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$L^p$ estimates for some Schrödinger type operators

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Abstract

We consider the Schrödinger operator $L = -\Delta + V$ with non-negative potentials $V$ on $\mathbb{R}^n$, $n \geq 3$. We assume that the potential $V$ belongs to the reverse Hölder class which includes non-negative polynomials. We show the $L^p$ estimates for the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, where $k$ is a positive integer.

1 Introduction

In this paper we consider the Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^n$, $V \geq 0$, $n \geq 3$. When $V$ is a non-negative polynomial, Zhong ([Zh]) proved that the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, $k \in \mathbb{N}$, are bounded on $L^p$, $1 < p \leq \infty$. For the potential $V$ which belongs to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh]) generalized Zhong's results. Actually, he proved that the operators $V L^{-1}$ and $V^{1/2} \nabla L^{-1}$ are bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

For the operator $L$ with potentials $V$ which belong to the reverse Hölder class, Kurata and the author generalized Shen's results as follows. In [KS1], we replace $\Delta$ by the second order uniformly elliptic operator $L_0 = -\sum_{i,j=1}^n (\partial/\partial x_i) \{a_{ij}(x)(\partial/\partial x_j)\}$ and assume certain assumptions for $a_{ij}$. Then we showed that the operators $V(L_0 + V)^{-1}$ and $V^{1/2} \nabla (L_0 + V)^{-1}$ are bounded on weighted $L^p$ space ($1 < p < \infty$) and Morrey spaces. Moreover, in [Su], the author showed weighted $L^p-L^q$ estimates of the operators $V^\alpha L^{-\beta}$ and $V^\alpha \nabla L^{-\beta}$ ($\alpha, \beta \in (0,1]$) and their boundedness on Morrey spaces.

The purpose of this paper is to show the $L^p$ boundedness of the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, $k \in \mathbb{N}$, where $V$ belongs to the reverse Hölder class.

We shall repeat the definitions of the reverse Hölder class (e.g.[Sh]). Throughout this paper we denote by $B_r(x)$ the ball centered at $x$ with radius $r$, and the letter $C$ stands for a constant not necessarily the same at each occurrence.
**Definition 1** (Reverse Hölder class) Let $V \geq 0$.

1. For $1 < p < \infty$ we say $V \in (RH)_p$, if $V \in L^p_{loc}(\mathbb{R}^n)$ and there exists a constant $C$ such that
   \[
   \left( \frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \right)^{1/p} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy
   \]  
   (1)
   holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$.

2. We say $V \in (RH)_{\infty}$, if $V \in L^\infty_{loc}(\mathbb{R}^n)$ and there exists a constant $C$ such that
   \[
   \|V\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy
   \]  
   (2)
   holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$.

**Remark 1** If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^\alpha$ belongs to $(RH)_{\infty}$ ([Fe]). For $1 < p < \infty$, it is easy to see $(RH)_{\infty} \subset (RH)_p$.

In [Zh], Zhong proved the $L^p$ estimates of the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$ with non-negative polynomials $V$ by using the $k$ times composition of the Hardy-Littlewood maximal operator $M$. In [KS1] we considered the uniformly elliptic operators $L_0$ and proved a pointwise bound $|Tf(x)| \leq CM(|f|)(x)$ where $Mf$ is Hardy-Littlewood maximal function and $T$ is either $V(L_0 + V)^{-1}$ or $V^{1/2} \nabla (L_0 + V)^{-1}$. Pointwise estimates are also used by Zhong in the polynomial case. Once we have these pointwise estimates the boundedness of these operators in any spaces on which the Hardy-Littlewood maximal operator is known to be bounded. Examples are weighted $L^p$ space and Morrey spaces.

In this paper we establish pointwise estimates (see Lemma 3) which generalize Zhong’s estimates we mentioned above. By using them we show the $L^p$ boundedness of these operators (see Theorem 1).

We denote by $\Gamma(x, y)$ the fundamental solution for $L$. The operator $L^{-1}$ is the integral operator with $\Gamma(x, y)$ as its kernel. Let $f \in C_0^\infty(\mathbb{R}^n)$. Then we have $L^{-1} f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. For any integer $k \geq 2$, we define $L^{-k}$ as follows.

\[
L^{-k} f(x) = \int_{\mathbb{R}^n} \Gamma(x, y)L^{-(k-1)} f(y) dy.
\]

Now we state our theorem.

**Theorem 1** Suppose $V \in (RH)_{\infty}$. Then there exist constants $C, C'$ such that

\[
\|V^k L^{-k} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad f \in C_0^\infty(\mathbb{R}^n),
\]  
(3)
\[ \|V^{k-1/2} \nabla L^{-k} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad f \in C_0^\infty(\mathbb{R}^n), \]  
(4)

where $1 < p \leq \infty$ and $k \in \mathbb{N}$.

**Remark 2** In Theorem 1 the case $k = 1$ was shown in [Sh, Remark 2.9, Theorem 4.13].

The plan of this paper is as follows. In section 2, we recall Shen's lemmas which we use to prove Theorem 1. In section 3, we prove Theorem 1.

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## 2 Preliminaries

In [Sh], Shen defined the auxiliary function $m(x, V)$ and established the estimates of the fundamental solution of $L$ (see Lemma 1). By using the estimates he proved $L^p$ boundedness of the operators $VL^{-1}$ and $V^{1/2} \nabla L^{-1}$. We also need them to prove our theorem.

We recall the definition of the function $m(x, V)$.

**Definition 2** ([Sh, Definition 1.3]) Let $V \in (RH)_{n/2}$ and $V \not\equiv 0$. Then it is well-known that there exists $\epsilon > 0$ such that $V \in (RH)_{n/2+\epsilon}$ ([Ge]). Then the function $m(x, V)$ is well-defined by

\[
\frac{1}{m(x, V)} = \sup \left\{ r > 0 : \frac{r^2}{|B_r(x)|} \int_{B_r(x)} V(y)dy \leq 1 \right\}
\]

and satisfies $0 < m(x, V) < \infty$ for every $x \in \mathbb{R}^n$.

**Remark 3** If $V \in (RH)_\infty$ then there exists a constant $C$ such that $V(x) \leq Cm(x, V)^2$ ([Sh, Remark 2.9]).

We recall the estimates of the fundamental solution for $L$.

**Lemma 1** ([Sh])

1. Suppose $V \in (RH)_{n/2}$. Then for any positive integer $N$ there exists a constant $C_N$ such that

\[
(0 \leq) \Gamma(x, y) \leq \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2}}.
\]
(2) Suppose $V \in (RH)_n$. Then for any positive integer $N$ there exists a constant $C_N$ such that

\[
|\nabla_x \Gamma(x, y)| \leq \frac{C_N}{(1 + m(x, V)|x - y|)^N} \cdot \frac{1}{|x - y|^{n-1}}.
\]

The following Lemma is also needed to prove our theorem.

**Lemma 2** ([Sh, Lemma 1.4(c)]) Suppose $V \in (RH)_{n/2}$. Then there exist positive constants $C, k_0$ such that

\[
m(y, V) \geq \frac{Cm(x, V)}{(1 + m(x, V)|x - y|)^{k_0/(k_0 + 1)}}.
\]

### 3 Proof

Theorem 1 is easily proved by the following pointwise estimates. These estimates generalize the results in [Zh, Lemma 3.2] to the Schrödinger operators with reverse Hölder class potentials.

**Lemma 3** Let $k$ be a positive integer. The operator $M^k$ stands for the $k$ times composition of the Hardy-Littlewood maximal operator $M$.

1. Suppose $V \in (RH)_{n/2}$. Then there exist a constant $C$ such that

\[
|m(x, V)^{2k}L^{-k}f(x)| \leq CM^k(|f|)(x) \quad \text{for} \quad f \in C^\infty_0(\mathbb{R}^n).
\]  

2. Suppose $V \in (RH)_n$. Then there exist a constant $C$ such that

\[
|m(x, V)^{2k-1}\nabla L^{-k}f(x)| \leq CM^k(|f|)(x) \quad \text{for} \quad f \in C^\infty_0(\mathbb{R}^n).
\]

**Remark 4** In Lemma 3 the case $k = 1$ was shown in [KS, Theorem 1.3].

**Proof of Theorem 1.** Since $V(x) \leq Cm(x, V)^2$, estimate (3) immediately follows from (5) and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. The proof of (4) can be done in the same way as above by using (6). \qed
Proof of Lemma 3. Let \( f \in C_0^\infty(\mathbb{R}^n) \). We prove estimate (5) by induction on \( k \). For the proof of the case \( k = 1 \), see [KS1, Theorem 1.3]. We assume it is true for \( k = l \), that is, there exists a constant \( C \) such that
\[
|m(x, V)^{2l}L^{-l}f(x)| \leq CM^l(|f|)(x)
\]
and show the case \( k = l + 1 \). It follows from Lemma 1 (1) and Lemma 2 that
\[
\begin{align*}
|m(x, V)^{2(l+1)}L^{-(l+1)}f(x)| &\leq Cm(x, V)^2 \int_{\mathbb{R}^n} \Gamma(x, y)m(x, V)^{2l}L^{-l}f(y)dy \\
&\leq CC_Nm(x, V)^2 \int_{\mathbb{R}^n} \frac{\{1 + m(x, V)|x-y|\}^{2lk_0/(k_0+1)}m(y, V)^{2l}L^{-l}f(y)}{\{1 + m(x, V)|x-y|\}^N|x-y|^{n-2}}dy.
\end{align*}
\]
Therefore we obtain the desired estimate in the same way as the case \( k = 1 \).

The proof of (6) can be done in the same way as the proof of (5) by using Lemma 1 (2). \( \square \)

Remark 5 Let \( s \in (0, \infty) \). We can obtain the estimate for the operator \( V^sL^{-s} \) as follows. Suppose \( V \in (RH)_{n/2} \) and \( \alpha \in (0, 1] \). Then there exists a constant \( C \) such that
\[
|m(x, V)^{2\alpha}L^{-\alpha}f(x)| \leq CM(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbb{R}^n)
\]
(see [Su Theorem 1].) Combining (8) and the argument in the proof of Lemma 3, we arrive at the following pointwise estimate:
\[
|m(x, V)^{2s}L^{-s}f(x)| \leq CM^s(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbb{R}^n),
\]
where \( s \in (0, \infty) \) and
\[
s^* = \begin{cases} 
  s, & \text{if } s \text{ is an integer}, \\
  \lfloor s \rfloor + 1, & \text{otherwise},
\end{cases}
\]
where \( \lfloor s \rfloor \) is the largest integer smaller than or equal to \( s \). We should remark that, for the case \( V \) is a non-negative polynomial, Zhong proved the \( L^p \) boundedness (only for \( 1 < p < \infty \)) of the operator \( V^sL^{-s}, s \in (0, \infty) \) ([Zh, Corollary 1.5]).

Remark 6 Zhong also showed that the \( L^p \) estimate of the operator \( V^{k-q/2} \Delta^{q/2}L^{-k} \) with non-negative polynomials \( V \), where \( q \) and \( k \) are positive integers and \( 2 \leq q \leq 2k \) ([Zh, Theorem 1.3]). He proved this results by using the fact that the functions
$m(x, V)^{2k} L^{-k} f(x)$ and $m(x, V)^{2k-1} \nabla L^{-k} f(x)$ are bounded by the $k$ times composition of the Hardy-Littlewood maximal function and there exists a constant $C$ such that

$$|\Delta^{q/2} V(x)| \leq C m(x, V)^{q+2}$$

(10)

which holds for non-negative polynomials $V$. Hence if we assume the inequality (10), we can obtain the $L^p$ estimate of the operator $V^{k-q/2} \Delta^{q/2} L^{-k}$ with potentials $V$ which belong to the reverse Hölder class in the same way as for polynomial potentials by using Lemma 3 and the assumption (10).

**Remark 7** Shen proved that the operator $\nabla^2 L^{-1}$ is bounded on $L^p$, $1 < p < \infty$ ([Sh]). In [KS1] Kurata and the author extended this result to the uniformly elliptic operators. They also showed that the estimate for the kernel of the operator $\nabla^2 L^{-1}$ ([KS2]). However, it is not known that whether the operator $V^{k-1} \nabla^2 L^{-k}$, $k \geq 2$ is bounded on $L^p$ or not.

**References**


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