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An effective surjectivity of mod $l$ Galois representation of 1- and 2-dimensional abelian varieties with trivial endomorphism ring

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1 Introduction and main results

Let $A$ be a principally polarized abelian variety of dimension $n$ over an algebraic number field $K$. For a prime $l$ let $A_l$ be the group of $l$-division points of $A$, which is a vector space of dimension $2n$ over $\mathbb{F}_l$. Let $\mu_l$ be the group of $l$-th roots of unity in the algebraic closure $\bar{K}$ of $K$, and let $\varepsilon_l : G_K := \text{Gal}(\bar{K}/K) \to \mathbb{F}_l^* \cong \text{Aut}(\mu_l)$ be the cyclotomic character. As $A$ is principally polarized, the Weil pairing $W : A_l \times A_l \to \mu_l$, written additively, defines a symplectic form with $2n$ variables, satisfying $W(\sigma(P), \sigma(Q)) = \varepsilon_l(\sigma)W(P, Q)$ for $(P, Q) \in A_l \times A_l$ and $\sigma \in G_K$. Hence a Galois representation $\rho_l : G_K \to GSp_{2n}(\mathbb{F}_l)$ is obtained, where
$GSp_{2n}(F_l)$ is the group of symplectic similitudes of dimension $2n$ with entries in $F_l$.

Serre [1] proved that when $n = 2, 6$ or odd, and $\text{End}_K(A) = \mathbb{Z}$, $\rho_l$ is surjective for sufficiently large $l$. The proof uses Faltings’ theorem and standard theorems of algebraic groups. Though the result is general, it does not give an effective lower bound of $l_0$ such that $\rho_l$ is surjective for $l > l_0$.

Le Duff [2] gives a sufficient condition for the surjectivity of $\rho_l$ when $n = 2$ under some assumption on the reduction of abelian varieties. He also suggested that the explicit calculation of the constants in the refinement of Faltings’ theorem by Masser and Wüstholz [3] should enable one to evaluate $l_0$ effectively. But no details are given.

The purpose of this paper is to supply an “elementary” proof of the surjectivity for $n = 1$ or 2, which also gives an effective evaluation of $l_0$. The proof uses Masser-Wüstholz theorem [3] and Kleidman and Liebeck’s [4] detailed results about the classification of the maximal subgroups of the finite classical groups, especially of $GSp_2(F_l) \cong GL_2(F_l)$ and $GSp_4(F_l)$.

**Main Theorem 1.** Let $E$ be an elliptic curve over an algebraic number
field $K$ of degree $d$ with $\operatorname{End}_K(E) = \mathbb{Z}$. For a prime $l$ let $E_l$ be the group of $l$-division points of $E$, and let $G_l$ be the image of the representation $\rho_l$ of $G_K := \text{Gal}(\overline{K}/K)$ on $E_l$. If $l > \max(49, |D(K)|, C(1)[\max\{2d, h(E)\}]^{\tau(1)})$, then $G_l = \text{GL}_2(\mathbb{F}_l)$, where $D(K)$ is the discriminant of $K$, $h(E)$ is the Faltings height of $E$, $C(1)$ is a constant $C(n)$ in Theorem 2 of Section 2 when $n = 1$, and $\tau(1)$ is the constant $\tau$ given in Theorem 1 of Masser and Wüstholz [3] when $n = 1$. Explicitly $\tau(1) = 2^{277} \cdot 3^4 \cdot 5^2 \cdot 136! \times (2^{276} \cdot 3^3 \cdot 5 \cdot 136! + 1)^7 + 2^{1066} \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 528! \times (2^{1061} \cdot 17 \cdot 31 \cdot 528! + 1)^{15}$.

**Main Theorem 2.** Let $A$ be a two-dimensional principally polarized abelian variety over an algebraic number field $K$ of degree $d$ with $\operatorname{End}_K(A) = \mathbb{Z}$. If $l > \max(3841, |D(K)|, C(2)[\max\{2d, h(E)\}]^{\tau(2)})$, then $G_l = \text{GSp}_4(\mathbb{F}_l)$, where $C(2)$ is a constant $C(n)$ in Theorem 2 of Section 2 when $n = 2$, and $\tau(2)$ is the constant $\tau$ given in Theorem 1 of Masser and Wüstholz [3] when $n = 2$. Explicitly $\tau(2) = 2^{1064} \cdot 17 \cdot 31^2 \cdot 528! \times (2^{1061} \cdot 17 \cdot 31 \cdot 528! + 1)^{15} + 2^{4176} \cdot 3^6 \cdot 7^3 \cdot 11 \cdot 19 \cdot 2080! \times (2^{4166} \cdot 3^3 \cdot 7 \cdot 11 \cdot 2080! + 1)^{31}$. 
2 Proof of Main Theorems

Masser and Wüstholz [5, Theorem II] (see also the note at the end of [5]) estimated the degree of an isogeny between abelian varieties over a number field effectively.

Theorem 1. Given positive integers $n$ and $d$, there are constants $\kappa(n)$ and $C(n)$ depending only on $n$ with the following property. Let $A$ and $A'$ be abelian varieties of dimension $n$ defined over a number field $K$ of degree $d$. Then if they are isogenous over $K$, there is an isogeny over $K$ from $A$ to $A'$ of degree at most $C(n)[\max\{d, h(A)\}]^{\kappa(n)}$, where $h(A)$ is the Faltings height of $A$, which is invariant under extension of the ground field.

Using Theorem 1, they [3, Theorem 1] (see also the note at the end of [3]) refined Faltings' theorem in the following effective way.

Theorem 2. Given positive integers $n$ and $d$, there are constants $\tau(n)$ and $C(n)$ depending only on $n$ with the following property. Let $A$ be an abelian variety of dimension $n$ defined over a number field $K$ of degree $d$. then there is a positive integer $M \leq C(n)[\max\{d, h(A)\}]^{\tau(n)}$ such that for any positive integer $m$ the natural map $\text{End}_K(A) \to \text{End}_{G_K}(A_m)$ has
cokernel killed by $M$.

**Corollary.** Suppose $M$ as in Theorem 2. Then for any prime $l$ not dividing $M$ the natural map $\text{End}_K(A) \otimes \mathbb{Z} F_l \rightarrow \text{End}_{G_K}(A_l)$ is an isomorphism.

Explicitly $\tau(n) = n^2 \lambda(8n) + 3\kappa(2n)$ by [3, Section 6], where $\lambda(n) = 4\text{rank}_\mathbb{Z}\{\text{End}_K(A)\}n(2n-1)k(n)\{2nk(n)+1\}^{n-1}$ by [6, Section 5], $k(n)$ being $(2n^2+n-1)4^{n(2n+1)}\{n(2n+1)\}$, and $\kappa(n) = 10n^3\lambda(8n)+32n^2\mu(8n)$ by [5, Section 7], $\mu(n)$ being $[\text{rank}_\mathbb{Z}\{\text{End}_K(A)\}]^{-1}n\lambda(n)$ by [6, Section 6].

Let us recall another material. Aschbacher [7] obtained the classification theorem of the maximal subgroups of the finite classical groups. Kleidman and Liebeck [4] decided the structure of the maximal subgroups more precisely. After that the Main Theorem and Table 3.5.C of [4, Ch. 3, pp. 57, 70 and 72] imply the following Propositions about the maximal subgroups of $GL_2(F_l)$ and $GSp_4(F_l)$.

**Proposition 1.** When $l \geq 5$, a maximal subgroup of $GL_2(F_l)$ is conjugate to one of the following five subgroups.

1. $SL_2(F_l) \times \langle \delta_1 \rangle$,
2. maximal parabolic subgroup,
(3) normalizer of the split Cartan subgroup $\cong \mathbf{F}_l^\times \rtimes S_2 \rtimes \langle \delta_1 \rangle$, 

(4) normalizer of the nonsplit Cartan subgroup $\cong \mathbf{F}_{l^2}^\times \rtimes \mathbb{Z}_2$, and 

(5) $\mathbb{Q}_8 \rtimes D_6$,

where $\delta_1$ is the element expressed as diag($\mu$, 1) with respect to a basis of $\mathbf{F}_l^2$, $\mu$ being a generator of $\mathbf{F}_l^\times$. For groups $G$ and $H$, $G \rtimes H$ denotes the extension of $G$ by $H$. $D_n$ is the dihedral group of order $n$, $\mathbb{Z}_2$ is the cyclic group of order 2, and $\mathbb{Q}_8$ is the quaternion group.

**Proposition 2.** When $l \geq 3$, a maximal subgroup of $GSp_4(\mathbf{F}_l)$ is conjugate to one of the following seven subgroups.

(1) $Sp_4(\mathbf{F}_l) \rtimes (\text{maximal subgroup of } \langle \delta_2 \rangle)$,

(2) maximal parabolic subgroup,

(3) $SL_2(\mathbf{F}_l) \rtimes S_2 \rtimes \langle \delta_2 \rangle$,

(4) $GL_2(\mathbf{F}_l) \rtimes \mathbb{Z}_2 \rtimes \langle \delta_2 \rangle$,

(5) $SL_2(\mathbf{F}_{l^2}) \rtimes \langle \delta_2 \rangle$,

(6) $GU_2(\mathbf{F}_{l^2}) \rtimes \langle \delta_2 \rangle$, and

(7) $D_8 \circ \mathbb{Q}_8 \rtimes O_4^-(\mathbf{F}_2)$,

where $\delta_2$ is the element expressed as diag($\mu$, $\mu$, 1, 1) with respect to a symplectic basis of $\mathbf{F}_l^4$. $\circ$ denotes the central product, and $O_4^-$ is the
4-dimensional orthogonal group with defect 1.

Let \( \zeta_l \) be a primitive \( l \)-th root of unity. If \( K \cap \mathbb{Q}(\zeta_l) = \mathbb{Q} \), then \( \epsilon_l \) is surjective. The condition on \( l \) is given by the following Lemma.

**Lemma.** If \( l > |D(K)| \), then \( K \cap \mathbb{Q}(\zeta_l) = \mathbb{Q} \).

**Proof.** The discriminant of \( \mathbb{Q}(\zeta_l) \), \( D(\mathbb{Q}(\zeta_l)) \), is \( l^{l-2} \) when \( l = 2 \) or \( \equiv 1 \) \((\text{mod} \ 4)\), and \(-l^{l-2} \) when \( l \equiv 3 \) \((\text{mod} \ 4)\). The discriminant of \( K \cap \mathbb{Q}(\zeta_l) \) divides the greatest common divisor of \( D(K) \) and \( D(\mathbb{Q}(\zeta_l)) \), which is 1 if \( l > |D(K)| \). By Minkowski's theorem \( K \cap \mathbb{Q}(\zeta_l) = \mathbb{Q} \). q. e. d.

**Proof of Main Theorem 1.** We prove that \( G_l \) is not contained in any maximal subgroups of \( GL_2(\mathbb{F}_l) \) in Proposition 1.

As \( l > |D(K)| \), \( \epsilon_l \) is surjective by Lemma, so that \( G_l \not\subset SL_2(\mathbb{F}_i) \times \) (maximal subgroup of \( \langle \delta_1 \rangle \)).

The Borel subgroup stabilizes a one-dimensional subspace \( W_1 \) of \( V_1 := \mathbb{F}_i^2 \). If \( G_l \) is contained in it, there is a \( K \)-isogeny \( f \) : \( E \to E/W_1 \) of degree \( l \). By Theorem 1 it should be a composition of isogenies of degree at most \( C(1)[\max\{d, h(E)\}]^{\kappa(1)} \), contradicting the fact that \( l \) is a prime.

Next if \( G_l \subset \mathbb{F}_l^* \times S_2 \rtimes \langle \delta_1 \rangle \), then there exists a surjective homomorphism \( \varphi \) from \( G_l \) to \( S_2 \). Let \( L \) be \( \overline{K^\ker(\varphi)} \), then \([L : K] \leq 2\), and
\[ \rho_l(G_L := \text{Gal}(\bar{K}/L)) \subset \mathbf{F}_l^* \times \langle \delta_1 \rangle. \] Thus \( \text{End}_{G_L}(E_l) \supset \mathbf{F}_l^2 \). On the other hand, as \( l > C(1)[\max\{2d, h(E)\}]^{\tau(1)} \), \( \text{End}_{G_L}(E_l) \cong \text{End}_{L}(E) \otimes \mathbb{Z}^F \mathbf{F}_l \cong \mathbf{F}_l \) by Corollary. This is a contradiction.

If \( G_l \subset \mathbf{F}_l^* \bullet \mathbb{Z}_2 \), then there exists a quadratic extension \( L' \) of \( K \) such that \( \rho_l(G_{L'} := \text{Gal}(\bar{K}/L')) \subset \mathbf{F}_l^* \). Thus \( \text{End}_{G_{L'}}(E_l) \supset \mathbf{F}_l^2 \). On the other hand, as \( l > C(1)[\max\{2d, h(E)\}]^{\tau(1)} \), \( \text{End}_{G_{L'}}(E_l) \cong \text{End}_{L'}(E) \otimes \mathbb{Z} F \mathbf{F}_l \cong \mathbf{F}_l \) by Corollary. Hence a contradiction.

Lastly assume that \( G_l \subset Q_8 \bullet D_6 \). As \( \epsilon_l \) is surjective by Lemma, \( |G_l| \geq |\mathbf{F}_l^*| = l - 1 > 48 = |Q_8 \bullet D_6| \). This is a contradiction.

When \( \text{End}_K(E) = \mathbb{Z} \), \( \tau(1) = 2^{277} \cdot 3^4 \cdot 5^2 \cdot 136! \times (2^{276} \cdot 3^3 \cdot 5 \cdot 136! + 1)^7 + 2^{1066} \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 528! \times (2^{1061} \cdot 17 \cdot 31 \cdot 528! + 1)^{15} \).

**Proof of Main Theorem 2.** We prove that \( G_l \) is not contained in any maximal subgroups of \( GSp_4(\mathbf{F}_l) \) in Proposition 2.

\[ G_l \not\subset Sp_4(\mathbf{F}_l) \times (\text{maximal subgroup of } \langle \delta_2 \rangle) \text{, for } \epsilon_l \text{ is surjective.} \]

Maximal parabolic subgroups stabilize a one- or two-dimensional subspace of \( V_2 := \mathbf{F}_l^4 \) [4, p. 72, Table 3.5.C]. So \( G_l \) is not contained in them similarly as the case of the Borel subgroup in Main Theorem 1.

\( SL_2(\mathbf{F}_l) \times S_2 \times \langle \delta_2 \rangle \) stabilizes a two-dimensional subspace of \( V_2 \). In fact,
let \( \{e_i|1 \leq i \leq 4\} \) be a symplectic basis of \( V_2 \). Let \( H := SL_2(\mathbb{F}_l) \times S_2 \),

\[
H_0 := \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \left| \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in SL_2(\mathbb{F}_l) \right\},
\]

and

\[
w := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Then \( H = H_0 \cup H_0 w \). We consider the action of \( H \) on \( W_2 := \mathbb{F}_l(e_1 \oplus e_2) \oplus \mathbb{F}_l(e_3 \oplus e_4) \). For \( k_1 \) and \( k_2 \in \mathbb{F}_l \)

\[
\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_1 \\ k_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} ak_1 + bk_2 \\ ak_1 + bk_2 \\ ck_1 + dk_2 \\ ck_1 + dk_2 \end{pmatrix},
\]
\[
\begin{pmatrix}
a & 0 & b & 0 \\
0 & a & 0 & b \\
c & 0 & d & 0 \\
0 & c & 0 & d \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_1 \\
k_2 \\
k_2 \\
\end{pmatrix}
= \begin{pmatrix}
ak_1 + bk_2 \\
aka_1 + bk_2 \\
ck_1 + dk_2 \\
ck_1 + dk_2 \\
\end{pmatrix}.
\]

So \( H_0 W_2 \subset W_2 \) and \( H_0 w W_2 \subset W_2 \). Thus \( W_2 \) is a nontrivial invariant subspace of \( V_2 \) under the action of \( H \). As \( \langle \delta_2 \rangle \) acts on \( F_l(e_1 \oplus e_2) \) by multiplication by scalars, and on \( F_l(e_3 \oplus e_4) \) trivially, \( W_2 \) is invariant also under the action of \( H \rtimes \langle \delta_2 \rangle = SL_2(F_l) \rtimes S_2 \rtimes \langle \delta_2 \rangle \). Thus \( G_l \not\subset SL_2(F_l) \rtimes S_2 \rtimes \langle \delta_2 \rangle \) similarly as the case of maximal parabolic subgroups.

\[ G_l \not\subset GL_2(F_l) \rtimes \langle \delta_2 \rangle \] similarly as the case of \( F_l^* \rtimes S_2 \rtimes \langle \delta_1 \rangle \) in Main Theorem 1.

If \( G_l \subset SL_2(F_{l^2}) \rtimes \langle \delta_2 \rangle \) or \( G_l \subset GU_2(F_{l^2}) \rtimes \langle \delta_2 \rangle \), then \( G_l \) commutes with \( F_{l^2} \). On the other hand, as \( l > C(2)[\max\{d, h(A)\}] \tau(2) \), \( \text{End}_{K}(A_l) \cong \text{End}_{K}(A) \otimes \mathbb{Z} F_l \) by Corollary. Hence a contradiction.

\[ G_l \not\subset D_8 \circ Q_8 \rtimes (\mathbb{F}_2) \] similarly as the case of \( D_8 \circ Q_8 \) in Main Theorem 1, for \(|D_8 \circ Q_8 \rtimes (\mathbb{F}_2)| = 3840\).

When \( \text{End}_{K}(A) = \mathbb{Z}, \tau(2) = 2^{1064} \cdot 17 \cdot 31^2 \cdot 528! \times (2^{1061} \cdot 17 \cdot 31 \cdot 528! + \ldots) \)
\[ 1^{15} + 2^{4176} \cdot 3^6 \cdot 7^3 \cdot 11 \cdot 19 \cdot 2080! \times (2^{4166} \cdot 3^3 \cdot 7 \cdot 11 \cdot 2080! + 1)^{31}. \]

Remarks. (a) The effective dependence of \( C(n) \) on the dimension \( n \) remains an interesting problem.

(b) When \( \dim A = 3 \), the classification of maximal subgroups of \( GSp_6(F_l) \) is also known [4, p. 72, Table 3.5.C]. When \( l \geq 5 \), they are

1. \( Sp_6(F_l) \rtimes (\text{maximal subgroup of } \langle \delta_3 \rangle) \),
2. maximal parabolic subgroup,
3. \( SL_2(F_l) \times Sp_4(F_l) \rtimes \langle \delta_3 \rangle \),
4. \( SL_2(F_l) \rtimes S_3 \rtimes \langle \delta_3 \rangle \),
5. \( GL_3(F_l) \bullet \mathbb{Z}_2 \rtimes \langle \delta_3 \rangle \),
6. \( SL_2(F_{l^3}) \rtimes \langle \delta_3 \rangle \),
7. \( GU_3(F_{l^2}) \rtimes \langle \delta_3 \rangle \), and
8. \( SL_2(F_l) \circ O_3(F_l) \rtimes \langle \delta_3 \rangle \),

where \( \delta_3 \) is the element expressed as \( \text{diag}(\mu, \mu, \mu, 1, 1, 1) \) with respect to a symplectic basis of \( F_l^6 \). The first seven are handled similarly as the 2-dimensional case, for (3) is also reducible. Only the case (8) seems to be difficult to treat.

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