom of the page) and $\underline{G}(j\varphi)$ is given by (15) with $\Lambda(z)$ replaced by $\Lambda_{22}(z)$.

B. H_2-Norm of a Hybrid Input/Output Sampled-Data System

Now, let us partition (65) as

$$\underline{\tilde{G}}(j\varphi) = [\underline{\tilde{G}}_c(j\varphi) \quad \underline{\tilde{G}}_d(j\varphi)] \tag{67}$$

where $\underline{\tilde{G}}_c(j\varphi)$ is for the continuous-time input w_1 and $\underline{\tilde{G}}_d(j\varphi)$ is for the discrete-time input ρ_2. Now, let us introduce the following definition of an H_2-norm for the hybrid input/output sampled-data system of Fig. 3.

Definition 2: The quantity

$$\begin{aligned} ||\tilde{\mathcal{G}}||_2 &:= \left\{ \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{\tilde{G}}_c(j\varphi)^* \underline{\tilde{G}}_c(j\varphi))\, d\varphi \right. \\ &\quad \left. + \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{\tilde{G}}_d(j\varphi)^* \underline{\tilde{G}}_d(j\varphi))\, d\varphi \right\}^{1/2} \\ &= \left\{ \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{\tilde{U}}^* \underline{\tilde{G}}(j\varphi)^* \underline{\tilde{G}}(j\varphi) \underline{\tilde{U}})\, d\varphi \right\}^{1/2} \end{aligned} \tag{68}$$

is the H_2-norm of the hybrid FR-operator $\tilde{\mathcal{G}}(j\varphi)(\varphi \in (-\omega_s/2, \omega_s/2])$ associated with the input $[w_1^T, \rho_2^T]^T$ and output $[z_1^T, \zeta_2^T]^T$ of the sampled-data system shown in Fig. 3, where

$$\underline{\tilde{U}} := \begin{bmatrix} I & 0 \\ 0 & h^{1/2} I \end{bmatrix}. \tag{69}$$

It is easy to verify that the above definition reduces to Definition 1 when $\dim(\rho_2)$ and $\dim(\zeta_2)$ are both set to zero so that the system of Fig. 3 reduces to that of Fig. 1. Also, if we set $\dim(w_1) = \dim(z_1) = 0$ so that the system of Fig. 3 reduces to the discrete-time system it reduces to the H_2-norm of the discrete-time system $\Lambda_{11}(z)$. Moreover, we can show that the square of the above H_2-norm gives the sum of the square of the steady-state power of z_1 (as a continuous-time signal) and that of ζ_2 (as a discrete-time signal) for the independent continuous-time and discrete-time white noise inputs w_1 and ρ_2 both with the unit intensity (i.e., with the identity covariance matrices).

Remark 1: When we associate the H_2-norm of a system with control performances, we are usually (implicitly) taking the standpoint that each input is likely to be equally excited and that each output is equally important as a measure of control performances. If this is not the case, we should make appropriate scaling of input and output signals. Similarly, in our setting here, appropriate scaling of the signals w_1, ρ_2, z_1, and ζ_2 [or, equivalently, $P(s)$ and $\Psi(z)$] should be made beforehand so that the conditions

a) w_1 and ρ_2 are both white noises (or both impulses) with the same intensity and
b) direct comparison (i.e., without any weighting) of z_1 and ζ_2 is meaningful

are satisfied in accordance with the description just above this remark. How to scale discrete-time signals with respect to continuous-time signals, however, is actually somewhat discretionary. For example, even if we were to replace a) with

a′) the intensity of ρ_2 should be $h^{-1/2}$ times that of w_1

we could develop an equally valid theory. In that case, (68) would take a simpler (and presumably more natural) form

$$||\tilde{\mathcal{G}}||_2 = \left\{ \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \operatorname{trace}(\underline{\tilde{G}}(j\varphi)^* \underline{\tilde{G}}(j\varphi))\, d\varphi \right\}^{1/2}. \tag{70}$$

We do not dare to do this in this paper.

C. H_2-Norm Computation

The following theorem gives a way to compute the H_2-norm of the system of Fig. 3.

Theorem 5: The H_2-norm $||\tilde{\mathcal{G}}||_2$ of the hybrid FR-operator $\tilde{\mathcal{G}}(j\varphi)(\varphi \in (-\omega_s/2, \omega_s/2])$ associated with the input $[w_1^T, \rho_2^T]^T$ and output $[z_1^T, \zeta_2^T]^T$ of the stable hybrid input/output sampled-data system of Fig. 3 is given by

$$||\tilde{\mathcal{G}}||_2 = \{||\operatorname{diag}[h^{-1/2}I, I]\Theta(z)||_2^2 + \gamma\}^{1/2} \tag{71}$$

$$\begin{bmatrix} \Lambda_{11}(z) & \Lambda_{12}(z) \\ \Lambda_{21}(z) & \Lambda_{22}(z) \end{bmatrix} := \begin{bmatrix} \Psi_{11} + \Psi_{12}\Pi_{22}(I - \Psi_{22}\Pi_{22})^{-1}\Psi_{21} & \Psi_{12}(I - \Pi_{22}\Psi_{22})^{-1} \\ (I - \Psi_{22}\Pi_{22})^{-1}\Psi_{21} & (I - \Psi_{22}\Pi_{22})^{-1}\Psi_{22} \end{bmatrix} \tag{66}$$

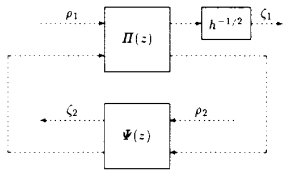

Fig. 4. Discrete-time control system corresponding to Fig. 3.

where $\Theta(z)$ is given by

$$\Theta(z) = \begin{bmatrix} \Pi_{11}(z) + \Pi_{12}(z)\Lambda_{22}(z)\Pi_{21}(z) & \Pi_{12}(z)\Lambda_{21}(z) \\ \Lambda_{12}(z)\Pi_{21}(z) & \Lambda_{11}(z) \end{bmatrix} \tag{72}$$

γ is given by (57), and $||\cdot||_2$ in the right-hand side denotes the H_2-norm of a stable pulse transfer function matrix [see Fig. 4 for the system corresponding to diag $[h^{-1/2}I, I]\Theta(z)$].

Proof: From (65) and (69), we have

$$\begin{aligned} \underline{\tilde{U}}^* \underline{\tilde{G}}(j\varphi)^* \underline{\tilde{G}}(j\varphi)\underline{\tilde{U}} \overset{\text{tr}}{=} & \underline{G}(j\varphi)^* \underline{G}(j\varphi) + \Lambda_{21}(e^{j\varphi h})^* \\ & \cdot \frac{1}{h}\underline{P_{12}H}(j\varphi)^* \underline{P_{12}H}(j\varphi) \cdot \Lambda_{21}(e^{j\varphi h}) \\ & + h\Lambda_{12}(e^{j\varphi h}) \cdot \frac{1}{h}\underline{P_{21}}(j\varphi)\underline{P_{21}}(j\varphi)^* \\ & \cdot \Lambda_{12}(e^{j\varphi h})^* + h\Lambda_{11}(e^{j\varphi h})^* \Lambda_{11}(e^{j\varphi h}). \end{aligned} \tag{73}$$

Therefore, using (29), (35), and (51) with $\Lambda(z)$ replaced by $\Lambda_{22}(z)$, we can easily verify that

$$||\tilde{\mathcal{G}}||_2^2 = ||\text{diag}\,[h^{-1/2}I, I]\hat{\Theta}(z)||_2^2 + \hat{\gamma} \tag{74}$$

where

$$\hat{\Theta}(z) := \begin{bmatrix} \hat{\Pi}_{11}(z) + \Pi_{12}(z)\Lambda_{22}(z)\Pi_{21}(z) & \Pi_{12}(z)\Lambda_{21}(z) \\ \Lambda_{12}(z)\Pi_{21}(z) & \Lambda_{11}(z) \end{bmatrix} \tag{75}$$

and $\hat{\gamma}$ is given by (53). As in the preceding sections, we can see that $\hat{\Theta}(z)$ is stable. Therefore, noting that the direct-feedthrough term of $\hat{\Theta}(z)$ is

$$\hat{\Theta}(\infty) = \begin{bmatrix} \Delta & \Pi_{12}(\infty)\Lambda_{21}(\infty) \\ 0 & \Lambda_{11}(\infty) \end{bmatrix} =: [\hat{\Theta}_1(\infty) \quad \hat{\Theta}_2(\infty)] \tag{76}$$

we obtain by successive application of Lemma 1 that

$$\begin{aligned} & ||\text{diag}\,[h^{-1/2}I, I]\hat{\Theta}(z)||_2^2 \\ & = ||\text{diag}\,[h^{-1/2}I, I](\hat{\Theta}(z) - \hat{\Theta}(\infty))||_2^2 \\ & \quad + \text{trace}\,(\hat{\Theta}(\infty)^T \text{diag}\,[h^{-1}I, I]\hat{\Theta}(\infty)) \\ & = ||\text{diag}\,[h^{-1/2}I, I][(\hat{\Theta}(z) - \hat{\Theta}(\infty) \\ & \quad + [0, \hat{\Theta}_2(\infty)])||_2^2 + h^{-1}\,\text{trace}\,(\Delta^T\Delta) \\ & = ||\text{diag}\,[h^{-1/2}I, I]\Theta(z)||_2^2 \\ & \quad + h^{-1}\,\text{trace}\,(\Delta^T\Delta). \end{aligned} \tag{77}$$

Therefore, in the same manner as in the preceding section, from (B3), (74) and (77), we obtain

$$||\tilde{\mathcal{G}}||_2^2 = ||\text{diag}\,[h^{-1/2}I, I]\Theta(z)||_2^2 + \gamma. \tag{78}$$

Q.E.D.

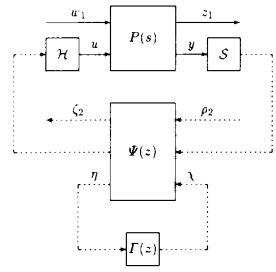

Fig. 5. Sampled-data control system with continuous-time input and output (with an explicit representation of a controller).

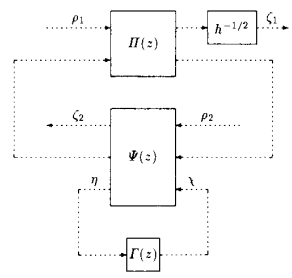

Fig. 6. Discrete-time control system corresponding to Fig. 5.

D. Hybrid Input/Output Sampled-Data H_2 Optimal Control Problem

Next, let us study an H_2 optimal control problem of a sampled-data system with hybrid input and output signals. For this purpose, consider the sampled-data system shown in Fig. 5, where $P(s)$ satisfies the same assumptions A1)–A5) as before, $\Psi(z)$ is a proper FDLTI pulse transfer function matrix representing an interconnection structure (e.g., computational delay), and $\Gamma(z)$ is a proper FDLTI discrete-time controller to be designed. For this sampled-data system, we pose the following H_2 control problem.

Problem 2: Suppose that the generalized plant (9) and the interconnection structure $\Psi(z)$ are given, where Assumptions A1)–A5) are satisfied. For a given sampling period h, find, if one exists, an optimal proper FDLTI discrete-time controller $\Gamma(z)$ such that the closed-loop system of Fig. 5 is internally stable and the H_2-norm of the hybrid FR-operator from $[w_1^T, \rho_2^T]^T$ to $[z_1^T, \zeta_2^T]^T$ is minimized.

Then, from the above arguments, we can immediately obtain the following theorem.

Theorem 6: The task of finding an optimal discrete-time controller $\Gamma(z)$ for Problem 2 is equivalent to the discrete-time H_2 control problem of finding an optimal controller $\Gamma(z)$ such that the closed-loop system of Fig. 6 is internally stable and the H_2-norm of the pulse transfer function matrix from $\rho := [\rho_1^T, \rho_2^T]^T$ to $\zeta := [\zeta_1^T, \zeta_2^T]^T$ (i.e., diag $[h^{-1/2}I, I]\Theta(z)$) is minimized, where $\Pi(z)$ is given by (38).

Remark 2: The above equivalent discrete-time problem can be solved by forming a new discrete-time generalized plant $\Pi_\Psi(z)$ such that

$$[\zeta^T, \eta^T]^T = \Pi_\Psi[\rho^T, \chi^T]^T \tag{79}$$

using a state-space formula so that an unstable pole will not be canceled by an unstable zero.

E. Time-Domain Equivalent of the H_2-Norm

In [7] and [9]–[11], the following time-domain definition of an H_2-norm was employed for the sampled-data system of Fig. 3

$$\left\{\frac{1}{h}\int_0^h \sum_{i=1}^{\dim(w_1)} \int_0^\infty \|g_{cc}^i(t:\sigma)\|^2\,dt\,d\sigma + \sum_{i=1}^{\dim(\rho_2)} \int_0^\infty \|g_{cd}^i(t:0)\|^2\,dt + \frac{1}{h}\int_0^h \sum_{i=1}^{\dim(w_1)} \sum_{k=0}^{\infty} \|g_{dc}^i(k:\sigma)\|^2\,d\sigma + \sum_{i=1}^{\dim(\rho_2)} \sum_{k=0}^{\infty} \|g_{dd}^i(k:0)\|^2\right\}^{1/2}. \tag{80}$$

Here, $g_{cc}^i(t:\sigma)$ denotes the response of z_1 at time t, and $g_{dc}^i(k:\sigma)$ denotes the response of ζ_2 at the kth sampling instant, both for the continuous-time unit impulse applied to the ith entry of w_1 at time σ. Similarly, $g_{dd}^i(k:\nu)$ denotes the response of ζ_2 at the kth sampling instant, and $g_{cd}^i(t:\nu)$ denotes the response of z_1 at time t, both for the discrete-time unit impulse applied to the ith entry of ρ_2 at the νth sampling instant. On the other hand, a time-domain equivalent of our H_2-norm defined by (68) turns out to be as follows

$$\left\{\frac{1}{h}\int_{-h}^0 \sum_{i=1}^{\dim(w_1)} \int_{-\infty}^0 \|g_{cc}^i(t:\sigma)\|^2\,d\sigma\,dt + \frac{1}{h}\int_{-h}^0 \sum_{i=1}^{\dim(\rho_2)} \sum_{\nu=-\infty}^{0} \|g_{cd}^i(t:\nu)\|^2\,dt + \sum_{i=1}^{\dim(w_1)} \int_{-\infty}^0 \|g_{dc}^i(0:\sigma)\|^2\,d\sigma + \sum_{i=1}^{\dim(\rho_2)} \sum_{\nu=-\infty}^{0} \|g_{dd}^i(0:\nu)\|^2\right\}^{1/2}. \tag{81}$$

In spite of the 'h-periodicity' of the responses such as $g_{cd}^i(t:\nu) = g_{cd}^i(t-\nu h:0)$, etc., (81) and (80) differ slightly in the coefficients of the second and third terms (the difference corresponds to moving the factor $h^{-1/2}$ in Fig. 4 to the side of ρ_1).

To clarify the source of this discrepancy, let us recall that definition (80) was employed in the previous studies (simply) because it is a natural extension of a time-domain representation of the H_2-norm of a stable continuous-time (or discrete-time) linear time-invariant system: if we denote by $g(t:\sigma)$ the response at time t of such a scalar system for the unit impulse applied at time σ, the H_2-norm of its transfer function coincides with

$$\left\{\int_0^\infty |g(t:0)|^2\,dt\right\}^{1/2}. \tag{82}$$

Now, note that the above quantity can be written also as

$$\left\{\int_{-\infty}^0 |g(0:\sigma)|^2\,d\sigma\right\}^{1/2} \tag{83}$$

in which the observation instant, rather than the excitation instant, is fixed. Then, we can realize that (81) is an equally natural extension of (83).

In some sense, the difference is actually not essential and is only a matter of scaling between continuous-time and discrete-time signals (cf., Remark 1). An important consequence of this difference, however, is that Assumptions a) and b) in Remark 1 are not the appropriate assumptions for the results of the previous studies [7], [9]–[11] to be valid, if they are to be applied to the stochastic (as opposed to deterministic) setting. This fundamental fact has not been clear so far, and thus no alternative (scaling) assumptions that should be made in the stochastic setting have been described in the previous studies (in other words, because of the ambiguity in the scaling, it was not clear how to formulate a mathematical H_2 control problem with their H_2-norm, given a practical design specification in the stochastic setting). An advantage of our definition is that the normalizing assumptions (i.e., a) and b) in Remark 1) is "consistent" both with the deterministic and stochastic settings. Actually, our definition is more compatible with other stochastic studies of sampled-data systems such as [22]. A sampled-data LQG problem has also been discussed from this viewpoint in the preliminary version of this paper [25].

VII. Conclusion

Based on the recently introduced notion called FR-operator and its generalized notion of hybrid FR-operator introduced in this paper, we investigated the H_2 control problems of sampled-data systems. It was shown that they can be reduced to equivalent discrete-time H_2 control problems as in [6]–[11], but using only such elementary frequency-domain notions as (conventional) z-transformation, pulse transfer function, and impulse modulation formula. This demonstrates that (hybrid) FR-operators can really be a tool not only for the analysis but also for the synthesis of sampled-data systems in the frequency domain. Compared with the previous frequency-domain studies for the synthesis of sampled-data systems [8], [12]–[13], our study is dealing with a more general problem in the sense that the hybrid case (i.e., the case where both the continuous-time and discrete-time input and output signals are there) is also studied. In [12]–[13], confining the arguments to the case of scalar control systems, the study was carried out fully in the frequency domain including the design of the optimal controller, while in our method, as well as [8], the design step was converted into a state-space form so that modern sophisticated design tools such as MATLAB can be employed.

For the hybrid case, the H_2-norm defined by a hybrid FR-operator turned out to differ slightly from that defined in the

previous studies [7], [10], and the source of the discrepancy was identified. The study of this paper has a very close connection with [22], where a covariance matrix is defined for a sampled-data system. By introducing a "blockwise trace" (which is a finite-dimensional matrix) of a partitioned infinite dimensional matrix, we can also define a covariance matrix of a sampled-data system using a technique quite similar to that exploited in this paper.

Appendix A
Derivation of (26)

In this appendix, we derive (26). It readily follows that

$$P_{21}(s)\tilde{P}_{21}(s) = \left[\begin{array}{cc|c} -A^T & 0 & C_2^T \\ -B_1B_1^T & A & 0 \\ \hline 0 & C_2 & 0 \end{array}\right]. \tag{A1}$$

Therefore, we obtain

$$\begin{aligned} z^{-1}\cdot Z[P_{21}(s)\tilde{P}_{21}(s)] &= \left[\begin{array}{cc|c} \exp(-A^Th) & 0 & C_2^T \\ -WW^T\exp(-A^Th) & \exp(Ah) & 0 \\ \hline 0 & C_2 & 0 \end{array}\right] \end{aligned} \tag{A2}$$

where W is such that (28) holds. On the other hand, let

$$\Pi_{21}(z) = \left[\begin{array}{c|c} A_d & B_{d1} \\ \hline C_{d2} & 0 \end{array}\right] \tag{A3}$$

be a state-space description of $\Pi_{21}(z)$. Then, we have

$$\begin{aligned} z^{-1}\tilde{\Pi}_{21}(z) &= \left[\begin{array}{c|c} A_d^{-T} & C_{d2}^T \\ \hline -B_{d1}^TA_d^{-T} & 0 \end{array}\right], \\ z^{-1}\Pi_{21}(z)\tilde{\Pi}_{21}(z) &= \left[\begin{array}{cc|c} A_d^{-T} & 0 & C_{d2}^T \\ -B_{d1}B_{d1}^TA_d^{-T} & A_d & 0 \\ \hline 0 & C_{d2} & 0 \end{array}\right]. \end{aligned} \tag{A4}$$

By setting $A_d = \exp(Ah), B_{d1} = W$ and $C_{d2} = C_2$ as in (27), and comparing (A2) and (A4), we can see that (26) is true.

Appendix B
Proof of Theorem 2

This section gives the proof of Theorem 2. Let us begin by rewriting (52) as

$$||\mathcal{G}||_2^2 = ||h^{-1/2}\mathcal{F}_l(\Pi(z),\Psi(z)) + h^{-1/2}\Delta||_2^2 + \hat{\gamma} \tag{B1}$$

using (50). Since $\mathcal{F}_l(\Pi(z),\Psi(z))$ is strictly proper and stable, it follows from Lemma 1 that

$$\begin{aligned} &||h^{-1/2}\mathcal{F}_l(\Pi(z),\Psi(z)) + h^{-1/2}\Delta||_2^2 \\ &\quad = ||h^{-1/2}\mathcal{F}_l(\Pi(z),\Psi(z))||_2^2 + h^{-1}\operatorname{trace}(\Delta^T\Delta). \end{aligned} \tag{B2}$$

Therefore, from (53), (55), and (B1) and from the above equation, γ is given by

$$\begin{aligned} \gamma &= \hat{\gamma} + h^{-1}\operatorname{trace}(\Delta^T\Delta) && \text{(B3)} \\ &= ||\Pi_0(z)||_2^2 - h^{-1}||\hat{\Pi}_{11}(z)||_2^2 + h^{-1}\operatorname{trace}(\Delta^T\Delta) \\ &= \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2}\operatorname{trace}(\Omega(e^{j\varphi h}))\,d\varphi && \text{(B4)} \end{aligned}$$

where

$$\begin{aligned} \Omega(e^{j\varphi h}) := \Pi_0(e^{j\varphi h})^*\Pi_0(e^{j\varphi h}) - h^{-1}\hat{\Pi}_{11}(e^{j\varphi h})^*\hat{\Pi}_{11}(e^{j\varphi h}) \\ + h^{-1}\Delta^T\Delta. \end{aligned} \tag{B5}$$

Now, substituting (43) and (46) into the above equation, and rearranging the result, we obtain

$$\begin{aligned} \Omega(e^{j\varphi h}) \stackrel{\text{tr}}{=} &(e^{j\varphi h}I - \exp(Ah))^{-*}V_1^T \\ &\cdot V_1(e^{j\varphi h}I - \exp(Ah))^{-1} \\ &\cdot (B_1B_1^T - h^{-1}WW^T) \\ &- h^{-1}(e^{j\varphi h}I - \exp(Ah))^{-*}V_1^T\Delta W^T \\ &- h^{-1}W\Delta^TV_1(e^{j\varphi h}I - \exp(Ah))^{-1} \end{aligned} \tag{B6}$$

where X^{-*} denotes $(X^{-1})^*$. Here, note that $W\Delta^TV_1$ is given from (47) by

$$\begin{aligned} W\Delta^TV_1 = &\int_0^h \exp(A\sigma)B_1B_1^T\int_0^\sigma \exp(A^T(\sigma-t))C_1^TC_1 \\ &\cdot \exp(A(\sigma-t))\,dt\exp(A(h-\sigma))\,d\sigma. \end{aligned} \tag{B7}$$

Now, for simplicity, we assume that A has no eigenvalues symmetric with respect to the imaginary axis, so that the continuous-time Lyapunov equation

$$V_\infty A + A^TV_\infty + C_1^TC_1 = 0 \tag{B8}$$

has a unique solution $V_\infty = V_\infty^T$. Then

$$\begin{aligned} &\int_0^\sigma \exp(A^T(\sigma-t))C_1^TC_1\exp(A(\sigma-t))\,dt \\ &\quad = V_\infty - \exp(A^T\sigma)V_\infty\exp(A\sigma) \end{aligned} \tag{B9}$$

is true for any σ [27, p. 349]. Substituting the above into (B7), we have

$$\begin{aligned} W\Delta^TV_1 = &\int_0^h \exp(A\sigma)B_1B_1^TV_\infty\exp(A(h-\sigma))\,d\sigma \\ &- WW^TV_\infty\exp(Ah). \end{aligned} \tag{B10}$$

Substituting the above equation into (B6) and noting that $\exp(A\sigma)$ commutes with $(e^{j\varphi h}I - \exp(Ah))^{-1}$, we obtain

$$\Omega(e^{j\varphi h}) \stackrel{\text{tr}}{=} \hat{\Omega}(e^{j\varphi h})(B_1B_1^T - h^{-1}WW^T) \tag{B11}$$

where

$$\begin{aligned} \hat{\Omega}(e^{j\varphi h}) := &(e^{j\varphi h}I - \exp(Ah))^{-*}V_1^TV_1 \\ &\cdot (e^{j\varphi h}I - \exp(Ah))^{-1} \\ &- (e^{j\varphi h}I - \exp(Ah))^{-*} \\ &\cdot \exp(A^Th)V_\infty - V_\infty\exp(Ah) \\ &\cdot (e^{j\varphi h}I - \exp(Ah))^{-1}. \end{aligned} \tag{B12}$$

Now, setting $\sigma = h$ in (B9), we have

$$V_1^T V_1 = V_\infty - \exp(A^T h) V_\infty \exp(Ah). \tag{B13}$$

Substituting this into (B12) and rearranging the results, we obtain

$$\hat{\Omega}(e^{j\varphi h}) \equiv V_\infty. \tag{B14}$$

This, together with (B4) and (B11), implies

$$\gamma = \operatorname{trace}(V_\infty(B_1 B_1^T - h^{-1} W W^T)). \tag{B15}$$

(This actually gives a state-space method for computing γ.) Now, it easily follows from (28) and (B15) that

$$\gamma \stackrel{\text{tr}}{=} h^{-1} \int_0^h B_1^T V_\infty B_1 \, d\sigma - h^{-1} \int_0^h B_1^T \cdot \exp(A^T \sigma) V_\infty \exp(A\sigma) B_1 \, d\sigma. \tag{B16}$$

Applying (B9) to the above equation, we obtain (57).

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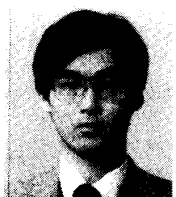

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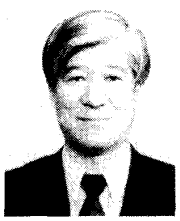

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