A SURVEY ON GENERALIZED HERMITE CONSTANTS

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This is an expository note on Hermite's constant. We give an account of a recent development of some generalizations of Hermite's constant.

1. Hermite–Rankin's constant. Let \( \mathcal{L}^n \) be the set of all lattices of rank \( n \) in the Euclidean space \( \mathbb{R}^n \). For \( L \in \mathcal{L}^n \), \( d(L) \) stands for the volume of the fundamental parallelepiped of \( L \). It was proved by Hermite that

\[
\min_{0 \neq x \in L} txx \leq \left( \frac{2}{\sqrt{3}} \right)^{n-1} d(L)^{2/n}
\]

holds for all \( L \in \mathcal{L}^n \). Thus \( \min_{0 \neq x \in L} txx/d(L)^{2/n} \) is bounded and there exists the maximum

\[
\gamma_n = \max_{L \in \mathcal{L}^n} \min_{0 \neq x \in L} \frac{txx}{d(L)^{2/n}}.
\]

The constant \( \gamma_n \) is called Hermite's constant. A well-known example of its appearance is the lattice sphere packing problem, namely the density of the densest lattice packing of spheres in \( \mathbb{R}^n \) equals

\[
\delta_n = \gamma^{n/2} \frac{V(n)}{2^n},
\]

where \( V(n) \) denotes the volume of the unit ball in \( \mathbb{R}^n \), i.e., \( V(n) = \pi^{n/2}/\Gamma(1 + n/2) \). Originally, \( \gamma_n \) arose from the reduction theory of positive definite quadratic forms initiated by Lagrange, Seeber and Gauss. In terms of quadratic forms, \( \gamma_n \) is represented as

\[
(1) \quad \gamma_n = \max_{g \in GL_n(\mathbb{R})} \min_{0 \neq x \in \mathbb{Z}^n} \frac{tx^tggx}{(\det g)^{2/n}}.
\]

The exact value of \( \gamma_n \) is known only for \( n \leq 8 \), i.e., \( \gamma_2 = 2/\sqrt{3}, \gamma_3 = \sqrt[3]{2}, \gamma_4 = \sqrt{2}, \gamma_5 = \sqrt[5]{8}, \gamma_6 = \sqrt[6]{64}/3, \gamma_7 = \sqrt[7]{64}, \gamma_8 = 2 \). One has the estimate

\[
(2) \quad \left( \frac{2\zeta(n)}{V(n)} \right)^{2/n} \leq \gamma_n \leq 4 \left( \frac{1}{V(n)} \right)^{2/n}.
\]

This upper bound was given by Minkowski and follows from \( \delta_n \leq 1 \). The lower bound was first stated by Minkowski and was proved by Hlawka.
The next step of Hermite’s constant is the following extension due to Rankin. For every $1 \leq d \leq n - 1$, define

$$
\gamma_{n,d} = \max_{L \in \mathbb{C}^n} \min_{z_1, \ldots, z_d \in L, z_1 \wedge \cdots \wedge z_d \neq 0} \frac{\det(t^i z_j)_{1 \leq i, j \leq d}}{d(L)^{2d/n}}.
$$

Obviously, $\gamma_{n,1}$ equals $\gamma_n$. Rankin ([R]) proved $\gamma_{n,d}$ satisfies the inequality

$$
\gamma_{n,d} \leq \gamma_{m,d}(\gamma_{n,m})^{d/m}
$$

for $1 \leq d < m \leq n - 1$, and he showed $\gamma_{4,2} = 3/2$. Rankin’s inequality and the duality $\gamma_{n,d} = \gamma_{n,n-d}$ yield Mordell’s inequality $\gamma_n^{n-2} \leq \gamma_{n-1}^{n-1}$.

2. Icaza–Thunder’s generalization. As a generalization of Hermite–Rankin constant, Thunder defined the constant $\gamma_{n,d}(k)$ for any algebraic number field $k$ of finite degree $r$ over $\mathbb{Q}$ in 1997. At first, we recall a definition of twisted heights. Let $e_1, \ldots, e_n$ be a standard basis of $k^n$. For any extension field $L$ over $k$, $W_{n,d}(L)$ stands for the $d$-th exterior product of $L^n$. A basis of $W_{n,d}(k)$ is formed by the elements $e_I = e_{i_1} \wedge \cdots \wedge e_{i_d}$ with $I = \{1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$. For each place $v$ of $k$, let $k_v$ be the completion of $k$ at $v$ and $| \cdot |_v$ the usual normalized absolute value of $k_v$. We define the local height on $W_{n,d}(k_v)$ by

$$
H_v(\sum_I a_I e_I) = \begin{cases} 
(\sum_I |a_I|_v^{[\mathbb{C}:k_v]})^{1/([\mathbb{C}:k_v]r)} & \text{(if } v \text{ is infinite)} \\
(\sup_I |a_I|_v)^{1/r} & \text{(if } v \text{ is finite)}
\end{cases}
$$

Then the global height $H$ on $W_{n,d}(k)$ is defined to be the product of $H_v$:

$$
H(x) = \prod_v H_v(x) \quad (x \in W_{n,d}(k)).
$$

Let $A$ be the adele ring of $k$ and $| \cdot |_A$ the idele norm on $A^\times$. Since $H(ax) = |\alpha|_A^{1/r}H(x) = H(x)$ for $\alpha \in k^\times$, $H$ defines a height on the projective space $\mathbb{P}W_{n,d}(k)$. By the Plücker embedding, $H$ is regarded as a height on the Grassmanian $\text{Gr}_{n,d}(k)$ of all $d$-dimensional subspaces of $k^n$. For $X \in \text{Gr}_{n,d}(k)$, $H(X)$ is precisely given by $H(x_1 \wedge \cdots \wedge x_d)$, where $x_1, \ldots, x_d$ is an arbitrary $k$-basis of $X$. More generally, for each $g = (g_v)$ in $GL_n(A)$, the twisted height $H_g$ on $\text{Gr}_{n,d}(k)$ is defined as

$$
H_g(X) = \prod_v H_v(g_v x_1 \wedge \cdots \wedge g_v x_d).
$$

Now the constant $\gamma_{n,d}(k)$ is defined to be

$$
\gamma_{n,d}(k) = \max_{g \in GL_n(A)} \min_{X \in \text{Gr}_{n,d}(k)} \frac{H_g(X)^2}{\det g_A^{2d/(nr)}}.
$$

In the case of $k = \mathbb{Q}$, this definition is identical with (1) and (3), so that one has $\gamma_{n,d}(\mathbb{Q}) = \gamma_{n,d}$. As generalizations of Minkowski – Hlawka bound and Rankin’s inequality, Thunder showed
Theorem. ([T]) One has

\[
\left( \frac{n|D_k|^{d(n-d)/2}}{\text{Res}_{s=1} \zeta_k(s)} \prod_{j=n-d+1}^{n} Z_k(j) \right)^{2/(nr)} \leq \gamma_{n,d}(k) \leq \left( \frac{2^{r_1+r_2}|D_k|^{1/2}}{V(n)^{r_1/n}V(2n)^{r_2/n}} \right)^{2d/r}
\]

and

\[
\gamma_{n,d}(k) \leq \gamma_{m,d}(k)(\gamma_{n,m}(k))^{d/m} \quad (1 \leq d < m \leq n-1).
\]

Here \(Z_k(s) = (\pi^{-s/2} \Gamma(s/2))^{r_1}((2\pi)^{1-s} \Gamma(s))^{r_2} \zeta_k(s)\) denotes the zeta function of \(k\), \(D_k\) the discriminant of \(k\) and \(r_1\) (resp. \(r_2\)) the number of real (resp. imaginary) places of \(k\).

We particularly write \(\gamma_{n}(k)\) for \(\gamma_{n,1}(k)\). Newman ([N, XI]) and Icaza ([I]) also considered \(\gamma_{n}(k)\) based on Humbert's reduction theory. Newman gave exact values of \(\gamma_2(k)\) for some Euclidean imaginary quadratic fields. To be precise, one has \(\gamma_2(\mathbb{Q}(\sqrt{-1})) = \sqrt{2}, \gamma_2(\mathbb{Q}(\sqrt{-2})) = 2, \gamma_2(\mathbb{Q}(\sqrt{-3})) = \sqrt{6}/2, \gamma_2(\mathbb{Q}(\sqrt{-7})) = \sqrt{21}/3\), and \(\gamma_2(\mathbb{Q}(\sqrt{-11})) = \sqrt{22}/2\). As for \(\gamma_2(k)\) of real quadratic fields, some numerical examples and conjectures were given by Cohn [C]. Recently, Coulangeon proved a part of Cohn's conjecture, i.e., \(\gamma_2(\mathbb{Q}(\sqrt{2})) = 2/\sqrt{2\sqrt{6} - 3}, \gamma_2(\mathbb{Q}(\sqrt{3})) = 4\) and \(\gamma_2(\mathbb{Q}(\sqrt{5})) = 2/\sqrt{5}\), by using the Voronoi reduction. In a general \(k\), Ohno and the author obtained an upper bound of \(\gamma_n(k)\) better than (5).

Theorem. ([O-W]) One has

\[
\gamma_{n}(k) \leq |D_k|^{1/r} \frac{\gamma_{nr}(\mathbb{Q})}{r}.
\]

Combining this with (5), one obtains

\[
\frac{r}{\pi} \left\{ \frac{n w_k \Gamma(n/2)^{r_1} \Gamma(n)^{r_2} \zeta_k(n)}{2^{r_1+n r_2} h_k R_k} \right\}^{2/(nr)} \leq \gamma_{nr}(\mathbb{Q})
\]

for any algebraic number field \(k\) of degree \(r\). Here \(h_k, R_k\) and \(w_k\) denote the class number of \(k\), the regulator of \(k\) and the number of the roots of unity in \(k\), respectively.

If a small \(n\) is fixed, there are some numerical examples that (6) for a suitable \(k\) is better than the Minkowski-Hlawka bound of \(\gamma_{nr}(\mathbb{Q})\).
3. Generalized Hermite constants of flag varieties. Thunder's definition of Hermite's constant can be extended to flag varieties. In order to do this, we use a theory of linear algebraic groups. Let $G$ be a connected reductive linear algebraic group defined over $k$ and $\pi: G \to GL(V_\pi)$ a $k$-rational absolutely irreducible representation. We denote by $D_\pi$ the highest weight line in $V_\pi$ with respect to a fixed Borel subgroup of $G$. The stabilizer $Q_\pi$ of $D_\pi$ in $G$ is a parabolic subgroup of $G$. The representation $\pi$ is said to be strongly $k$-rational if $Q_\pi$ is defined over $k$. Then the flag variety $G/Q_\pi$ is defined over $k$ and is embedded in the projective space $\mathbb{P}V_\pi$. Let $G(A)$ be the adele group of $G$ and $G(A)^1$ the group consisting of $g \in G(A)$ such that $|\chi(g)|_\lambda = 1$ for any $k$-rational character $\chi$ of $G$. For each $g \in GL(V_\pi(A))$, a twisted height $H_g$ on $\mathbb{P}V_\pi(k)$ is defined similarly to §2. Then we can prove that the following maximum exists for any strongly $k$-rational $\pi$ ([W, Proposition 2]):

$$\gamma^G_\pi = \max_{g \in G(A)^{1}} \min_{\gamma \in G(k)} H_\pi(g\gamma)(D_\pi)^2,$$

where we regard $D_\pi$ as a $k$-rational point in $\mathbb{P}V_\pi$. If $G = GL_n$ and $\pi$ is a $d$-th exterior representation $\pi_d$ of $G$, then one sees $\gamma^G_{\pi_d} = \gamma_{n,d}(k)$. A mean value argument used to prove Minkowski–Hlawka bound works well in this general setting (cf. [M-W, §3.3]).

Theorem. ([W]) If $Q = Q_\pi$ is a maximal parabolic subgroup of $G$, we have a lower estimate of the form

$$(7) \quad \left(\frac{C_Qd_Ge_Q\tau(G)}{C_Gd_Q\tau(Q)}\right)^{2\pi/(e_Qr)} \leq \gamma^G_\pi.$$

Here $\tau(G)$ and $\tau(Q)$ denote the Tamagawa numbers of $G$ and $Q$, respectively, $d_G$, $d_Q$, $e_Q$ and $e_\pi$ are some elementary positive rational numbers depending on $G$, $Q$ and $\pi$, and furthermore $C_G$ and $C_Q$ are the volumes of some maximal compact subgroups of $G(A)$ and $Q(A)$, respectively.

If $G$ is split over $k$, both constants $C_G$ and $C_Q$ are described by special values of the Dedekind zeta function. Particularly, the estimate (7) in the case of $G = GL_n$ and $\pi = \pi_d$ coincides with the lower bound of (5). An upper bound of $\gamma^G_\pi$ is not yet known in general.

4. Some examples. We show two examples. First, let $F: k^n \times k^n \to k$ be a nondegenerate symmetric bilinear form of Witt index $q \geq 1$ and $G = SO_F$ be the special orthogonal group of $F$. For $1 \leq d \leq q$, the $d$-th exterior representation $\pi_d: G(k) \to GL(W_{n,d}(k))$ yields a strongly $k$-rational representation of $G$. (The case $q = n/2 = d$ is exceptional since $\pi_q$ is not irreducible.) We write $\gamma^F_d$ for the generalized Hermite constant $\gamma^G_{\pi_d}$. As an analogue of (4), $\gamma^F_d$ has the following geometrical representation:

$$\gamma^F_d = \max_{g \in G(A)} \min_{X \in \text{Gr}_{n,d}(k,F)} H_g(X)^2,$$
where $\text{Gr}_{n,d}(k,F)$ denotes a subset of $\text{Gr}_{n,d}(k)$ consisting of $d$-dimensional totally isotropic subspaces of $k^n$ with respect to $F$. In particular, $\gamma^F_1$ is related to an existence of a nontrivial small integral solution of the homogeneous quadratic equation $F(x,x) = 0$. If $2q = n$ or $2q + 1 = n$, (7) gives

$$\gamma^F_1 \geq \begin{cases} 
\frac{|D_k|^{q-1}(2q-2)}{\text{Res}_{s=1} \zeta_k(s)} Z_k(2q) & (2q = n) \\
\frac{|D_k|^{q-1/2}(2q-1)}{\text{Res}_{s=1} \zeta_k(s)} Z_k(2q) & (2q + 1 = n)
\end{cases}$$

Moreover, we can show the following estimate and an analogue of Rankin's inequality.

**Theorem.** ([O-W],[W2]) For any nondegenerate $F$, one has

$$\gamma^F_d \leq \gamma_{n-d}(k)^{n-d}(2H(F))^{n-d} \quad (1 \leq d \leq q)$$
$$\gamma^F_d \leq \gamma_{m,d}(k)(\gamma^F_m)^{d/m} \quad (1 \leq d < m \leq q).$$

Here $H(F)$ denotes a height of the symmetric matrix corresponding to $F$.

Second, let $D$ be a central simple division algebra of dimension $q^2$ over $k$ and $G$ be an inner $k$-form of $GL_{q^n}$ whose group of $k$-rational points equals $GL_n(D)$. If a cyclic extension $L$ of degree $q$ over $k$ contained in $D$ is fixed, then $GL_n(D)$ is realized as a subgroup of $GL_{q^n}(L)$. Since the $qd$-th exterior representation of $GL_{q^n}(L)$ gives rise to a fundamental $k$-rational representation $\pi_d$ of $G$ for $1 \leq d \leq n-1$, one has the generalized Hermite constant $\gamma^G_{\pi_d}$. We write $\gamma_{n,d}(D)$ for $\gamma^G_{\pi_d}$. Geometrically, $\gamma_{n,d}(D)$ has the following representation similar to (4):

$$\gamma_{n,d}(D) = \max_{g \in G(k)} \min_{X \in \text{BS}_{n,d}(D)} \frac{H_g(X)^2}{|\text{Nr}(g)|^{2d/(nr)}},$$

where $\text{BS}_{n,d}(D)$ denotes the set of $d$-dimensional $D$-subspace in $D^n$ and $\text{Nr}$ the reduced norm on $M_n(D)$. The set $\text{BS}_{n,d}(D)$ is called the generalized Brauer–Severi variety and is realized as a subset of the Grassmanian $\text{Gr}_{q^n,qd}(L)$. The twisted height $H_g$ on $\text{BS}_{n,d}(D)$ is defined as the restriction of that on $\text{Gr}_{q^n,qd}(L)$. By using this expression, we can prove the following.

**Theorem.** ([W3]) One has

$$\gamma_{n,d}(D) \leq \varepsilon_D \left( \frac{2^{r_1(L)+r_2(L)}|D_L|^{1/2}}{V(qn)^{r_1(L)/(qn)} V(2qn)^{r_2(L)/(qn)}} \right)^{2d/r}$$

and

$$\gamma_{n,d}(D) \leq \gamma_{m,d}(D)(\gamma_{n,m}(D))^{d/m} \quad (1 \leq d < m \leq n - 1).$$
Here $D_L$ denotes the discriminant of $L$ and $r_1(L)$ (resp. $r_2(L)$) the number of real (resp. imaginary) places of $L$. The constant $\epsilon_D$ is given by

$$\epsilon_D = \left( \prod_w \max(1, |a_w|) \right)^{2(q-1)n/(qr)} \quad (w \text{ runs over all places of } L)$$

if we realize $D$ as a cyclic algebra $[L/k, \sigma, a]$ by a generator $\sigma$ of the Galois group of $L/k$ and an element $a \in k^\times$.

REFERENCES


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