ON THE UNIFICATION OF KUMMER AND ARTIN-SCHREIER-WITT THEORIES (Algebraic number theory and related topics)

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ON THE UNIFICATION OF KUMMER AND ARTIN-SCHREIER-WITT THEORIES

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1. Motivation

Our aim of this report is to give an explanation of the final version of our theory which unifies the Kummer theory and Artin-Schreier-Witt theory. The details of this report can be seen in the Bordeaux preprint [15].

First, we review the Kummer theory.

Let \( n \) be an integer with \( n \geq 2 \), and \( K \) be a field of characteristic \( q \) with \( q \nmid n \) and \( K \supset \mu_n = \{ \zeta | \zeta^n = 1 \} \).

Theorem 1.1 (Kummer Theory).

\[
\frac{L}{K}: n\text{-cyclic Galois extension} \\
\iff \exists a \in K^* \text{ s.t. } L = K(\sqrt[n]{a})
\]

\[
\begin{array}{c}
L = K \otimes_{K[X, X^{-1}]} K[X, X^{-1}] \\
\uparrow \\
K \leftarrow K[X, X^{-1}] \\
a \leftarrow X
\end{array}
\]

\[
\begin{array}{c}
\text{Spec } L \\
\downarrow
\end{array} \rightarrow 
\begin{array}{c}
\text{Spec } K
\end{array} \rightarrow 
\begin{array}{c}
\text{Spec } L
\end{array}
\]

where \( \theta_n : G_{m,K} \rightarrow G_{m,K}; \ x \mapsto x^n \).

Namely, the Kummer theory implies that the following exact sequence (so-called the Kummer exact sequence) of sheaves on the fppf (or étale) site on Spec \( K \) is essential in the world of cyclic coverings of \( K \):

\[
1 \rightarrow \mu_{n,K} \rightarrow G_{m,K} \xrightarrow{\theta_n} G_{m,K} \rightarrow 1
\]

In fact, from the exact sequence, for any \( K \)-scheme \( X \) we can deduce the exact sequence:

\[
G_{m,K}(X) \xrightarrow{\theta_n} G_{m,K}(X) \xrightarrow{\partial} H^1(X, \mu_{n,K}) \rightarrow H^1(X, G_{m,K}) \rightarrow H^1(X, G_{m,K}).
\]

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Here

\( H^1(X, \mu_{n,K}) = \) the set of isomorphism classes of unramified \( \mu_n \) coverings of \( X \)

\( H^1(X, G_{m,K}) = 0 \) for suitable \( X \)'s by Hilbert Theorem 90

Next we review the Artin-Schreier-Witt theory.

Let \( k \) be a field of positive characteristic \( p \).

\( W_{n,k} \): the group scheme of Witt vectors of length \( n \)

\( \wp : W_{n,k} \to W_{n,k} ; x \mapsto x^{(p)} - x \)

**Theorem 1.2 (Artin-Schreier-Witt Theory).**

\[ K/k : p^n \text{-cyclic Galois extension} \]
\[ \iff \exists a \in W_n(k) \text{ s.t. } K = k(\wp^{-1}(a)) \]
\[ K = k \otimes_{k[X]} k[X] \]
\[ \wp : W_{n,k} \]
\[ \wp^* \]
\[ k \]
\[ a_i \]
\[ \Spec K \to \]
\[ W_{n,k} \]
\[ \Spec k \to \]
\[ W_{n,k} \]

where \( X = (X_0, X_1, \ldots, X_{n-1}) \).

Namely, the Artin-Schreier-Witt theory implies that the following exact sequence (so-called the Artin-Schreier-Witt exact sequence) of sheaves on the fppf (or étale) site on the Spec \( k \) is essential:

\[ 0 \to \mathbb{Z}/p^n \mathbb{Z} \to W_{n,k} \xrightarrow{\wp} W_{n,k} \to 0. \]

In fact, from the exact sequence, for any \( k \)-scheme \( X \) we can deduce the exact sequence:

\[ W_{n,k}(X) \xrightarrow{\wp} W_{n,k}(X) \xrightarrow{\delta} H^1(X, \mathbb{Z}/p^n) \to H^1(X, W_{n,k}) \to H^1(X, W_{n,k}). \]

Here

\( H^1(X, \mathbb{Z}/p^n) = \) the set of isomorphism classes of unramified \( \mathbb{Z}/p^n \mathbb{Z} \) coverings of \( X \)

\( H^1(X, W_{n,k}) = 0 \) for affine schemes \( X \)

Therefore, the Kummer theory implies that in the world of unramified \( p^n \)-cyclic coverings in characteristic 0, the Kummer exact sequence is the Buddha, and any such coverings is deduced from the sequence. On the other hand, the Artin-Schreier-Witt theory implies that in the world of unramified \( p^n \)-cyclic coverings in characteristic \( p \), the
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Artin-Schreier-Witt exact sequence is the Buddha, and any such covering is deduced from the sequence. But our religion asserts that every Buddha should be deduced from the unique essential Buddha (Mahāvairocanah). Hence, behind the two Buddhas, there should exist a more essential Buddha unifying them.

So we arrive at the following problems:

- Search for the Buddha unifying the Kummer and ASW sequences.
- Construct the deformations of the group schemes of Witt vectors of finite length to tori.
- Such deformations should keep the filtrations of the group schemes of Witt vectors.

2. 1 DIMENSIONAL CASE

Let $(A, \mathfrak{m})$ be a DVR with f.f.$A = K$ and $A/\mathfrak{m} = k$, and $\lambda \in \mathfrak{m} \setminus \{0\}$. Now we look at the plane curve over $A$:

$$C : \quad Y^2Z - \lambda XYZ - X^3 = 0 \subset \mathbb{P}^2,$$

whose generic fibre is a nodal curve and the special fibre is a cuspidal curve. Therefore the Picard scheme of the curve gives a deformation of an additive group scheme to a torus:

$$\text{Pic}^0(C/A) \cong \text{Spec} \, A[X, 1/(1 + \lambda X)],$$

with group law $x \cdot y = \lambda xy + x + y$. Hereafter we denote this group scheme by $G^{(\lambda)}$:

$$G^{(\lambda)} := \text{Spec} \, A[X, 1/(1 + \lambda X)].$$

The important fact is that any deformations of $G_a$ to $G_m$ over $A$ are only the type of $G^{(\lambda)}$'s. In fact, we have the following.

**Theorem 2.1** ([17, Th. 2.5]). Let $G$ be a flat group scheme over $\text{Spec} \, A$ with generic fibre $G_m$ and special fibre $G_a$. Then there exists a non-zero element $\lambda$ of $\mathfrak{m}$, uniquely up to unit factors, such that

$$G \cong G^{(\lambda)}.$$

3. HIGHER DIMENSIONAL CASE

If we obtain a deformation $\mathcal{W}_{n-1}$ of $W_{n-1}$ to $G^{n-1}_{m,K}$, then since the Witt vectors has the filtration

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0,$$

we can expect the next one fits into an extension

$$0 \to G^{(\lambda)} \to \mathcal{W}_{n+1} \to \mathcal{W}_n \to 0 \in \text{Ext}^1(\mathcal{W}_n, G^{(\lambda)}).$$
Definition 3.1. Let \((A, m)\) be a DVR, and \(\lambda_1, \lambda_2, \ldots, \lambda_n \in m \setminus \{0\}\). If \(\mathcal{W}_n\) is given by the extensions

\[
0 \to \mathcal{G}^{(\lambda_2)} \to \mathcal{W}_2 \to \mathcal{G}^{(\lambda_1)} \to 0 \in \text{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)})
\]
\[
0 \to \mathcal{G}^{(\lambda_3)} \to \mathcal{W}_3 \to \mathcal{W}_2 \to 0 \in \text{Ext}^1(\mathcal{W}_2, \mathcal{G}^{(\lambda_3)})
\]
\[
\ldots \ldots
\]
\[
0 \to \mathcal{G}^{(\lambda_n)} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0 \in \text{Ext}^1(\mathcal{W}_{n-1}, \mathcal{G}^{(\lambda_n)}),
\]
we call it a group scheme of type \((\lambda_1, \lambda_2, \ldots, \lambda_n)\).

To compute the group \(\text{Ext}^1(\mathcal{W}_\ell, \mathcal{G}^{(\lambda_{\ell+1})})\) for a group scheme \(\mathcal{W}_\ell\) of type \((\lambda_1, \lambda_2, \ldots, \lambda_\ell)\), the following exact sequence of sheaves on each the small Zariski, fpf or étale site on \(\text{Spec} A\) is essential:

\[
0 \to \mathcal{G}^{(\lambda)} \xrightarrow{\alpha^{(\lambda)}} \mathbb{G}_{m, A} \xrightarrow{\rho^{(\lambda)}} \iota_* \mathbb{G}_{m, A/\lambda} \to 0.
\]

where \(\iota : \text{Spec}(A/\lambda) \hookrightarrow \text{Spec} A\) is the canonical inclusion.

By an explicit computation of cocycles, we have

Proposition 3.1.

\[
\text{Ext}^1(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m, A}) = 0.
\]

Therefore inductively we have

\[
\text{Ext}^1(\mathcal{W}_\ell, \mathbb{G}_{m, A}) = 0,
\]

for any group scheme of type \((\lambda_1, \lambda_2, \ldots, \lambda_\ell)\).

Hence, by using the above exact sequence we obtain the following.

Theorem 3.2. Let \(\mathcal{W}_n\) be a group scheme of type \((\lambda_1, \lambda_2, \ldots, \lambda_n)\), and \(\lambda \in m \setminus \{0\}\). Then we have

\[
\text{Ext}^1(\mathcal{E}, \mathcal{G}^{(\lambda)}) \cong \text{Hom}(\mathcal{E}, \iota_* \mathbb{G}_{m, A/\lambda}) / (\rho^{(\lambda)})_* (\text{Hom}(\mathcal{E}, \mathbb{G}_{m, A})).
\]

From this theorem, we can deduce the following.

Theorem 3.3. Let \(\mathcal{W}_n\) be a group scheme of type \((\lambda_1, \lambda_2, \ldots, \lambda_n)\). Then there exists a homomorphism

\[
D_\ell : \mathcal{W}_\ell \to \iota_* \mathbb{G}_{m, A/\lambda_{\ell+1}}
\]

for each \(\ell (2 \leq \ell \leq n - 1)\), and each \(\mathcal{W}_\ell\) is given by

\[
\mathcal{W}_\ell \cong \text{Spec} A[X_0, \ldots, X_{\ell-1}, \frac{1}{1 + \lambda_1 X_0},
\frac{1}{D_1(X_0) + \lambda_2 X_1}, \ldots, \frac{1}{D_{\ell-1}(X_0, \ldots, X_{\ell-2}) + \lambda_\ell X_{\ell-1}}].
\]
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Moreover, the group law of $\mathcal{W}_\ell$ is the one which makes the morphism

$$\alpha^{(\ell)}: \mathcal{W}_\ell \to (\mathbb{G}_{m,A})^\ell$$

$$(X_0, \ldots, X_{\ell-1}) \mapsto (1 + \lambda_1 X_0, D_1(X_0) + \lambda_2 X_1, \ldots, D_{\ell-1}(X_0, \ldots, X_{\ell-2}) + \lambda_\ell X_{\ell-1})$$

a group-schematic homomorphism.

Definition 3.2. Suppose that $A$ dominates $\mathbb{Z}(p)[\mu_{p^n}]$, and put $\lambda = \lambda(1)$. We call a group scheme $\mathcal{W}_1 = \mathcal{G}^{(\lambda)}, \mathcal{W}_2, \ldots, \mathcal{W}_n$ over $A$ of type $\lambda = (\lambda, \lambda, \ldots, \lambda)$ a KASW group scheme over $A$, if there exists an inclusion $i_\ell: \mathbb{Z}/p^\ell \to \mathcal{W}_\ell$ for each $\ell$ satisfying a commutative diagram

$$0 \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to \cdots \to \mathcal{W}_2 \to \mathcal{W}_1 \to 0.$$
for $x \in W(A)$. Then we have $T_a = \sum_{k \geq 0} V^k \cdot \tilde{a}_k$.

If $A$ is a ring (not necessarily a $\mathbb{Z}_{(p)}$-algebra),

$$
\tilde{W}_n(A) = \{ (a_0, a_1, \ldots, a_{n-1}) \in W_n(A) ; \ a_i \text{ is nilpotent for all } i \} \\
$$

and

$$
\tilde{W}(A) = \{ (a_0, a_1, a_2, \ldots) \in W_n(A) ; \ a_i \text{ is nilpotent for all } i \text{ and } a_i = 0 \text{ for all but a finite number of } i \}.
$$

Moreover we need to deform the Artin-Hasse exponential series

$$
E_p(X) := \exp \left( X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots \right) = e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \cdots \in \mathbb{Z}_{(p)}[[X]].
$$

The well-known formula $\lim_{\lambda \to 0} (1 + \lambda x)^{a/\lambda} = e^{\alpha x}$ can be seen that $(1 + \lambda x)^{a/\lambda}$ is a deformation of $e^{\alpha x}$. From this point of view, we obtain the deformations of Artin-Hasse exponential series:

$$
E_p(U, \Lambda; X) := (1 + \Lambda X)^{Y} \prod_{k=1}^{\infty} \left( 1 + \Lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k \lambda^{p^k} \Phi_{k-1}(F^{(\lambda)}a)}} \in \mathbb{Z}_{(p)}[U, \Lambda][[X]].
$$

Moreover for a Witt vector $a \in W(A)$, we define a formal power series as follows:

$$
E_p(a, \lambda; X) := \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) = (1 + \lambda X)^{\frac{1}{p^k \lambda^{p^k} \Phi_{k-1}(F^{(\lambda)}a)}} \in \mathbb{Z}_{(p)}[a, \lambda][[X]].
$$

The boundary of this power series $E_p(a, \lambda; X)$ is given by the following.

$$
(\partial E_p(a, \lambda, \cdot))(X, Y) = \frac{E_p(a, \lambda; X) E_p(a, \lambda; Y)}{E_p(a, \lambda; X + Y + \lambda XY)} = \prod_{k=1}^{\infty} \left( \frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k} \Phi_{k-1}(F^{(\lambda)}a)}}.
$$

Now replacing $F^{(\lambda)}a$ with a Witt vector $b = (b_0, b_1, \ldots)$ in the right hand side of this equation, we define a cocycle as follows.

$$
F_p(b, \lambda; X, Y) := \prod_{k=1}^{\infty} \left( \frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k} \Phi_{k-1}(b)}} \in \mathbb{Z}_{(p)}[b, \lambda][X, Y]].
$$

Using these deformed Artin-Hasse exponential series, we can obtain the following.
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Theorem 4.1 (Explicit Formula in 1 Dimensional Case).

\[ \xi_0^1 : \text{Ker} \left( \overline{W}(A/\lambda_2) \xrightarrow{F(\lambda_1)} \overline{W}(A/\lambda_2) \right) \xrightarrow{a} \text{Hom}(G^{(\lambda_1)}, \iota_*G_{m,A/\lambda_2}), \]

\[ \xi_0^1 : \text{Coker} \left( \overline{W}(A/\lambda_2) \xrightarrow{F(\lambda_1)} \overline{W}(A/\lambda_2) \right) \xrightarrow{b} H_0^2(G^{(\lambda_1)}, \iota_*G_{m,A/\lambda_2}). \]

Therefore

\[ \xi_1^0 : \frac{\text{Ker} \left( \overline{W}(A/\lambda_2) \xrightarrow{F(\lambda_1)} \overline{W}(A/\lambda_2) \right)}{<1>} \xrightarrow{c} \text{Hom}(G^{(\lambda_1)}, \iota_*G_{m,A/\lambda_2}) \xrightarrow{d} \text{Ext}^1(G^{(\lambda_1)}, G^{(\lambda_2)}). \]

In higher dimensional case, we need more notations. For a vector \( \mathbf{U} = (U_0, U_1, \ldots) \), we define

\[ [p]E_p(\mathbf{U}, \Lambda; X) := E_p([p]\mathbf{U}, \Lambda; X), \]

\[ [p]F_p(\mathbf{U}, \Lambda; X, Y) := F_p([p]\mathbf{U}, \Lambda; X, Y). \]

Moreover

\[ H(X, Y) := \frac{1}{\Lambda_2} \{F_p(\mathbf{U}, \Lambda_1; X, Y) - 1\}, \]

\[ G_p(A, \Lambda_2; E) := \prod_{\ell \geq 1} \left( 1 + \frac{(E - 1)^{p^\ell}}{[p]^\ell E} \right)^{\frac{1}{p^\ell \Lambda_2} \Phi_{\ell-1}(A)}, \]

\[ G_p(A, \Lambda_2; F) := \prod_{\ell \geq 1} \left( 1 + \frac{(F - 1)^{p^\ell}}{[p]^\ell F} \right)^{\frac{1}{p^\ell \Lambda_2} \Phi_{\ell-1}(A)}, \]

\( \in \mathbb{Z}_{(p)}[A, \frac{\mathbf{U}}{\Lambda_2}, \Lambda_1, \Lambda_2][[X, Y]] \)

For a series of variables \( \Lambda_1, \Lambda_2, \ldots \), and a series of vectors \( A_j^i = (A_{j0}^i, A_{j1}^i, \ldots) \) \( (1 \leq i; 1 \leq j \leq i) \), we denote

\[ A^i = (A_{\ell}^i)_{1 \leq \ell \leq i} := \begin{pmatrix} A_1^i \\ \vdots \\ A_i^i \end{pmatrix} \quad \text{and} \quad (\Lambda_{\ell})_{1 \leq \ell \leq i} := \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_i \end{pmatrix}. \]

We define vectors \( B_j^i \) \( (1 \leq j < i) \) inductively by

\[ B_1^i := \frac{1}{\Lambda_2} F^{(\Lambda_1)} A_1^i, \]

and for \( k \geq 2, \)

\[ \begin{cases} B_{j+1}^k := \frac{1}{\Lambda_{k+1}} \left( F^{(\Lambda_k)} A_j^k - \sum_{\ell=j+1}^k T_{B_j^\ell} A_{\ell}^k \right) & 1 \leq j \leq k - 1 \\ B_k^k := \frac{1}{\Lambda_{k+1}} F^{(\Lambda_k)} A_k^k. \end{cases} \]
Using these symbols, we define triangle matrices $U^n$'s by

$$U^n := \begin{pmatrix}
F^{(\Lambda_1)} & -T_{B_2} & -T_{B_3} & \cdots & -T_{B_7} \\
0 & F^{(\Lambda_2)} & -T_{B_3} & \cdots & -T_{B_2} \\
0 & 0 & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & -T_{B_{n-1}} \\
0 & 0 & \cdots & 0 & F^{(\Lambda_n)}
\end{pmatrix}.$$ 

We define inductively a series of formal power series $D_k(X_0, X_1, \ldots, X_{k-1})$'s by

$$D_0 = 1,$$

$$D_1(X_0) = E_p(A_1^1; X_0),$$

and for $k \geq 1$,

$$D_{k+1}(X_0, X_1, \ldots, X_k) = E_p(A_1^{k+1}; \Lambda_{1\leq i \leq k+1}; X_0, X_1, \ldots, X_k)$$

$$:= \prod_{i=1}^{k+1} E_p(A_1^{k+1}, \Lambda_i; \frac{X_i - 1}{D_i(X)} \cdot \frac{X_{i-1}}{D_i(X)}).$$

Hereafter, we put $X = (X_0, X_1, \ldots)$, $Y = (Y_0, Y_1, \ldots)$ and $\Sigma := X \oplus Y \in W$. We define

$$F^{(k)} := \partial(D_k(X)) = \frac{D_k(X)D_k(Y)}{D_k(\Sigma)},$$

$$H_k(X, Y) := \frac{1}{\Lambda_{k+1}}(F^{(k)} - 1),$$

$$F_p(V_i, \Lambda_i; X, Y) := F_p(V_i, \Lambda_i; X_0, Y_0)$$

$$= \prod_{i=1}^{n} F_p(V_i, \Lambda_i; \frac{X_i - 1}{D_i(X)} \cdot \frac{Y_i - 1}{D_i(Y)}),$$

$$\times \prod_{i=2}^{n} G_p(V_i, \Lambda_i; F^{(i-1)})^{-1}.$$

Then the important thing is the following result.

**Theorem 4.2.** For each $n \geq 1$, we have

$$F^n = \frac{D_n(X)D_n(Y)}{D_n(\Sigma)} = F_p(U^n A^n; \Lambda_{1\leq i \leq n}; X, Y).$$

By using this theorem, we can obtain the explicit determination of $\text{Ext}^1(W_n, \mathcal{G}^{(\lambda)})$. In fact, let $(A, m)$ be a DVR dominating $\mathbb{Z}(p)$, and $\lambda, \lambda_1, \lambda_2, \ldots$ be non-zero elements of $m$. We choose Witt vectors

$$\vec{a} = (\overline{a}_j)_{1 \leq j \leq i} \in \text{Ker} \left( U^i : \overline{W}(A/\lambda_{i+1})^i \to \overline{W}(A/\lambda_{i+1})^i \right).$$
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inductively by the following recursive conditions:

\[ U^1 = F^{(\lambda_1)}, \]

\[ \overline{a}^1 = \overline{a}_1^1 \in \text{Ker} \left( U^1 : \overline{W}(A/\lambda_2) \to \overline{W}(A/\lambda_2) \right), \]

\[ b^2_i = \frac{1}{\lambda_2} a^1_i, \quad U^2 = \begin{pmatrix} F^{(\lambda_1)} & -T_{\beta_2} \\ 0 & F^{(\lambda_2)} \end{pmatrix}, \]

and for \( k \geq 2 \), we choose

\[ \overline{a}^k = (\overline{a}_i^k)_{1 \leq i \leq k} \in \text{Ker} \left( U^k : \overline{W}(A/\lambda_{k+1})^k \to \overline{W}(A/\lambda_{k+1})^k \right), \]

and we define

\[
\begin{cases}
  b_j^{k+1} := \frac{1}{\lambda_{k+1}} \left( F^{(\lambda_j)} a^k_j - \sum_{\ell=j+1}^{k} T_{\mu_{\ell}} a^k_{\ell} \right) & 1 \leq j \leq k - 1 \\
  b_k^{k+1} := \frac{1}{\lambda_{k+1}} F^{(\lambda_k)} a^k_k
\end{cases}
\]

\[ U^{k+1} := \begin{pmatrix}
  F^{(\lambda_1)} & -T_{\beta_2} & & -T_{\beta_1}^{k+1} \\
  0 & F^{(\lambda_2)} & -T_{\beta_2} & & -T_{\beta_1}^{k+1} \\
  0 & 0 & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & -T_{\beta_1}^{k+1} \\
  0 & 0 & \cdots & \cdots & 0 & F^{(\lambda_{k+1})}
\end{pmatrix}, \]

\[ \overline{a}^{k+1} = (\overline{a}_i^{k+1})_{1 \leq i \leq k+1} \in \text{Ker} \left( U^{k+1} : \overline{W}(A/\lambda_{k+2})^{k+1} \to \overline{W}(A/\lambda_{k+2})^{k+1} \right). \]

We define formal power series \( D_k(X) = D_k(X_0, \ldots, X_{k-1}) \) \((k \geq 1)\) by

\[ D_0 = 1, \]

\[ D_1(X_0) = E_p(a_1^1, \lambda_1; X_0), \]

and for \( k \geq 1, \)

\[ D_{k+1}(X_0, X_1, \ldots, X_k) = E_p(a^{k+1}, (\lambda_i)_{1 \leq i \leq k+1}; X_0, X_1, \ldots, X_k) \]

\[ := \prod_{i=1}^{k+1} E_p(a_i^{k+1}, \lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, \ldots, X_{i-1})}). \]

We put

\[ \mathcal{W}_n := \text{Spec} A[X_0, \ldots, X_{n-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{1 + \lambda_2 X_1}, \ldots, \frac{1}{1 + \lambda_n X_{n-1}}]. \]

**Theorem 4.3 (Explicit Formula in General Case).** Let \( B = A/\lambda \). Then we have

\[ \xi^n : \ker(\overline{W}(B)^n) \xrightarrow{U^n} \overline{W}(B)^n \xrightarrow{\sim} \text{Hom}(\mathcal{W}_n, \mathbb{G}_{m, B}); \]

\[ \overline{v}^n = (\overline{v}_i^n)_{1 \leq i \leq n} \mapsto E_p(\overline{v}^n, (\lambda_i)_{1 \leq i \leq n}; X_0, X_1, \ldots, X_{n-1}) \]

\[ \xi^n : \text{coker}(\overline{W}(B)^n) \xrightarrow{U^n} \overline{W}(B)^n \xrightarrow{\sim} H^2_0(\mathcal{W}_n, \mathbb{G}_{m, B}); \]

\[ \overline{w}^n = (\overline{w}_i^n)_{1 \leq i \leq n} \mapsto F_p(\overline{w}^n, (\lambda_i)_{1 \leq i \leq n}; X, Y). \]
Theorem 4.4.

$$\overline{\xi}_{0}^{n} : \frac{\text{Ker}(U^{n} : \overline{W}(B)^{n} \to \overline{W}(B/\lambda)^{n})}{<c^{0}, c^{1}, \ldots, c^{n-1}>} \sim \text{Ext}^{1}(\mathcal{W}_{n,B}, \mathcal{G}_{B}^{(\lambda)})$$

where $$<c^{0}, c^{1}, \ldots, c^{n-1}>$$ is the subgroup generated by the vectors $$c^{0} = (\tilde{\lambda}_{1}, 0, \ldots, 0), c^{1} = (a^{t}, \tilde{\lambda}_{2}, 0, \ldots, 0), \ldots, c^{n-1} = (a^{n-1}, \tilde{\lambda}_{n}).$$

5. REDUCTIONS OF EXTENSIONS

The special fibres of the group schemes of type $$(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})$$ can be decided as follows.

Theorem 5.1. Let

$$\mathcal{W}_{n} = \text{Spec} A[X_{0}, X_{1}, \ldots, X_{n-1}, \frac{1}{1+\lambda_{1}X_{0}}, \frac{1}{D_{1}(X_{0})+\lambda_{2}X_{1}}, \ldots, \frac{1}{D_{n-1}(X_{0}, \ldots, X_{n-2})+\lambda_{n}X_{n-1}}]$$

be the group scheme of type $$(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n-1})$$ defined by

$$D_{1}(X_{0}) = E_{p}(a_{1}^{i}, \lambda_{1}; X_{0})$$

and for $$1 \leq k \leq n-2,$$

$$D_{k+1}(X) = E_{p}(a^{k+1}, (\lambda_{\ell})_{1 \leq \ell \leq k+1}; X_{0}, X_{1}, \ldots, X_{k}),$$

and

$$\overline{a}^{k} \in \text{Ker}(U^{k} : \overline{W}(A/\lambda_{k+1})^{k} \to \overline{W}(A/\lambda_{k})^{k}).$$

Here

$$b^{i} = t(b_{1}^{i}, b_{2}^{i}, \ldots, b_{i-1}^{i}) = \frac{1}{\lambda_{i}^{i}} U^{i-1} a^{i-1} \quad (i = 2, \ldots, n).$$

If $$b_{k}^{\ell} \equiv 0 \pmod{m}$$ for $$3 \leq k \leq n, 1 \leq \ell \leq k-2,$$ and $$b_{k-1}^{\ell} \equiv (1, 0, \ldots) \pmod{m},$$ then we have

$$\mathcal{W}_{n,k} = \mathcal{W}_{n} \otimes_{A} k = W_{n,k}.$$
Then we have

\[ i^n : \text{Ext}^1_A(\mathcal{W}_n, G^{(\lambda)}) \rightarrow \text{Ext}^1_A((\mathbb{Z}/p^n)_A, G^{(\lambda)}) \]

\[ \ker(\overline{W}(A/\lambda)^n \to \overline{W}(A/\lambda)) \twoheadrightarrow (1 + \lambda A) / (1 + \lambda A)^p^n \]

\[ a^n \mapsto \Pi_{r>0} \left( E_p(a_{r^n}, \lambda_{(1)}^{p^n}; 1)^{p^n} \Pi_{i=2}^{n} E_p(a_{r^n}, \lambda_{(1)}^{p^n}; \left(\frac{c_{i-1}}{D_{i-1}(n(1))}\right)^{p^n})^{p^n} \right) \]

Under these notations, we have the following.

**Theorem 6.1.** Let \( \mathcal{W}_{n+1} \in \text{Ext}^1(\mathcal{W}_n, G^{(\lambda)}) \) be the extension corresponding to a vector \( a^n = (a_i^n)_{1 \leq i \leq n} \) by the isomorphism

\[ \frac{\ker(U^n : \overline{W}(A/\lambda)^n \to \overline{W}(A/\lambda))}{(c^1, c^2, \ldots, c^{n-1})} \sim \text{Ext}^1(\mathcal{W}_n, G^{(\lambda)}). \]

Then there exists an inclusion \( (\mathbb{Z}/p^{n+1})_A \subset \mathcal{W}_{n+1} \) fitting into a commutative diagram

\[ \begin{array}{cccccc}
0 & \longrightarrow & (\mathbb{Z}/p)_A & \longrightarrow & (\mathbb{Z}/p^{n+1})_A & \longrightarrow & (\mathbb{Z}/p^n)_A & \longrightarrow & 0 \\
& & \downarrow i_1 & & \downarrow i_{n+1} & & \downarrow i_n & & \\
0 & \longrightarrow & G^{(\lambda)} & \longrightarrow & \mathcal{W}_{n+1} & \longrightarrow & \mathcal{W}_n & \longrightarrow & 0,
\end{array} \]

if and only if

\[ E_p(a^n, (\lambda, \ldots, \lambda); i_n(1))^{p^n} = \zeta_1. \]

Using these results, we construct explicitly the KASW group schemes.

**Theorem 6.2 (Main Theorem).** For each positive integer \( n \), we construct explicitly a standard KASW group scheme \( \mathcal{W}_n \) over \( \mathbb{Z}[\mu_p^n] \).

Finally, we remark that for a KASW group scheme \( \mathcal{W}_n \), the quotient \( \mathcal{V}_n := \mathcal{W}_n / (\mathbb{Z}/p^n) \) is a group scheme of type \( (\lambda^{p^n}_{(1)})^n \), and which is given explicitly.

**REFERENCES**


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