Analytic Study on the Intrinsic Zeros of Sampled-Data Systems

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Abstract—**This paper investigates the properties of the mapping from the simple zero γ of a scalar continuous-time system to the corresponding zero $\Gamma(T)$ of the sampled-data system that results by its discretization using a zero-order hold, where T is the sampling period. It is shown that $\Gamma(T)$ admits a Taylor expansion with respect to T, and that it coincides with that of $\exp(\gamma T)$ at least up to the second-order term, in general, and at least up to the third-order term if the relative degree of the continuous-time system is greater than or equal to two. The result is applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small sampling periods.**

I. Introduction

It is widely recognized that a zero-order hold is one of the basic elements in the implementation of digital control systems. Thus, it has been of fundamental interest to clarify the properties of the sampled-data system $G_T(z)$ obtained by the discretization of the continuous-time system $G(s)$ using a zero-order hold [4], [6], [7], [15]–[17], where T is the sampling period. As is well known, by such discretization, the pole λ of $G(s)$ is mapped to the pole $\Lambda(T) = \exp(\lambda T)$ of $G_T(z)$. However, the mapping of a zero is not so simple that it is generally impossible to derive a closed-form expression of the zero $\Gamma(T)$ of $G_T(z)$ that corresponds to the zero γ of $G(s)$ in terms of the parameters of $G(s)$ and T. Thus, many studies have been carried out about the zeros of $G_T(z)$ [1], [3], [5], [8]–[14].

In this paper, confining ourselves to the case of scalar systems, we show that $\Gamma(T)$ admits a Taylor expansion with respect to T if γ is a simple zero of $G(s)$. Furthermore, we show that the expansion coincides with that of $\exp(\gamma T)$ at least up to the second-order term, in general, and at least up to the third-order term if the relative degree of $G(s)$ is greater than or equal to two. The result is applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small T. Some comments are also given on the case where γ is a multiple zero of $G(s)$.

In the following, let (c, A, b) be a minimal realization of $G(s)$:

$$G(s) = c(sI - A)^{-1}b \tag{1}$$

where $A \in \boldsymbol{R}^{n\times n}, b \in \boldsymbol{R}^{n\times 1}, c \in \boldsymbol{R}^{1\times n}$. Then, it is well known (see, e.g., [6] and [10]) that the zeros of $G(s)$ and $G_T(z)$ are, respectively, given by the roots of the polynomials

$$N(s) = \det\begin{bmatrix} sI - A & -b \\ c & 0 \end{bmatrix} \tag{2}$$

and

$$N_T(z) = \det\begin{bmatrix} zI - A_T & -b_T \\ c & 0 \end{bmatrix} \tag{3}$$

where

$$A_T = \exp(AT), \quad b_T = \int_0^T \exp(At)b\, dt. \tag{4}$$

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II. Main Results—Taylor Expansion of $\Gamma(T)$

Suppose that $s = \gamma$ is a simple zero of $G(s)$, and let $\mathcal{S}$ be a simply-connected bounded domain containing γ but no other zeros of $G(s)$. The following result is a direct consequence of [10, Theorem 3].

Lemma: There exists $T_S(>0)$ such that for every T with $0<T<T_S, G_T(z)$ has exactly one zero in the domain $\exp(\mathcal{S}T) := \{\exp(sT)|s \in \mathcal{S}\}(\ni \exp(\gamma T))$.

The above lemma justifies us to say that $G_T(z)$ has a zero corresponding to the zero γ of $G(s)$ [8]–[10]. Specifically, it is called the intrinsic zero[1] of $G_T(z)$ corresponding to γ, which we denote by $\Gamma(T)$.

The above lemma means that $\Gamma(T)$ can be approximated by $\exp(\gamma T)$ in some sense, but it is not very clear how close $\Gamma(T)$ is to $\exp(\gamma T)$. On the other hand, it was shown in [13] that $\Gamma(T)$ can be approximated by $1 + \gamma T$. The purpose of this paper is to get a more accurate approximation for $\Gamma(T)$. For this purpose, let us suppose that $\Gamma(T)$ admits a power series expansion of the form

$$\Gamma(T) = 1 + \gamma T + \eta T^2 + \xi T^3 + O(T^4). \tag{5}$$

Since $\Gamma(T)$ is a zero of $G_T(z)$, it must satisfy

$$\psi(T) := \det\begin{bmatrix} \Gamma(T)I - A_T & -b_T \\ c & 0 \end{bmatrix} = 0. \tag{6}$$

Therefore, our purpose is to find the coefficients η and ξ such that the Taylor expansion of $\psi(T)$ with respect to T becomes as close to zero as possible. More specifically, we are to find η and ξ such that $(d/dT)^k\psi(T)|_{T=0} = 0 (k = 0, \cdots, K)$ for as large K as possible.

The following equation is readily obtained as in [6], [8]–[10] irrespective of η and ξ, using a formula for the derivative of a determinant:

$$(d/dT)^k\psi(T)|_{T=0} = 0 \qquad (k = 0, \cdots, n). \tag{7}$$

Next, from the condition $(d/dT)^k\psi(T)|_{T=0} = 0$ for $k = n + 1$, we obtain

$$\det\begin{bmatrix} \gamma I - A & \hat{b}_\eta \\ c & 0 \end{bmatrix} = 0 \tag{8}$$

where $\hat{b}_\eta$ is given by

$$\hat{b}_\eta = (\gamma I - A)^{-1}(\eta I - A^2/2)b - Ab/2. \tag{9}$$

Furthermore, from the condition $(d/dT)^k\psi(T)|_{T=0} = 0$ for $k = n + 2$, we obtain

$$\det\begin{bmatrix} \gamma I - A & \hat{b}_{\eta\xi} \\ c & 0 \end{bmatrix} = 0 \tag{10}$$

where

$$\begin{aligned} \hat{b}_{\eta\xi} = &-A^2b/6 + (\xi I - A^3/6)(\gamma I - A)^{-1}b \\ &- [(\eta I - A^2/2)(\gamma I - A)^{-1}]^2 b \\ &+ (\eta I - A^2/2)(\gamma I - A)^{-1}Ab/2 \\ &+ \text{trace}\,((\eta I - A^2/2)(\gamma I - A)^{-1}) \\ &\cdot [(\eta I - A^2/2)(\gamma I - A)^{-1}b - Ab/2]. \end{aligned} \tag{11}$$

The conditions (8) and (10), and even higher order conditions, can be derived using essentially the same technique as that employed in the proof of [8, Lemma 1] and [10, Lemma 1] (basically, differentiate

[1] A zero of $G_T(z)$ is called an intrinsic zero if it corresponds to a zero of $G(s)$. $G_T(z)$ often has a zero that has no continuous-time counterpart [1], which we call a discretization zero of $G_T(z)$. See [8]–[10] for more details.

the matrix in (6) row by row repeatedly and add and subtract appropriate terms to arrange the results using the Laplace expansion of a determinant). The lengthy derivations are not repeated here.

Since (8) is equivalent to $c(\gamma I-A)^{-1}\hat{b}_\eta = 0$, we obtain from (9) the following equation for η:

$$c(\gamma I-A)^{-2}b\cdot\eta = c(\gamma I-A)^{-2}A^2b/2+c(\gamma I-A)^{-1}Ab/2. \quad (12)$$

Now, by the assumption that γ is a simple zero of $G(s)$, we have

$$c(\gamma I-A)^{-2}b = -G'(\gamma) \neq 0 \quad (13)$$

where $G'(s)$ denotes $(d/ds)G(s)$. Therefore, η can be obtained as

$$\begin{aligned}\eta &= \frac{c(\gamma I-A)^{-2}A^2b + c(\gamma I-A)^{-1}Ab}{2c(\gamma I-A)^{-2}b}\\ &= \frac{\gamma c(\gamma I-A)^{-2}Ab}{2c(\gamma I-A)^{-2}b}\\ &= \gamma^2/2\end{aligned} \quad (14)$$

where we added $\gamma c(\gamma I-A)^{-1}b = 0$ to the numerator to get the last expression.

Substituting the above equation into (11), $\hat{b}_{\eta\xi}$ reduces to $\hat{b}_\xi$, where

$$\begin{aligned}\hat{b}_\xi &= -A^2b/6 + (\xi I - A^3/6)(\gamma I-A)^{-1}b\\ &\quad - \gamma(\gamma I+A)b/4 + \gamma\ \text{trace}\ (\gamma I+A)b/4.\end{aligned} \quad (15)$$

Then, since (10) is equivalent to $c(\gamma I-A)^{-1}\hat{b}_\xi = 0$, we obtain from (15) and $c(\gamma I-A)^{-1}b = 0$ the following equation for ξ:

$$c(\gamma I-A)^{-2}b\cdot\xi = c(\gamma I-A)^{-1}\hat{b} \quad (16)$$

where

$$\begin{aligned}\hat{b} &= A^2b/6 + (\gamma I-A)^{-1}A^3b/6 + \gamma Ab/4\\ &= \gamma(\gamma I-A)^{-1}A^2b/6 + \gamma Ab/4\\ &= \gamma Ab/12 + \gamma^2(\gamma I-A)^{-1}Ab/6.\end{aligned} \quad (17)$$

Therefore, ξ can be obtained as

$$\xi = \frac{\gamma c(\gamma I-A)^{-1}Ab/12 + \gamma^2 c(\gamma I-A)^{-2}Ab/6}{c(\gamma I-A)^{-2}b}. \quad (18)$$

Here, since $sG(s) = c(sI-A)^{-1}Ab+cb$, we have $G(s)+sG'(s) = -c(sI-A)^{-2}Ab$. From these equations and from $G(\gamma) = 0$, we obtain $c(\gamma I-A)^{-1}Ab = -cb$ and $c(\gamma I-A)^{-2}Ab = -\gamma G'(\gamma)$. Substituting these and (13) into (18), we obtain

$$\xi = \gamma^3/6 + \gamma cb/12G'(\gamma). \quad (19)$$

Continuing the above manner, it is easily seen that we can derive the Taylor expansion[2] of $\Gamma(T)$ which justifies (5). To summarize the above arguments, we have shown that

$$\Gamma(T) = 1+\gamma T + \frac{\gamma^2}{2}T^2 + \left(\frac{\gamma^3}{6} + \frac{\gamma cb}{12G'(\gamma)}\right)T^3 + O(T^4). \quad (20)$$

Noting that $cb = 0$ if the relative degree of $G(s)$ is greater than or equal to two, we obtain the following theorem.

Theorem 1: Suppose that γ is a simple zero of $G(s)$. Then, $\Gamma(T)$ admits a Taylor expansion with respect to T, and it coincides with that of $\exp(\gamma T)$ at least up to the second-order term. In particular, if the relative degree of $G(s)$ is greater than or equal to two, they coincide at least up to the third-order term.

[2] The expansion is possible in principle, but to express its coefficients in an explicit compact form seems nontrivial.

Remark 1: Even if the relative degree of $G(s)$ is one, the third-order terms still coincide if $\gamma = 0$. Actually, $\Gamma(T) = 1$ for any $T(>0)$ if $\gamma = 0$, regardless of the relative degree of $G(s)$ (see, e.g., [6]), and thus $\Gamma(T) = \exp(\gamma T)$ is always true if $\gamma = 0$.

Remark 2: If the relative degree of $G(s)$ is greater than or equal to two, $\Gamma(T) = \exp(\gamma T)$ can be the case. For example, for

$$G(s) = \frac{s-\gamma}{(s-p)(s-q)(s-2\gamma)} \qquad (\gamma = (p+q)/2) \quad (21)$$

the zeros of $G_T(z)$ are given by $\pm\exp(\gamma T)$.

III. Application to the Stability Condition of $\Gamma(T)$

In this section, we study the stability of $\Gamma(T)$, where it is said to be stable if it lies inside the unit circle. From the lemma, the following result is immediate [8]–[10].

Corollary: For any zero γ of $G(s), |\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if $\Re(\gamma) < 0$(respectively $\Re(\gamma) > 0$).

From this result, we can check the stability of $\Gamma(T)$ if the zero γ of $G(s)$ is not on the imaginary axis. However, if it is on the imaginary axis, the lemma is not helpful to examine stability of the corresponding zero $\Gamma(T)$, because $\exp(ST)$ necessarily contains the points both inside and outside the unit circle. From this difficulty, no stability condition of $\Gamma(T)$ has been obtained for the case of $\Re(\gamma) = 0$ (except the special case of $\gamma = 0$ as described in remark 1). In the following, we give a stability condition for such a case using the results of the preceding section.

Now, suppose that $\gamma = j\beta(\neq 0)$ so that γ is on the imaginary axis. Then, from (5) and (14), we obtain

$$\begin{aligned}\Gamma(T) &= \left(1 - \frac{\beta^2}{2}T^2 + \sigma T^3 + O(T^4)\right)\\ &\quad + j(\beta T + \omega T^3 + O(T^4))\end{aligned} \quad (22)$$

where

$$\sigma := \Re(\xi), \qquad \omega := \Im(\xi). \quad (23)$$

Therefore, we obtain

$$\begin{aligned}|\Gamma(T)|^2 &= \left(1 - \frac{\beta^2}{2}T^2 + \sigma T^3\right)^2\\ &\quad + (\beta T + \omega T^3)^2 + O(T^4)\\ &= 1 + 2\sigma T^3 + O(T^4).\end{aligned} \quad (24)$$

From this equation, we can conclude that $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if $\sigma < 0$(respectively, $\sigma > 0$). Here, from (19) and $\gamma = j\beta$, we have

$$\sigma = \Re(\xi) = \Re(\gamma cb/12G'(\gamma)). \quad (25)$$

In the following, we assume that the relative degree of $G(s)$ is one so that $cb \neq 0$. Then, $|\Gamma(T)| < 1$(respectively, $|\Gamma(T)| > 1$) if cb and $\Re(\gamma/G'(\gamma))$ have opposite signs (respectively, the same sign). Here, let us rewrite $G(s)$ in the form

$$G(s) = \tilde{N}(s)(s^2-\gamma^2)/D(s) \quad (26)$$

where $\tilde{N}(s)$ and $D(s)$ are coprime polynomials. Then, we can easily verify that

$$\gamma/G'(\gamma) = D(\gamma)/2\tilde{N}(\gamma). \quad (27)$$

Next, let us rewrite $1/G(s)$ in the form

$$\frac{1}{G(s)} = (p_1s+p_0) + \frac{q(s)}{\tilde{N}(s)} + \frac{r_1s+r_0}{s^2-\gamma^2} \quad (28)$$

TABLE I
$|\Gamma(T)|$ FOR EXAMPLE

T	$\lvert\Gamma(T)\rvert$ for $G_1(s)$
0.01	1.0000000417
0.1	1.0000413
0.5	1.00407
1	0.9987

T	$\lvert\Gamma(T)\rvert$ for $G_2(s)$
0.01	0.9999999583
0.1	0.9999578
0.5	0.99323
1	0.9119

where $q(s)$ is an appropriate polynomial whose degree is less than that of $\tilde{N}(s)$. Then, we can easily show that $cb = 1/p_1$. Furthermore, substituting (26) into (28), multiplying the both sides by $s^2 - \gamma^2$, and letting $s = \gamma = j\beta$, we readily obtain $\Re(D(\gamma)/\tilde{N}(\gamma)) = r_0$.

Combining the above arguments, we are led to the following stability condition of $\Gamma(T)$.

Theorem 2: Suppose that the relative degree of $G(s)$ is one and let $\gamma(\neq 0)$ be a simple zero of $G(s)$ on the imaginary axis. Then, the corresponding zero $\Gamma(T)$ of $G_T(z)$ satisfies $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if p_1 and r_0 have opposite signs (respectively, the same sign), where p_1 and r_0 are given by (28).

We study simple examples to illustrate the above theorem.

Example: For the stable minimum phase systems

$$G_1(s) = \frac{(s+1)(s^2+4)}{s^4+3s^3+10s^2+16s+13} \tag{29}$$

$$G_2(s) = \frac{(s+1)(s^2+4)}{s^4+3s^3+10s^2+14s+11} \tag{30}$$

we have

$$\frac{1}{G_1(s)} = (s+2) + \frac{1}{s+1} + \frac{3s+1}{s^2+4} \tag{31}$$

$$\frac{1}{G_2(s)} = (s+2) + \frac{1}{s+1} + \frac{3s-1}{s^2+4}. \tag{32}$$

Therefore from Theorem 2, we can conclude that for sufficiently small T, the $\Gamma(T)$ corresponding to $\gamma = 2j$ lies outside the unit circle for $G_1(s)$ and inside the unit circle for $G_2(s)$. This is demonstrated in Table I.

IV. Comments on the Case Where γ Is a Multiple Zero

When γ is a multiple zero of $G(s)$, it is easy to see that (8) becomes indefinite with respect to η and that (10) reduces to the quadratic equation for only η (i.e., ξ vanishes in the equation) given by

$$G''(\gamma)(\eta - \gamma^2/2)^2/2 - \gamma cb/12 = 0 \tag{33}$$

where $G''(s) := (d/ds)^2 G(s)$. Therefore, if γ is a zero with degree two so that $G''(\gamma) \neq 0$, then we can obtain two values of η from the above equation, each of which corresponds to one of the two "branches" of $\Gamma(T)$.

However, if the degree of γ as a zero of $G(s)$ is greater than two so that $G''(\gamma) = 0$, and if the relative degree of $G(s)$ is one so that $cb \neq 0$, then (33) admits no solution η unless $\gamma = 0$. This is because the expansion of $\Gamma(T)$ in (5) is not always adequate when γ is a multiple zero; the branches of $\Gamma(T)$ do not admit Taylor expansions, in general. This is not surprising in view of the theory of algebraic functions [2]; the expansion of $\Gamma(T)$ would require fractional power of T, in general, if γ is a multiple zero.

V. Conclusion

The properties of the zero $\Gamma(T)$ of $G_T(z)$ corresponding to the zero γ of $G(s)$ are investigated and are applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small T.

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