数理解析研究所講究録 1201 巻 2001 年 17-25

ON THE WELL POSEDNESS AND ILL POSEDNESS OF THE IVP FOR A CLASS OF NONLINEAR DISPERSIVE EQUATIONS

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§1. INTRODUCTION

This paper is concerned with the minimal regularity properties required on the initial data to guarantee the local well posedness of the IVP for some nonlinear evolution evolution equations. Most of the results mentioned here are joint work with Carlos E. Kenig and Luis Vega.

We should refer to Prof. H. Takaoka paper for a very exciting set of results concerning the global well posedness of some of the problems considered here.

We are mainly concerned with the following nonlinear dispersive equations on the real line,

$$\begin{array}{ll} \text{cubic NLS} & i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \\ \text{mKdV} & \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0, \\ \text{KdV} & \partial_t u + \partial_x^3 u + u\partial_x u = 0. \end{array}$$

The regularity of the data u_0 will be measured in classical Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, where

$$H^s(\mathbb{R}) = \{u_0 \in \mathcal{S}'(\mathbb{R}) \ : \ (1+|\xi|^2)^{s/2}\hat{u}_0(\xi) \in L^2\}.$$

The IVP is locally well posed (LWP) in $H^s(\mathbb{R})$ if there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u(t)$ of the corresponding IVP such that

(i) $u \in C([-T, T] : H^s) \cap Y_T = X_T$,

(ii) The map data-solution, $u_0 \to u(t)$, from H^s into X_T is uniformly continuous, i.e.

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad : \quad \|u_0^1 - u_0^2\|_{H^s} < \delta \ \Rightarrow \ \|u^1 - u^2\|_{X_T} < \epsilon,$$

$$\delta = \delta(\epsilon, M), \qquad \|u_0^1\|_{H^s},\ \|u_0^2\|_{H^s} \le M.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For the equations considered above the best known LWP results are :

(1.1) the IVP for the cubic NLS is LWP in $H^s(\mathbb{R})$ with $s \geq 0$, [Ts],

(1.2) the IVP for the mKdV is LWP in $H^s(\mathbb{R})$ with $s \geq 1/4$, [KePoVe2],

and

(1.3) the IVP for the KdV is LWP in $H^s(\mathbb{R})$ with $s > -3/4$, [KePoVe3].

It is interesting to observe that the proofs of these three results are quite different. Let us recall the main ideas in these proofs.

In the case of the cubic NLS the proof in [Ts] is based on the Strichartz estimates [St], (see also [GV]). In the case of the 1-D linear Schrödinger equation these estimates can be written as

$$(1.4) \qquad \|e^{it\partial_x^2}u_0\|_{L_t^q L_x^p} = \left(\int_{-\infty}^{\infty} \|e^{it\partial_x^2}u_0\|^q_{L^p(\mathbb{R})}dt\right)^{1/q} \leq c\|u_0\|_{L^2},$$

for $2/q = 1/2 - 1/p$ with $2 \leq p \leq \infty$. (1.4) can be seen as an estimate for the Fourier transform of a measure on $\mathbb{R}^2$ supported in the parabola $\tau = \xi^2$ with density $\hat{u}_0(\xi)$.

In the case of the mKdV the proof in [KePoVe2] follows by combining the following linear estimates

$$(1.5) \qquad \|e^{-t\partial_x^3}u_0\|_{L_x^4 L_t^\infty} = \left(\int_{-\infty}^{\infty} \sup_{t\in\mathbb{R}} |e^{t\partial_x^3}u_0(x)|^4 dx\right)^{1/4} \leq c\|D^{1/4}u_0\|_{L^2},$$

and

$$(1.6) \qquad \|\partial_x e^{-t\partial_x^3}u_0\|_{L_x^\infty L_t^2} = \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |\partial_x e^{-t\partial_x^3}u_0(x)|^2 dt\right)^{1/2} \leq c\|u_0\|_{L^2}.$$

The inequality (1.5), an estimate for the maximal function associated to the group $\{e^{-t\partial_x^3} : t \in \mathbb{R}\}$, was established in [KeRu] as part of the study of the pointwise behavior of $e^{-t\partial_x^3}u_0$ as $t \to 0$. It was also proven in [KeRu] that (1.5) is sharp in the sense that it fails, even locally in time, for $p \neq 4$ on the left hand side or with D^s, $s < 1/4$ on the right hand and any p on the left side.

The identity (1.6) was proven in [KePoVe1] and is a sharp version of the smoothing effect first deduced in [K] in solutions of the KdV equation.

Finally, the proof for the KdV in [KePoVe3] is based on the use of the space $X_{s,b}$. These spaces which heavily reflect the geometry of the symbol of the associated linear operator were first introduced in this context in [B1]. $X_{s,b}$ denotes the completion of the Schwartz space $S(\mathbb{R}^2)$ with respect to the norm

$$\|F\|_{X_{s,b}} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\widehat{F}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2}.$$

One of the key estimates in [KePoVe3] affirms that if $s \in (-3/4, 0]$ there exists $b \in (1/2, 1)$ such that

$$\|\partial_x(uv)\|_{X_{s,b-1}} \leq c\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}. \tag{1.7}$$

It was also proved in [KePoVe3] that (1.7) fails for $s < -3/4$ and any $b \in \mathbb{R}$. In [NaTaTs] this negative result was extended to the limiting case $s = -3/4$.

In [Bo2] it was established that the map data $\to$ solution is not analytic at 0, if (1.1)-(1.3) do not hold. In fact, it was shown that it is not C^2 for the cubic NLS, mKdV, and not C^3 for the KdV (in [Tz] the argument was extended to C^2). The idea is to compute the coefficients of the Taylor expansion at 0, which turn out to be the second and third Picard iteration in respectively.

In [KePoVe5] we showed the LWP results in (1.1)-(1.3) are sharp in a stronger sense than the one described above. More precisely, it was established that the IVP for the cubic NLS, mKdV and KdV are ill posed in $H^s(\mathbb{R})$ for any index s smaller than 0, 1/4 and $-3/4$, respectively.

Here we will recall some of the main ideas in [KePoVe5]. A different with the LWP results previously described one has that the proofs of the ill posedness for these equations are quite related.

Also it should be mentioned that the sharp index 0, 1/4 and $-3/4$ are larger than those suggested by the scaling argument described below, i.e. $-1/2$, $-1/2$ and $-3/2$ for the cubic NLS, mKdV and KdV respectively.

In the same vain one has that for the generalized KdV

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad k \in \mathbb{Z}^+,$$

the scaling argument suggests as a "critical" value $s_k = (k-4)/2k$. For the powers $k \geq 4$ LWP was established in $H^s(\mathbb{R})$ with $s \geq s_k$ in [KePoVe2], and ill posedness in $H^s(\mathbb{R})$, with $s < s_k$ in [BiKePoSvVe].

The first example of ill posedness above the Sobolev index suggested by the scaling was obtained in [Li] for a quadratic nonlinear perturbation of the classical wave equation in 3 space dimensions.

We shall start with the cubic NLS. The main idea is to use the Galilean invariance, i.e. if $u(x,t)$ is the solution to cubic NLS with initial data $u_0(x)$, then

$$u_N(x,t) = e^{-itN^2} e^{iNx} u(x - 2tN, t)$$

is also a solution of the cubic NLS with initial data

$$u_N(x,0) = e^{iNx} u_0(x).$$

Thus, taking $u_0 \in H^s(\mathbb{R})$ and assuming that the time of existence $T = T(\|u_0\|_{H^s})$ one has that the time of existence for u_N is also T, although if $s < 0$, $\|u_N(x,0)\|_{H^s} \downarrow 0$ as $N \uparrow \infty$. So we can say that $H^0 = L^2$ is "critical" for the cubic NLS.

For the focusing case, i.e. taking the $+$ sign in the cubic NLS, a more rigorous analysis can be obtained using the "ground state" solutions, i.e. solutions of the form

$$u(x,t) = e^{it} f(x).$$

Let $f = \sqrt{2}\, sech(x)$ which satisfies the equation

$$-f + f'' + f^3 = 0.$$

Using the scaling argument, i.e. if $u(x,t)$ is a solution of the cubic NLS then for any $\omega \in \mathbb{R}$, $u_\omega(x,t) = \omega u(\omega x, \omega^2 t)$ is also a solution of the cubic NLS, and the notation

$$f_\omega(x) = \omega f(\omega x),$$

one gets the family of solutions of the cubic (focusing) NLS

$$u_\omega(x,t) = e^{it\omega^2} f_\omega(x).$$

Now using the Galilean invariance we obtain the two parameter family of solutions to the cubic (focusing) NLS

$$u_{N,\omega}(x,t) = e^{-it(N^2-\omega^2)} e^{iNx} f_\omega(x - 2tN).$$

with initial data

$$u_{N,\omega}(x,0) = e^{iNx} f_\omega(x).$$

This will allow us to prove our first result concerning the IVP for the 1-D (focusing) cubic Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.8}$$

Theorem 1 [KePoVe5].

If $s \in (-1/2, 0)$, then the mapping data-solution, $u_0 \to u(t)$ where $u(t)$ solves the IVP associated to the 1-D (focusing) cubic Schrödinger (1.8) is not uniformly continuous.

Remark: In [To] it was shown that the IVP (1.8) with nonlinearity $u\bar{u}u$ by $\bar{u}\bar{u}\bar{u}$ (same homogeneity, but not Galilean invariant) is LWP in H^s, $s > -1/3$. Thus, the nonlinearity $u\bar{u}u$ is worse behaved from this point of view.

Remark: Consider the IVP

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases}$$

The results in [CW], [GV], [Ts] showed local well posedness in $H^s(\mathbb{R})$, $s \geq 0$, and the argument in the proof of Theorem 1 shows that this is the best possible result. This is in contrast with the results in [KePoVe4], where for the nonlinearities $\bar{u}\bar{u}$, uu local well posedness was shown in $H^s(\mathbb{R})$, $s \geq -3/4$, and for the nonlinearity $u\bar{u}$ in $H^s(\mathbb{R})$, $s \geq -1/4$.

Remark: The result in Theorem 1 can be extended to higher dimensions, for details see [KePoVe5].

Next we shall extend the result in Theorem 1 to the mKdV and KdV equations. However, these equations are not Galilean invariant in the sense described above. In this regard it is convenient to consider first the "complex mKdV", i.e.

$$\partial_t u + \partial_x^3 u + |u|^2 \partial_x u = 0.$$

It is easy to see that if f_ω is defined as above, i.e. $f_\omega(x) = \sqrt{2}\omega\, sech(\omega\, x)$, then

$$v_\omega(x,t) = \sqrt{3}\, f_\omega(x - t\omega^2)$$

is a solution of both complex mKdV and mKdV (a traveling wave solution). Moreover, we have the following remarkable fact concerning the IVP for the complex modified KdV

$$\begin{cases} \partial_t u + \partial_x^3 u + |u|^2 \partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.9}$$

Lemma [KePoVe5].

Let f_ω be defined as above. Then

$$u_{N,\omega}(x,t) = \sqrt{3}\, e^{-it(3N\omega^2 - N^3)} e^{ixN} f_\omega(x - t\omega^2 + 3tN^2)$$

solves the IVP (1.9) with initial data

$$u_{N,\omega}(x,0) = \sqrt{3}\, e^{ixN} f_\omega(x).$$

Combining the two parameter family of solutions of (1.9) described above and the argument in the proof of Theorem 1 one obtains the following result.

Theorem 2 [KePoVe5].

If $s < 1/4$, then the mapping data-solution, $u_0 \to u(t)$ where $u(t)$ solves the IVP for the focusing complex mKdV (1.9) is not uniformly continuous.

Remark: The local well posedness result in [KePoVe2] for the mKdV, $s \geq 1/4$, remains true, with identical proof, for the IVP (1.9) for the complex mKdV.

Next, we shall extend the result in Theorem 2 to the IVP associated to the modified KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.10}$$

First we observe that the mKdV has "breather" solutions, i.e. solutions that are periodic in the time variable and has exponential decay in the space variable. Up to translations the "breather" solutions are ([W], see also [L] and references therein)

$$u_{N,\omega}(x,t) = 2\sqrt{6}\, \omega sech(\omega x + \gamma t) \times$$

$$\left(\frac{cos(Nx + \delta t) - (\omega/N) sin(Nx + \delta t) tanh(\omega x + \gamma t)}{1 + (\omega/N)^2 sin^2(Nx + \delta t) sech(\omega x + \gamma t)} \right),$$

with

$$\delta = N(N^2 - 3\omega^2)\,, \qquad \gamma = \omega(3N^2 - \omega^2).$$

Hence, if $\omega/N \ll 1$, then

$$u_{N,\omega}(x,t) \approx 2\sqrt{6}\, cos(Nx + N(N^2 - 3\omega^2)t)\, \omega\; sech(\omega x + \omega(3N^2 - \omega^2)t),$$

which is basically a multiple of the real part of the function in the statement of the Proposition above. Therefore, using the argument in the proof of Theorem 2 we obtain the following result.

Theorem 3 [KePoVe5].

If $s < 1/4$, then the mapping data-solution, $u_0 \to u(t)$ where $u(t)$ solves the IVP for the (real) modified mKdV (1.10) is not uniformly continuous.

Now we turn our attention to the IVP for the KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u \partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.11}$$

We shall use Miura's transformation [M], which relates solutions of mKdV with solutions of KdV. Assume that v solves the mKdV equation, then

$$u(x,t) = -(v^2 + i\partial_x v)(x,t)$$

is a solution of the KdV equation.

A combination of Miura's transformation and Theorem 3 yields the following

Theorem 4 [KePoVe5].

If $s < -3/4$, then the mapping data-solution, $u_0 \to u(t)$ where $u(t)$ solves the IVP for the (complex) KdV (1.10) is not uniformly continuous.

Remark: The local well posedness result in [KePoVe2] for the KdV for H^s with $s > -3/4$, remains true with identical proof for complex valued solutions.

To complete the exposition we will sketch the proof of Theorem 1

§2. PROOF OF THEOREM 1

Using the ground state solution and the Galilean invariant property as described above it follows that the IVP for the 1-D (focusing) Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_{\omega,N}(x,0) = e^{iNx} f_\omega(x), \end{cases}$$

has two parameter family of solutions of the form

$$u_{N,\omega}(x,t) = e^{-it(N^2-\omega^2)} e^{iNx}\, f_\omega(x - 2tN),$$

where $f_\omega(x) = \omega f(\omega x)$, with $f_1(x) = f(x) = \sqrt{2}\, sech(x)$.

Now fixing s such that

$$s \in (-1/2, 0),$$

and taking

$$\omega = N^{-2s}, \qquad N_1, N_2 \simeq N,$$

we shall calculate

$$\|u_{N_1,\omega}(0) - u_{N_2,\omega}(0)\|_{H^s}^2 .$$

We observe that

$$\hat{f}_\omega(\xi) = \hat{f}(\xi/\omega)$$

so that $\hat{f}_\omega(\cdot)$ concentrates in $B_\omega(0) = \{\xi \in \mathbb{R} : |\xi| < \omega\}$. From to our choices above if $\xi \in B_\omega(\pm N)$, then $|\xi| \simeq N$. These observations combined with some computations show (for details see [KePoVe5]) that

$$\|u_{N_1,\omega}(0) - u_{N_2,\omega}(0)\|_{H^s}^2 = \|(1+|\xi|^2)^{s/2}(\hat{f}_\omega(\xi - N_1) - \hat{f}_\omega(\xi - N_2))\|_{L^2}^2$$

$$\leq cN^{2s}(N_1 - N_2)^2 \frac{1}{\omega^2}\, \omega = c(N^{2s}(N_1 - N_2))^2,$$

and that

$$\|u_{N_j,\omega}(0)\|_{H^s}^2 \simeq cN^{2s}\omega = c, \quad j = 1, 2.$$

Now we need to performe a similar computation for the corresponding solutions $u_{N_1,\omega}(t)$, $u_{N_2,\omega}(t)$ at time $t = T$, i.e. we need to estimate

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{H^s}.$$

Note first that

$$\|u_{N_j,\omega}(T)\|_{H^s}^2 = \|u_{N_j,\omega}(0)\|_{H^s}^2 \simeq c, \quad j = 1, 2.$$

Also we observe that the frequencies of both $u_{N_j,\omega}(T)$, $j = 1, 2$, are localized around $|\xi| \simeq N$, hence

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{H^s}^2 \simeq N^{2s}\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{L^2}^2.$$

Since

$$u_{N_j,\omega}(x,T) = e^{-i(TN_j^2 - N_j x - T\omega^2)}\omega f(\omega(x - 2TN_j)), \qquad j = 1, 2,$$

the support of $u_{N_j,\omega}(T)$ is concentrated in $B_{\omega^{-1}}(2TN_j)$, $j = 1, 2$. Therefore, if for T fixed, N_1, N_2 are chosen such that

$$T(N_1 - N_2) \gg \omega^{-1} = N^{2s},$$

there is not interaction and

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{L^2}^2 \simeq \|u_{N_1,\omega}(T)\|_{L^2}^2 + \|u_{N_2,\omega}(T)\|_{L^2}^2 \simeq \omega.$$

Hence, we have that

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{H^s}^2 \geq cN^{2s}\omega = c.$$

Finally, taking

$$N_1 = N \quad \text{and} \quad N_2 = N - \frac{\delta}{N^{2s}},$$

it follows that

$$c(N^{2s}(N_1 - N_2))^2 = c\delta^2,$$

and

$$T(N_1 - N_2) = T\frac{\delta}{N^{2s}} \gg N^{2s}, \quad \text{i.e.} \quad T \gg \frac{N^{4s}}{\delta}.$$

Since $s < 0$, given δ, $T > 0$, we can choose N so large that the last inequalities hold, which violates the uniform continuity and completes the proof of Theorem 1.

References

[BiKePoSvVe] Birnir, B., Kenig, C. E., Ponce, G., Svanstedt, N., and Vega, L., *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, J. London Math. Soc. (1996), 551–559.

[Bo1] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geometric and Functional Anal. **3** (1993), 107–156, 209–262.

[Bo2] Bourgain, J., *Periodic Korteweg-de Vries equation with measure as initial data*, Selecta Math. (N.S.) **3** (1997), 115–159.

[CW] Cazenave, T. and Weissler, F., *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. TMA **14** (1990), 807–836.

[GV] Ginibre, J. and Velo, G., *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J. Math. Pure Appl. **64** (1985), 363–401.

[K] Kato, T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8** (1983), 93-128.

[KePoVe1] Kenig, C. E., Ponce, G., and Vega, L., *Oscillatory integrals and regularity of dispersive equations*, Indiana U. Math. J. **40** (1991), 33-69.

[KePoVe2] Kenig, C. E., Ponce, G., and Vega, L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527-620.

[KePoVe3] Kenig, C. E., Ponce, G., and Vega, L., *A bilinear estimate with applications to the KdV equation*, Journal Amer. Math. Soc. **9** (1996), 573–603.

[KePoVe4] Kenig, C. E., Ponce, G., and Vega, L., *Quadratic forms for the 1-D semilinear Schrödinger equation*, Trans. Amer. Math. Soc. **348** (1996), 3323–3353.

[KePoVe5] Kenig, C. E., Ponce, G., and Vega, L., *On the ill posedness of some canonical dispersive equations*, to appear in Duke Math. J..

[KeRu] Kenig, C. E. and Ruiz, A., *A strong type (2,2) estimate for the maximal function associated to the Schrödinger equation*, Trans. Amer. Math. Soc. **230** (1983), 239–246.

[L] Lamb, G. L. jr., *Elements of soliton theory*, John Wiley & Sons, New York (1980).

[Li] Lindblad, H., *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations.*, Duke Math. J. **72** (1993), 503-539.

[M] Miura, R. M., *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9** (1969), 1202-1209.

[NaTaTs] Nakanishi, K., Takaoka, H., and Tsutsumi, Y.,, *Counterexamples to bilinear estimates related with the KdV equation and the nonlinear Schrödinger equation*, to appear in the Proceedings of the IMS Conference on Differential Equations from Mechanics.

[Sr] Strichartz, R. S., *Restriction of Fourier transforms to quadratic surface and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–714.

[To] Tonegawa, S., *Solvability of the initial value problem and asymptotic behavior of solutions for nonlinear Schrödinger equations*, Doctor Thesis, University of Tokyo, March 2000.

[Ts] Tsutsumi, Y., *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funk. Ekva. **30** (1987), 115–125.

[Tz] Tzvetkov, N., *Remark on the local ill-posedness for KdV equation*, to appear in C. R. Acad. Sci. Paris.

[W] Wadati, M., *The modified Korteweg-de Vries equation*, J. Phys. Soc. Japan **34** (1973), 1289-1296.