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<th>A SURVEY ON GENERALIZED HERMITE CONSTANTS (Algebraic number theory and related topics)</th>
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<tr>
<td>Author(s)</td>
<td>Watanabe, Takao</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1200: 65-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/40927">http://hdl.handle.net/2433/40927</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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This is an expository note on Hermite's constant. We give an account of a recent development of some generalizations of Hermite's constant.

1. Hermite–Rankin's constant. Let $\mathcal{L}^n$ be the set of all lattices of rank $n$ in the Euclidean space $\mathbb{R}^n$. For $L \in \mathcal{L}^n$, $d(L)$ stands for the volume of the fundamental parallelepiped of $L$. It was proved by Hermite that

$$\min_{0 \neq x \in L} txx \leq \left( \frac{2}{\sqrt{3}} \right)^{n-1} d(L)^{2/n}$$

holds for all $L \in \mathcal{L}^n$. Thus $\min_{0 \neq x \in L} txx / d(L)^{2/n}$ is bounded and there exists the maximum

$$\gamma_n = \max_{L \in \mathcal{L}^n} \min_{0 \neq x \in L} \frac{txx}{d(L)^{2/n}}.$$ 

The constant $\gamma_n$ is called Hermite's constant. A well-known example of its appearance is the lattice sphere packing problem, namely the density of the densest lattice packing of spheres in $\mathbb{R}^n$ equals

$$\delta_n = \gamma^{n/2} \frac{V(n)}{2^n},$$

where $V(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, i.e., $V(n) = \pi^{n/2}/\Gamma(1 + n/2)$. Originally, $\gamma_n$ arose from the reduction theory of positive definite quadratic forms initiated by Lagrange, Seeber and Gauss. In terms of quadratic forms, $\gamma_n$ is represented as

(1) $$\gamma_n = \max_{g \in GL_n(\mathbb{R})} \min_{0 \neq x \in \mathbb{R}^n} \frac{txx ggx}{(\det g)^{2/n}}.$$ 

The exact value of $\gamma_n$ is known only for $n \leq 8$, i.e., $\gamma_2 = 2/\sqrt{3}, \gamma_3 = \sqrt{2}, \gamma_4 = \sqrt{2}, \gamma_5 = \sqrt{8}, \gamma_6 = \sqrt{64}/3, \gamma_7 = \sqrt{64}, \gamma_8 = 2$. One has the estimate

(2) $$\left( \frac{2\zeta(n)}{V(n)} \right)^{2/n} \leq \gamma_n \leq 4 \left( \frac{1}{V(n)} \right)^{2/n}.$$ 

This upper bound was given by Minkowski and follows from $\delta_n \leq 1$. The lower bound was first stated by Minkowski and was proved by Hlawka.
The next step of Hermite’s constant is the following extension due to Rankin. For every $1 \leq d \leq n - 1$, define

$$\gamma_{n,d} = \max_{L \in \mathbb{Z}^n} \min_{\substack{z_1, \ldots, z_d \in L \\text{such that } z_1 \wedge \cdots \wedge z_d \neq 0}} \frac{\det(t^{x_i}x_j)_{1 \leq i,j \leq d}}{d(L)^{2d/n}}.$$  

Obviously, $\gamma_{n,1}$ equals $\gamma_n$. Rankin ([R]) proved $\gamma_{n,d}$ satisfies the inequality

$$\gamma_{n,d} \leq \gamma_{n,d}(\gamma_{n,m})^{d/m}$$

for $1 \leq d < m \leq n - 1$, and he showed $\gamma_{4,2} = 3/2$. Rankin’s inequality and the duality $\gamma_{n,d} = \gamma_{n,n-d}$ yield Mordell’s inequality $\gamma_n^{n-2} \leq \gamma_{n-1}^n$.

2. Icaza–Thunder’s generalization. As a generalization of Hermite–Rankin constant, Thunder defined the constant $\gamma_{n,d}(k)$ for any algebraic number field $k$ of finite degree $r$ over $\mathbb{Q}$ in 1997. At first, we recall a definition of twisted heights. Let $e_1, \ldots, e_n$ be a standard basis of $k^n$. For any extension field $L$ over $k$, $W_{n,d}(L)$ stands for the $d$-th exterior product of $L^n$. A basis of $W_{n,d}(k)$ is formed by the elements $e_I = e_{i_1} \wedge \cdots \wedge e_{i_d}$ with $I = \{1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$. For each place $v$ of $k$, let $k_v$ be the completion of $k$ at $v$ and $|\cdot|_v$ the usual normalized absolute value of $k_v$. We define the local height on $W_{n,d}(k_v)$ by

$$H_v(\sum_I a_I e_I) = \begin{cases} \left(\sum_I |a_I|_{v}^{[C:k_v]}\right)^{1/([C:k_v]r)} & \text{(if } v \text{ is infinite)} \\ \left(\sup_I |a_I|_{v}\right)^{1/r} & \text{(if } v \text{ is finite)} \end{cases}$$

Then the global height $H$ on $W_{n,d}(k)$ is defined to be the product of $H_v$:

$$H(x) = \prod_v H_v(x) \quad (x \in W_{n,d}(k)).$$

Let $A$ be the adele ring of $k$ and $| \cdot |_A$ the idele norm on $A^\times$. Since $H(\alpha x) = |\alpha|_A^{1/r}H(x)$ for $\alpha \in k^\times$, $H$ defines a height on the projective space $PW_{n,d}(k)$. By the Plücker embedding, $H$ is regarded as a height on the Grassmanian $Gr_{n,d}(k)$ of all $d$-dimensional subspaces of $k^n$. For $X \in Gr_{n,d}(k)$, $H(X)$ is precisely given by $H(x_1 \wedge \cdots \wedge x_d)$, where $x_1, \ldots, x_d$ is an arbitrary $k$-basis of $X$. More generally, for each $g = (g_v)$ in $GL_n(A)$, the twisted height $H_g$ on $Gr_{n,d}(k)$ is defined as

$$H_g(X) = \prod_v H_v(g_v x_1 \wedge \cdots \wedge g_v x_d).$$

Now the constant $\gamma_{n,d}(k)$ is defined to be

$$\gamma_{n,d}(k) = \max_{g \in GL_n(A)} \min_{X \in Gr_{n,d}(k)} \frac{H_g(X)^2}{\det g|_A^{2d/(nr)}}.$$  

In the case of $k = \mathbb{Q}$, this definition is identical with (1) and (3), so that one has $\gamma_{n,d}(\mathbb{Q}) = \gamma_{n,d}$. As generalizations of Minkowski – Hlawka bound and Rankin’s inequality, Thunder showed
Theorem. ([T]) One has

\[
\left( \frac{n|D_k|^{d(n-d)/2}}{\text{Res}_{s=1}\zeta_k(s)} \prod_{j=n-d+1}^n \frac{Z_k(j)}{Z_k(j)} \right)^{2/(nr)} \leq \gamma_{n,d}(k) \leq \left( \frac{2^{r_1+r_2}|D_k|^{1/2}}{V(n)^{r_1/n}V(2n)^{r_2/n}} \right)^{2d/r}
\]

and

\[
\gamma_{n,d}(k) \leq \gamma_{m,d}(k)(\gamma_{n,m}(k))^{d/m} \quad (1 \leq d < m \leq n-1).
\]

Here \( Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s) \) denotes the zeta function of \( k \), \( D_k \) the discriminant of \( k \) and \( r_1 \) (resp. \( r_2 \)) the number of real (resp. imaginary) places of \( k \).

We particularly write \( \gamma_n(k) \) for \( \gamma_{n,1}(k) \). Newman ([N, XI]) and Icaza ([I]) also considered \( \gamma_n(k) \) based on Humbert's reduction theory. Newman gave exact values of \( \gamma_2(k) \) for some Euclidean imaginary quadratic fields. To be precise, one has \( \gamma_2(\mathbb{Q}(\sqrt{-1})) = \sqrt{2}, \gamma_2(\mathbb{Q}(\sqrt{-2})) = 2, \gamma_2(\mathbb{Q}(\sqrt{-3})) = \sqrt{6}/2, \gamma_2(\mathbb{Q}(\sqrt{-7})) = \sqrt{21}/3 \) and \( \gamma_2(\mathbb{Q}(\sqrt{-11})) = \sqrt{22}/2 \). As for \( \gamma_2(k) \) of real quadratic fields, some numerical examples and conjectures were given by Cohn [C]. Recently, Coulangeon proved a part of Cohn's conjecture, i.e., \( \gamma_2(\mathbb{Q}(\sqrt{2})) = 2/\sqrt{2\sqrt{6}-3}, \gamma_2(\mathbb{Q}(\sqrt{3})) = 4 \) and \( \gamma_2(\mathbb{Q}(\sqrt{5})) = 2/\sqrt{5} \), by using the Voronoi reduction. In a general \( k \), Ohno and the author obtained an upper bound of \( \gamma_n(k) \) better than (5).

Theorem. ([O-W]) One has

\[
\gamma_n(k) \leq |D_k|^{1/r} \frac{\gamma_{nr}(\mathbb{Q})}{r}.
\]

Combining this with (5), one obtains

\[
\frac{r}{\pi} \left\{ \frac{n w_k \Gamma(n/2)^{r_1} \Gamma(n)^{r_2} \zeta_k(n)}{2^{r_1+n r_2} h_k R_k} \right\}^{2/(nr)} \leq \gamma_{nr}(\mathbb{Q})
\]

for any algebraic number field \( k \) of degree \( r \). Here \( h_k, R_k \) and \( w_k \) denote the class number of \( k \), the regulator of \( k \) and the number of the roots of unity in \( k \), respectively.

If a small \( n \) is fixed, there are some numerical examples that (6) for a suitable \( k \) is better than the Minkowski-Hlawka bound of \( \gamma_{nr}(\mathbb{Q}) \).
3. Generalized Hermite constants of flag varieties. Thunder's definition of Hermite's constant can be extended to flag varieties. In order to do this, we use a theory of linear algebraic groups. Let $G$ be a connected reductive linear algebraic group defined over $k$ and $\pi: G \to GL(V_\pi)$ a $k$-rational absolutely irreducible representation. We denote by $D_\pi$ the highest weight line in $V_\pi$ with respect to a fixed Borel subgroup of $G$. The stabilizer $Q_\pi$ of $D_\pi$ in $G$ is a parabolic subgroup of $G$. The representation $\pi$ is said to be strongly $k$-rational if $Q_\pi$ is defined over $k$. Then the flag variety $G/Q_\pi$ is defined over $k$ and is embedded in the projective space $PV_\pi$. Let $G(A)$ be the adele group of $G$ and $G(A)^1$ the group consisting of $g \in G(A)$ such that $|\chi(g)|_A = 1$ for any $k$-rational character $\chi$ of $G$. For each $g \in GL(V_\pi(A))$, a twisted height $H_g$ on $PV_\pi(k)$ is defined similarly to $\S 2$. Then we can prove that the following maximum exists for any strongly $k$-rational $\pi$ ([W, Proposition 2]):

$$
\gamma^G_\pi = \max_{g \in G(A)^1} \min_{\gamma \in G(k)} H_\pi(g\gamma)(D_\pi)^2,
$$

where we regard $D_\pi$ as a $k$-rational point in $PV_\pi$. If $G = GL_n$ and $\pi$ is a $d$-th exterior representation $\pi_d$ of $G$, then one sees $\gamma^G_{\pi_d} = \gamma_{n,d}(k)$. A mean value argument used to prove Minkowski-Hlawka bound works well in this general setting (cf. [M-W, $\S 3.3$]).

Theorem. ([W]) If $Q = Q_\pi$ is a maximal parabolic subgroup of $G$, we have a lower estimate of the form

$$
(7) \quad \left( \frac{C_Qd_GE_Q\tau(G)}{C_Gd_Q\tau(Q)} \right)^{2e_\pi/(e_Qe_\pi)} \leq \gamma^G_\pi.
$$

Here $\tau(G)$ and $\tau(Q)$ denote the Tamagawa numbers of $G$ and $Q$, respectively, $d_G$, $d_Q$, $e_Q$ and $e_\pi$ are some elementary positive rational numbers depending on $G$, $Q$ and $\pi$, and furthermore $C_G$ and $C_Q$ are the volumes of some maximal compact subgroups of $G(A)$ and $Q(A)$, respectively.

If $G$ is split over $k$, both constants $C_G$ and $C_Q$ are described by special values of the Dedekind zeta function. Particularly, the estimate (7) in the case of $G = GL_n$ and $\pi = \pi_d$ coincides with the lower bound of (5). An upper bound of $\gamma^G_\pi$ is not yet known in general.

4. Some examples. We show two examples. First, let $F: k^n \times k^n \to k$ be a nondegenerate symmetric bilinear form of Witt index $q \geq 1$ and $G = SO_F$ be the special orthogonal group of $F$. For $1 \leq d \leq q$, the $d$-th exterior representation $\pi_d: G(k) \to GL(W_{n,d}(k))$ yields a strongly $k$-rational representation of $G$. (The case $q = n/2 = d$ is exceptional since $\pi_q$ is not irreducible.) We write $\gamma^F_\pi$ for the generalized Hermite constant $\gamma^G_\pi$. As an analogue of (4), $\gamma^F_d$ has the following geometrical representation:

$$
\gamma^F_d = \max_{g \in G(A)} \min_{X \in Gr_{n,d}(k,F)} H_g(X)^2,
$$
where $\text{Gr}_{n,d}(k,F)$ denotes a subset of $\text{Gr}_{n,d}(k)$ consisting of $d$-dimensional totally isotropic subspaces of $k^n$ with respect to $F$. In particular, $\gamma^F_{1}$ is related to an existence of a nontrivial small integral solution of the homogeneous quadratic equation $F(x,x) = 0$. If $2q = n$ or $2q + 1 = n$, (7) gives

$$
\gamma^F_{1} \geq \left\{ \begin{array}{ll}
\left( \frac{|D_k|^{q-1}(2q-2)}{\text{Res}_{s=1} \zeta_k(s) Z_k(2q)} Z_k(2q-1) \right)^{1/((q-1)r)} & (2q = n) \\
\left( \frac{|D_k|^{q-1/2}(2q-1)}{\text{Res}_{s=1} \zeta_k(s) Z_k(2q)} \right)^{2/((2q-1)r)} & (2q + 1 = n)
\end{array} \right.
$$

Moreover, we can show the following estimate and an analogue of Rankin's inequality.

**Theorem.** ([O-W],[W2]) For any nondegenerate $F$, one has

$$
\gamma^F_d \leq \gamma_{n-d}(k)^n (2H(F))^{n-d} \quad (1 \leq d \leq q)
$$

$$
\gamma^F_d \leq \gamma_{m,d}(k) (\gamma^F_m)^{d/m} \quad (1 \leq d < m \leq q).
$$

Here $H(F)$ denotes a height of the symmetric matrix corresponding to $F$.

Second, let $D$ be a central simple division algebra of dimension $q^2$ over $k$ and $G$ be an inner $k$-form of $GL_{qn}$ whose group of $k$-rational points equals $GL_n(D)$. If a cyclic extension $L$ of degree $q$ over $k$ contained in $D$ is fixed, then $GL_n(D)$ is realized as a subgroup of $GL_{qn}(L)$. Since the $qd$-th exterior representation of $GL_{qn}(L)$ gives rise to a fundamental $k$-rational representation $\pi_d$ of $G$ for $1 \leq d \leq n-1$, one has the generalized Hermite constant $\gamma_{\pi_d}^G$. We write $\gamma_{n,d}(D)$ for $\gamma_{\pi_d}^G$. Geometrically, $\gamma_{n,d}(D)$ has the following representation similar to (4):

$$
\gamma_{n,d}(D) = \max_{g \in G(k)} \min_{X \in \text{BS}_{n,d}(D)} \frac{H_g(X)^2}{|\text{Nr}(g)|^{2d/(nr)}},
$$

where $\text{BS}_{n,d}(D)$ denotes the set of $d$-dimensional $D$-subspace in $D^n$ and $\text{Nr}$ the reduced norm on $M_n(D)$. The set $\text{BS}_{n,d}(D)$ is called the generalized Brauer–Severi variety and is realized as a subset of the Grassmanian $\text{Gr}_{qn,qd}(L)$. The twisted height $H_g$ on $\text{BS}_{n,d}(D)$ is defined as the restriction of that on $\text{Gr}_{qn,qd}(L)$. By using this expression, we can prove the following.

**Theorem.** ([W3]) One has

$$
\gamma_{n,d}(D) \leq \epsilon_D \left( \frac{2^{r_1(L)+r_2(L)}|D_L|^{1/2}}{V(qn)^{r_1(L)/(qn)}V(2qn)^{r_2(L)/(qn)}} \right)^{2d/r}
$$

and

$$
\gamma_{n,d}(D) \leq \gamma_{m,d}(D) (\gamma_{n,m}(D))^{d/m} \quad (1 \leq d < m \leq n-1).
$$
Here $D_L$ denotes the discriminant of $L$ and $r_1(L)$ (resp. $r_2(L)$) the number of real (resp. imaginary) places of $L$. The constant $\epsilon_D$ is given by

$$\epsilon_D = \left( \prod_w \max(1, |a|_w) \right)^{2(q-1)n/(qr)}$$

(w runs over all places of $L$)

if we realize $D$ as a cyclic algebra $[L/k, \sigma, a]$ by a generator $\sigma$ of the Galois group of $L/k$ and an element $a \in k^\times$.

REFERENCES


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