数理解析研究所講究録 1200 巻 2001 年 82-91

On central critical values of the degree four L-functions for GSp (4) and the matrix argument Kloosterman sums (joint work with J. A. Shalika)

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December 19, 2000

1. Motivation

Let Φ be a Siegel eigen cusp form of degree two of weight k with respect to $\mathrm{Sp}_4(\mathbb{Z})$ and let

$$\Phi(Z)=\sum_{T>0} a(T,\Phi)\exp\left[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\right]$$

be its Fourier expansion. Here T runs over $T=\begin{pmatrix} t_1 & t_2/2 \\ t_2/2 & t_3 \end{pmatrix}$ such that $t_1,t_2,t_3\in\mathbb{Z}$ and T is positive definite. For such T_1 and T_2, let

$$T_1\sim T_2 \overset{\mathrm{def}}{\Longleftrightarrow} \exists\gamma\in \mathrm{SL}_2(\mathbb{Z}) \text{ s.t. } T_2={}^t\gamma T_1\gamma.$$

Let E be an imaginary quadratic field and let D_E be its discriminant. Then we define

$$B_E(\Phi)\overset{\mathrm{def}}{=}\sum_{\{T\mid\det T=-D_E/4\}/\sim}\frac{a(T,\Phi)}{\varepsilon(T)}$$

where $\epsilon(T)=\#\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\mid {}^t\gamma T\gamma=T\}$. We recall that, by Gauss, there exists a bijection between the set $\{T\mid\det T=-D_E/4\}/\sim$ and the ideal class group of E.

Böcherer has proclaimed the following conjecture in 1986 (Preprint Math. Gottingensis Heft 68).

Böcherer's Conjecture . *There exists a constant c_Φ that depends only on Φ such that*

$$L\left(\frac{1}{2}, \Phi \otimes \chi_E\right) = c_\Phi \cdot |D_E|^{-k+1} \cdot |B_E(\Phi)|^2 \tag{1}$$

for any E. Here $L(s, \Phi \otimes \chi_E)$ denotes the spinor (degree four) L-function of Φ twisted by the quadratic character χ_E corresponding to the quadratic extension $E/\mathbb{Q}$, normalized so that its functional equation is with respect to $s \mapsto 1-s$.

Remarks

1. The *center* $s = 1/2$ is the only *critical point* in the sense of Deligne.
2. Böcherer, and, later he and Schulze-Pillot (Math. Z. **209** (1992)) verified the assertion for Eisenstein series, Saito-Kurokawa lifting and Yoshida lifting.
3. Kohnen and Kuss have made some numerical experiment on an eigenform of weight 20, which does not belong to the Saito-Kurokawa lifting.
4. We may normalize Φ so that $a(T, \Phi) \in \bar{\mathbb{Q}}$, hence $B_E(\Phi) \in \bar{\mathbb{Q}}$. Thus we may regard $|B_E(\Phi)|^2$ as the *algebraic part* of the special value. It is natural for us to fantasize about the generalized Birch&Swinnerton-Dyer conjecture, p-adic interpolation, etc.
5. Böcherer did not make any speculation about the constant c_Φ. It is important to identify c_Φ from the viewpoint of Deligne's conjecture (Proc. Sympos. Pure Math. **33**, 1979) since it is related to the *period part* of the special value.

Böcherer's conjecture reminds us of:

Waldspurger's Theorem . *(Compositio Math. **54** (1985)) Let F be a number field. Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. For Ω, a Hecke character of $\mathbb{A}_E^\times$ where E is a quadratic extension of F, let $\pi(\Omega)$ denote the theta series representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Assume that $\Omega|_{\mathbb{A}_F^\times} \cdot \omega_\pi = 1$ where ω_π denotes the central character of π. Then we have*

$$L\left(\frac{1}{2}, \pi \otimes \pi(\Omega)\right) \neq 0$$

if and only if there exists a quaternion algebra D over F containing E and an automorphic form φ^D in the space of π^D where π^D denotes the Jacquet-Langlands correspondent of π of $D^\times(\mathbb{A}_F)$ such that

$$\int_{\mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times} \varphi^D(t)\,\Omega(t)\,d^\times t \neq 0. \tag{2}$$

Remarks

1. Let $BC^E(\pi)$ denote the *base change* lifting of π to $\mathrm{GL}_2(\mathbb{A}_E)$. Then we have

$$L(s, \pi \otimes \pi(\Omega)) = L\left(s, BC^E(\pi) \otimes \Omega\right)$$

   and in particular, when Ω is trivial,

$$L\left(s, BC^E(\pi)\right) = L(s,\pi) \cdot L(s, \pi \otimes \chi_E)$$

   where χ_E denotes the quadratic character corresponding to E/F.
2. When Ω is trivial, there exists the following metaplectic version of the theorem which might be more familiar (Kohnen-Zagier, Invent. math. **64** (1981)): *Let f be a normalized eigenform of weight $2k$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ and let g be its Shimura correspondent, i.e.*

$$g(z) = \sum_{n \geq 1} b(n) \exp\left(2\pi\sqrt{-1}nz\right) \in S^+_{k+\frac{1}{2}}(4).$$

   Then for the fundamental discriminant D of $E = \mathbb{Q}\left(\sqrt{D}\right)$ such that $(-1)^k D > 0$, we have

$$\frac{|b(|D|)|^2}{(g,g)} = \frac{(k-1)!|D|^{k-1/2} L(k, f \otimes \chi_E)}{\pi^k (f,f)}. \tag{3}$$

   (Here we use the classical normalization for the L-function so that the functional equation is with respect to $s \mapsto 2k - s$.) We remark that (3) implies the non-negativity of the central value $L(k, f \otimes \chi_E)$ which is consistent with the generalized Riemann hypothesis (cf. Guo, Duke Math. J. **83** (1996)).
3. Similarly, in general, further analysis yields an identity that expresses the central critical value

$$L(1/2,\pi)\, L(1/2, \pi \otimes \chi_E) \text{ (resp. } L\left(1/2, BC^E(\pi) \otimes \Omega\right))$$

as the square norm of one of these period integrals in (2) multiplied by a constant C_π (resp. $C_{\pi,E}$) which depends only on π (resp. π and E), *not* on E (resp. Ω) (Chen and Jacquet, Bull Soc. Math. France, to appear).

4. The choice of the quaternion algebra which gives the non-zero period integral (2) is unique and is determined at each place by the local ε-factor of the L-function (see H. Saito, Compositio Math. **85** (1993)). This is a special case of the Gross-Prasad conjecture (Canad. J. Math. **44** (1992) and ibid **46** (1994)).

The original proof by Waldspurger was based on the Weil representation, i.e. theta correspondence. Later Jacquet has given another proof using the relative trace formula. Actually he proved *two* relative trace formulas, one corresponding to the case when Ω is trivial (Ann. scient. Éc. Norm. Sup. **19** (1986)) and the other corresponding to the case when Ω is arbitrary (Compositio Math. **63** (1987)).

2. Our Project

The *ultimate* goal of our project is to prove Böcherer's conjecture and its generalization by extending both of Jacquet's relative trace formulas to $\mathrm{GSp}(4) = \{g \in \mathrm{GL}_4 \mid {}^t gJg = \lambda J,\ \lambda \in \mathrm{GL}_1\}$, where $J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$.

Since our *conjectural* relative trace formulas themselves are too technical to state here, we refer to the two announcements (C.R. Acad. Sci. Paris **328** (1999), 105–110 and ibid **331** (2000), 593–598) for the details.

Instead let us explain the expected consequences of the trace formulas. First we need to introduce some notation. Let F be a number field and E be a quadratic extension of F and let $X(E : F)$ denote the set of isomorphism classes of the central quaternion algebras over F containing E. For $\epsilon \in F^\times$, let

$$D_\epsilon = \left\{ \begin{pmatrix} a & \epsilon b \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in E \right\}$$

where σ denotes the unique non-trivial element in the Galois group of E over F. Then $\epsilon \mapsto D_\epsilon$ induces a bijection between $F^\times / N_{E/F}(E^\times)$ and $X(E : F)$. Let $x \mapsto \bar{x}$ denote the involution of D_ϵ. Let

$$G_\epsilon = \left\{ g \in \mathrm{GL}_2(D_\epsilon) \mid g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda(g) \in \mathrm{GL}_1(F) \right\}$$

where $g^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Here we note that when $\epsilon = 1$, we have $D_1 \simeq \mathrm{M}_2(F)$ and $G_1 \simeq \mathrm{GSp}(4, F)$. Let us define the Bessel subgroup R_ϵ of G_ϵ by

$$R_\epsilon = \left\{ \begin{pmatrix} a & & & \\ & a^\sigma & & \\ & & a & \\ & & & a^\sigma \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mid a \in E^\times, \mathrm{tr} X = 0 \right\}.$$

Let ψ be a non-trivial character of $\mathbb{A}_F/F$ and let Ω be a character of $\mathbb{A}_E^\times/E^\times$. Then by abuse of notation we denote by Ω a character of $R_\epsilon(\mathbb{A}_F)$ defined by

$$\Omega\left[\begin{pmatrix} a & & & \\ & a^\sigma & & \\ & & a & \\ & & & a^\sigma \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right] = \Omega(a) \cdot \psi\left[\mathrm{tr}\left(\begin{pmatrix} -\eta & 0 \\ 0 & \eta \end{pmatrix} X\right)\right]$$

where $\eta \in E$ such that $E = F(\eta)$ and $\eta^2 \in F$.

Conjecture (Furusawa&Shalika) . *Let π be an irreducible cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A}_F)$. *Assume that the central character of π is equal to the inverse of $\Omega\mid_{\mathbb{A}_F^\times}$.*

Then we have

$$L\left(\frac{1}{2}, \pi \otimes \pi(\Omega)\right) \neq 0$$

if and only if there exists a triple $(\epsilon, \pi_\epsilon, \varphi_\epsilon)$, where $\epsilon \in F^\times$, π_ϵ an irreducible cuspidal representation of $G_\epsilon(\mathbb{A}_F)$, corresponding to π in the functorial sense, i.e. having the same L-function, and φ_ϵ a cusp form in the space of π_ϵ such that

$$\int_{\mathbb{A}_F^\times R_\epsilon(F)\backslash R_\epsilon(\mathbb{A}_F)} \varphi_\epsilon(r)\,\Omega(r)\,dr \neq 0. \qquad (4)$$

Moreover, the detailed analysis should yield an identity that expresses the central critical value of $L(s, \pi \otimes \pi(\Omega))$ as the square norm of one of these period integrals (4) multiplied by a constant $C'_{\pi,E}$ which depends only on π and E, *not* on the character Ω of $\mathbb{A}_E^\times$. Also when Ω is trivial, the central critical value of $L(s,\pi)\,L(s, \pi \otimes \chi_E)$ should be the square norm of the period integral (4) multiplied by a constant C_π which depends only on π and *not* on the quadratic extension E.

In particular when π is the cuspidal representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ corresponding to a holomorphic Siegel eigen cusp form $\Phi = \Phi_{\mathrm{hol}}$, by looking at the Fourier expansion of Φ_{hol}, it should follow as a corollary of the conjecture that there exists an imaginary quadratic field E and a *finite* order Hecke character of $\mathbb{A}_E^\times$ such that

$$L\left(\frac{1}{2}, \pi \otimes \pi(\Omega)\right) \neq 0.$$

We also speculate that the constant C_π mentioned above is given, in this case, essentially as a ratio of Petersson norms

$$\frac{(\Phi_{\mathrm{gen}}, \Phi_{\mathrm{gen}})}{(\Phi_{\mathrm{hol}}, \Phi_{\mathrm{hol}})}$$

where Φ_{gen} denotes a *generic* cusp form, i.e. having a non-zero Whittaker Fourier coefficient, corresponding to the same L-function as Φ_{hol}. It indicates that it is important to study the whole L-packet (i.e. all the automorphic representations giving the same L-function) in order to understand the nature of the special values of the L-function. We remark that the constant C_π here is essentially equal to $c_\Phi \cdot L(1/2, \pi)$ where c_Φ denotes the constant in Böcherer's original conjecture (1).

The first but crucial step to establish a trace formula is to prove the fundamental lemma, an equality between two local orbital integrals for the elements in the Hecke algebra. We have proved the fundamental lemma for the identity element in the Hecke algebra for both of the trace formulas. We have also proved the Plancherel formula for the Bessel model which reduces the fundamental lemma for the general element in the Hecke algebra to an equality between two finite sums of certain local orbital integrals for the identity element.

The proof of the fundamental lemma for the identity element essentially amounts to computing some matrix argument character sums, which we are going to discuss.

3. Matrix Argument Kloosterman Sums

From now on we denote by F a non-archimedean local field whose residual characteristic is not equal to *two*. Let ψ be a character of F whose conductor is $\mathcal{O}_F$, the ring of integers of F and let ϖ be a prime element of F. We denote by q the cardinality of the residue field $\mathcal{O}_F/\varpi\mathcal{O}_F$. Let

E be the unique unramified quadratic extension of F and $\mathcal{O}_E$ be its ring of integers.

First we recall the *classical* Kloosterman sum defined by

$$\mathcal{K}\ell(r,s) = \int_{\mathcal{O}^\times} \psi\left(r\varepsilon + s\varepsilon^{-1}\right) d\varepsilon$$

for $r, s \in F^\times$. Sometimes we call it the GL_2 Kloosterman sum since it is related to the Fourier coefficients of the Poincaré series for GL_2. For $a \in \mathcal{O}_F^\times$, let

$$\mathcal{H}_n(a) = \int_{\mathcal{Z}_a} \psi\left[\mathrm{tr}_{E/F}(\xi)\right] d\xi$$

where $\mathcal{Z}_a = \left\{\xi \in \mathcal{O}_E^\times \mid N_{E/F}(\xi) \equiv a \pmod{\varpi^n}\right\}$. Here we recall the *Davenport-Hasse relation*:

$$\mathcal{H}_n(a) = (-1)^n q^{-n} \cdot \mathcal{K}\ell\left(2\varpi^{-n}, 2\varpi^{-n}a\right).$$

Now let us consider the following matrix argument Kloosterman sums. For $A \in \mathrm{GL}_n(F)$, $S, T \in \mathrm{Sym}^n(F)$ and $\varepsilon \in \mathcal{O}_F^\times$, let

$$\mathcal{K}(A,S,T,\varepsilon) = \int_{\mathcal{X}_A} \psi\left[\mathrm{tr}\left(XS + \varepsilon TA^{-1}X^{-1}\,{}^tA^{-1}\right)\right] dX \tag{5}$$

where $\mathcal{X}_A = \{X \in \mathrm{Sym}^n(F) \mid XA \in \mathrm{GL}_n(\mathcal{O}_F)\}$. Here we remark that we may write the right hand side of (5) as

$$\sum_V \psi\left[\mathrm{tr}\left(YA^{-1}S + \varepsilon \cdot A^{-1}VT\right)\right] \tag{6}$$

where V runs over the set of representatives modulo $A \cdot \mathrm{Sym}^n(\mathcal{O}_F)$ of $V \in M_n(\mathcal{O}_F)$ such that

$$\exists\, U, Y \in M_n(\mathcal{O}_F) \quad \text{with} \quad \begin{pmatrix} Y & U \\ A & V \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathcal{O}_F).$$

As Kitaoka has shown, the Kloosterman sum (6) appears in the Fourier coefficients of the Poincaré series for the Siegel modular group (Nagoya Math. J. **93** (1984)).

Now we restrict ourselves to the case when $n = 2$. Let us define the *split* Kloosterman sum by

$$\mathcal{K}_{\text{spl}}(A,\varepsilon) = \mathcal{K}(A, S_1, S_1, \varepsilon) = \int_{\mathcal{X}_A} \psi\left[\text{tr}\left(S_1\left(X + \varepsilon A^{-1} X^{-1}\, {}^t A^{-1}\right)\right)\right] dX$$

where $S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem 1. *Suppose that*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \varpi \cdot \mathrm{M}_2(\mathcal{O}_F) \cap \mathrm{GL}_2(F).$$

Then we have

$$\mathcal{K}_{\text{spl}}(A,\varepsilon) = \frac{1}{|\Delta|}\left\{\mathcal{K}\ell\left(\frac{2\alpha}{\Delta}, \frac{2\varepsilon\delta}{\Delta}\right) + \mathcal{K}\ell\left(\frac{2\beta}{\Delta}, \frac{2\varepsilon\gamma}{\Delta}\right)\right\}$$

where $\Delta = \det A$.

Thus the two by two symmetric matrix argument split Kloosterman sum reduces to a sum of two classical Kloosterman sums.

Let us define the anisotropic Kloosteman sum. For our convenience let us employ another realization of $\mathrm{M}_2(F)$, namely,

$$D_1 = \left\{\begin{pmatrix} a & b \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in E\right\}.$$

Let us take $\eta \in \mathcal{O}_E^\times$ such that $E = F(\eta)$ and $\eta^2 = d \in F$. Then for $B \in D_1^\times$ and $\varepsilon \in \mathcal{O}_F^\times$, we define the *anisotropic* Kloosterman sum by

$$\mathcal{K}_{\text{an}}(B,\varepsilon) = \int_{\mathcal{Y}_B} \psi\left\{\text{tr}\left[\begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}\left(Y - \frac{\varepsilon}{\det B} B^{-1} Y^{-1} B\right)\right]\right\} dY$$

where $\mathcal{Y}_B = \{Y \in D_1 \mid \text{tr} Y = 0,\ YB \in \mathrm{GL}_2(\mathcal{O}_E)\}$. We call it anisotropic since the matrix $\begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}$ corresponds to the anisotropic symmetric matrix $\begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$ in the ordinary $\mathrm{M}_2(F)$ realization.

Theorem 2. *Suppose that $n > 0$ and $u \in \mathcal{O}_E \setminus \{0\}$ such that $uu^\sigma \neq 1$. We write $u = \varpi^m \varepsilon_u$ where $m = \mathrm{ord}(u)$. Let $A_u = \begin{pmatrix} 1 & u \\ u^\sigma & 1 \end{pmatrix}$.*

1. *When $m \geq n$, we have*

$$\mathcal{K}_{\mathrm{an}}(\varpi^n A_u, \varepsilon) = q^{2n}\left\{(-1)^n \mathcal{K}\ell\left(2\varpi^{-n}, -2\varpi^{-n} d\varepsilon\right) + 1 + q^{-1}\right\}.$$

2. *When $0 \leq m < n$, we have*

$$\begin{aligned}\mathcal{K}_{\mathrm{an}}(\varpi^n A_u, \varepsilon) &= \frac{(-1)^n q^{2n}}{|1 - uu^\sigma|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{-n}}{1 - uu^\sigma}, \frac{-2\varpi^{-n} d\varepsilon}{1 - uu^\sigma}\right) \\ &+ \frac{(-1)^{m-n} q^{2n}}{|1 - uu^\sigma|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{m-n}}{1 - uu^\sigma}, \frac{-2\varpi^{m-n} d\varepsilon\varepsilon_u\varepsilon_u^\sigma}{1 - uu^\sigma}\right).\end{aligned}$$

We also have a generalization of the Davenport-Hasse relation in our case. Let $T \in \mathrm{M}_2(E)$ such that ${}^tT^\sigma = T$ and $\det T \neq 0$. Then for $\varepsilon \in \mathcal{O}_E^\times$, let

$$\mathcal{H}(T, \varepsilon) = \int_{\mathcal{Z}_T} \psi\left\{\mathrm{tr}_{E/F}\left[\varepsilon \cdot \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z\right)\right]\right\} dZ$$

where $\mathcal{Z}_T$ consists of $Z \in \mathrm{Sym}^2(E)$ such that

$$TZ \in \mathrm{M}_2(\mathcal{O}_E) \text{ and } \left(T^{-1}\right)^\sigma - Z^\sigma T Z \in \mathrm{M}_2(\mathcal{O}_E).$$

Theorem 3. *Let $T \in \mathrm{M}_2(\mathcal{O}_E)$ such that ${}^tT^\sigma = T$ and $\det T \neq 0$. Let us write*

$$T = \begin{pmatrix} a & y \\ y^\sigma & b \end{pmatrix}, \qquad \Delta = \det T$$

and $y = \varpi^{\mathrm{ord}(y)}\varepsilon_y$ when $y \neq 0$. Let us define a non-negative integer m by

$$m = \min\{\mathrm{ord}(a), \mathrm{ord}(b)\}.$$

1. *Suppose that $m \leq \mathrm{ord}(y)$.*
   (a) *When $m = 0$ and $\mathrm{ord}(\Delta) = 0$, we have $\mathcal{H}(T, \varepsilon) = 1$.*
   (b) *When $m = 0$ and $0 < \mathrm{ord}(\Delta) \leq \mathrm{ord}(y)$, we have*

$$\mathcal{H}(T, \varepsilon) = |\Delta|^{-1}\left(1 + q^{-1}\right).$$

(c) When $m = 0$ *and* $0 \leq \operatorname{ord}(y) < \operatorname{ord}(\Delta)$, *we have*

$$\mathcal{H}(T,\varepsilon) = \frac{(-1)^{\operatorname{ord}(\Delta y^{-1})}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{\operatorname{ord}(y)}}{\Delta}, \frac{2\varpi^{\operatorname{ord}(y)}\varepsilon\varepsilon^{\sigma}\varepsilon_y\varepsilon_y^{\sigma}}{\Delta}\right).$$

(d) When $m > 0$ *and* $\operatorname{ord}(\Delta) \leq \operatorname{ord}(y)$, *we have*

$$\mathcal{H}(T,\varepsilon) = \frac{(-1)^{\operatorname{ord}(\Delta)}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2a}{\Delta}, \frac{2\varepsilon\varepsilon^{\sigma}b}{\Delta}\right) + \frac{1+q^{-1}}{|\Delta|}.$$

(e) When $m > 0$ *and* $\operatorname{ord}(y) < \operatorname{ord}(\Delta)$, *we have*

$$\mathcal{H}(T,\varepsilon) = \frac{(-1)^{\operatorname{ord}(\Delta)}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2a}{\Delta}, \frac{2\varepsilon\varepsilon^{\sigma}b}{\Delta}\right) + \frac{(-1)^{\operatorname{ord}(\Delta y)}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{\operatorname{ord}(y)}}{\Delta}, \frac{2\varpi^{\operatorname{ord}(y)}\varepsilon\varepsilon^{\sigma}\varepsilon_y\varepsilon_y^{\sigma}}{\Delta}\right).$$

2. Suppose that $m > \operatorname{ord}(y)$.

(a) When $\operatorname{ord}(y) = 0$, *we have* $\mathcal{H}(T,\varepsilon) = 1$.

(b) When $m \geq 2\operatorname{ord}(y)$, *we have*

$$\mathcal{H}(T,\varepsilon) = \frac{1-q^{-1}}{\Delta} + \frac{(-1)^{\operatorname{ord}(y)}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{\operatorname{ord}(y)}}{\Delta}, \frac{2\varpi^{\operatorname{ord}(y)}\varepsilon\varepsilon^{\sigma}\varepsilon_y\varepsilon_y^{\sigma}}{\Delta}\right).$$

(c) When $\operatorname{ord}(y) < m < 2\operatorname{ord}(y)$, *we have*

$$\mathcal{H}(T,\varepsilon) = \frac{\operatorname{sgn}\left(\varepsilon_y\varepsilon_y^{\sigma}\right)^m}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2a}{\Delta}, \frac{2b}{\Delta}\right) + \frac{(-1)^{\operatorname{ord}(y)}}{|\Delta|} \cdot \mathcal{K}\ell\left(\frac{2\varpi^{\operatorname{ord}(y)}}{\Delta}, \frac{2\varpi^{\operatorname{ord}(y)}\varepsilon\varepsilon^{\sigma}\varepsilon_y\varepsilon_y^{\sigma}}{\Delta}\right).$$

Here for $\zeta \in \mathcal{O}_F^{\times}$, *we have*

$$\operatorname{sgn}(\zeta) = \begin{cases} 1, & \text{if } \zeta \in \left(\mathcal{O}_F^{\times}\right)^2 \\ -1, & \text{if } \zeta \notin \left(\mathcal{O}_F^{\times}\right)^2. \end{cases}$$

These explicit formulas for the matrix argument Kloosterman sums might be of some independent interest. Also we mention that it is likely that there exists a geometric interpretation of these formulas, when the characteristic of F is positive, as in the case of Jacquet-Ye GL_n-Kloosterman sums proved by Ngô (Duke Math. J. **96**, 1999).