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On standing waves for nonlinear Schrödinger equations with potentials

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This is a joint work with Reika Fukuizumi (Tohoku University). We consider the instability of standing wave solution $u_\omega(t,x)=e^{i\omega t}\phi_\omega(x)$ for the nonlinear Schrödinger equation with potential $V(x)$:

$$iu_t=-\Delta u+V(x)u-|u|^{p-1}u, \qquad (t,x)\in\mathbb{R}^{1+n}. \tag{1}$$

We always assume $1<p<\infty$ if $n=1,2$, and $1<p<1+4/(n-2)$ if $n\geq 3$. Moreover, we suppose that $\omega\in\mathbb{R}$ and $\phi_\omega(x)$ is a ground state for

$$-\Delta\phi+\omega\phi+V(x)\phi-|\phi|^{p-1}\phi=0, \quad x\in\mathbb{R}^n. \tag{2}$$

In this note, under appropriate assumptions on $V(x)$, we will show that if $p>1+4/n$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable for sufficiently large $\omega>0$. Before stating our result precisely, we recall some known results. First, we consider the case $V(x)\equiv 0$. For any $\omega>0$, there exists a unique positive radial solution $\psi_\omega(x)$ of (2) with $V(x)\equiv 0$ in $H^1(\mathbb{R}^n)$, and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (1) with $V(x)\equiv 0$ is stable for any $\omega>0$ if $p<1+4/n$, and unstable for any $\omega>0$ if $p\geq 1+4/n$ (see, e.g., [1, 3, 6, 11, 12]). Meanwhile, when $-\Delta+V(x)$ has the first eigenvalue λ_1, it is shown in [10, 4] using standard bifurcation theory that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is stable for $\omega>-\lambda_1$ sufficiently close to $-\lambda_1$, even if $p\geq 1+4/n$.

For potential $V(x)$, we assume

(V1) $V(x)\in C^2(\mathbb{R}^n,\mathbb{R})$, and there exist $m\geq 0$ and $C>0$ such that $0\leq V(x)\leq C(1+|x|^m)$ on $\mathbb{R}^n$, and

(V2) $|x\cdot\nabla V(x)|+|\sum_{j,k=1}^n x_jx_k\partial_j\partial_k V(x)|\leq C(1+V(x))$ on $\mathbb{R}^n$.

Example. (i) (Harmonic potentials) Let $c_1, \cdots, c_n$ be positive constants. Then $V_1(x) = \sum_{j=1}^{n} c_j x_j^2$ satisfies (V1) and (V2).

(ii) $V_2(x) = 1 + \sin x_1$ satisfies (V1), but does not satisfy (V2).

(iii) For $c \geq 0$, $V_1(x) + cV_2(x)$ satisfies (V1) and (V2).

(iv) (V1) and (V2) are satisfied if $V(x) \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies

$$V(x) \geq 0, \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|} \quad (|\alpha| \leq 2).$$

We use the following notation.

$$\begin{aligned}
&X := \{v \in H^1(\mathbb{R}^n) : V(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}, \\
&E(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \\
&S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|_2^2, \\
&P(v) := \|\nabla v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^n} x \cdot \nabla V(x)|v(x)|^2 dx - \frac{n(p-1)}{2(p+1)}\|v\|_{p+1}^{p+1}, \\
&I_\omega(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \|v\|_{p+1}^{p+1}.
\end{aligned}$$

We note that

$$P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1}, \quad I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1},$$

where $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.

Assumption (A1). For any $u_0 \in X$, there exist $T = T(\|u_0\|_X) > 0$ and a unique solution $u(t) \in C([0,T), X)$ of (1) such that $u(0) = u_0$ and

$$E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0,T).$$

In addition, if $u_0 \in X$ satisfies $|x|u_0 \in L^2(\mathbb{R}^n)$, then we have

$$\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8P(u(t)), \quad t \in [0,T).$$

For sufficient conditions on $V(x)$ that (A1) holds, see, e.g., Section 9.2 in [2]. We remark that (A1) is satisfied for (iii) in Example above, and for $V(x) \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|} \quad (|\alpha| \leq 1).$$

Definition 1. We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $u_0 \in X$ satisfies

$$\inf_{\theta\in\mathbb{R}} \|u_0 - e^{i\theta}\phi_\omega\|_X < \delta,$$

then the solution $u(t)$ of (1) with $u(0) = u_0$ exists for all $t \geq 0$ and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi_\omega\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable.

Definition 2. $\mathcal{G}_\omega :=$ the set of minimizers for

$$\inf\{S_\omega(v) : \ v \in X \setminus \{0\}, \ I_\omega(v) = 0\}.$$

An element of $\mathcal{G}_\omega$ is called a ground state of (2).

Assumption (A2). There exists $\omega_0 \in (0, \infty)$ such that $\mathcal{G}_\omega$ is not empty for any $\omega \in (\omega_0, \infty)$.

If $V(x) \in C(\mathbb{R}^n, \mathbb{R})$ satisfies $\lim_{|x|\to\infty} V(x) = +\infty$, by the compactness of the embedding $X \subset L^q(\mathbb{R}^n)$ with $2 \leq q < 2n/(n-2)$, it is easy to see that (A2) is satisfied. However, for bounded $V(x)$, we may need some additional assumptions related to the concentration compactness principle (see [7, 8]).

Theorem 1. Assume (A1), (A2), (V1) and (V2). Let $p > 1 + 4/n$ and $\phi_\omega(x) \in \mathcal{G}_\omega$. Then there exists $\omega_* \in (\omega_0, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable for any $\omega \in (\omega_*, \infty)$.

By the general theory in [6], under an assumption on the spectrum of a linearized operator, the standing wave solution $e^{i\omega_1 t}\phi_{\omega_1}(x)$ of (1) is stable (resp. unstable) if the function $\|\phi_\omega\|_2^2$ is strictly increasing (resp. decreasing) at $\omega = \omega_1$. In case of $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, it is easy to check the monotonicity of $\|\psi_\omega\|_2^2$. However, it seems difficult to check this property for general $V(x)$. So, for the proof of Theorem 1, we use the following sufficient condition for the instability, which is a modification of Theorem 3 in [9] (see also [4, 5, 11]).

Proposition 2. Assume (A1), (A2), (V1) and (V2), and let $\phi_\omega(x) \in \mathcal{G}_\omega$. If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable. Here, $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ for $\lambda > 0$.

We note that $\|v^\lambda\|_2^2 = \|v\|_2^2$ and

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} V\left(\frac{x}{\lambda}\right)|v(x)|^2dx - \frac{\lambda^{n(p-1)/2}}{p+1}\|v\|_{p+1}^{p+1},$$

$$\partial_\lambda^2 E(v^\lambda)|_{\lambda=1} = \|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n}\left\{2x\cdot\nabla V(x) + \sum_{j,k=1}^n x_jx_k\partial_j\partial_k V(x)\right\}|v(x)|^2dx$$
$$-\frac{n(p-1)}{2(p+1)}\left\{\frac{n(p-1)}{2}-1\right\}\|v\|_{p+1}^{p+1}.$$

Since $P(\phi_\omega) = \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$, if we put

$$V^*(x) = 3x\cdot\nabla V(x) + \sum_{j,k=1}^n x_jx_k\partial_j\partial_k V(x),$$

then we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = \frac{1}{2}\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2dx - \frac{n(p-1)}{2(p+1)}\left\{\frac{n(p-1)}{2}-2\right\}\|\phi_\omega\|_{p+1}^{p+1}.$$

Thus, we see that the condition $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ is equivalent to

$$\frac{\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2dx}{\|\phi_\omega\|_{p+1}^{p+1}} < \frac{n(p-1)\{n(p-1)-4\}}{2(p+1)}. \tag{3}$$

We remark that the right hand side of (3) is a positive constant by the assumption $p > 1 + 4/n$ in Theorem 1. By using the variational characterization of the ground state $\phi_\omega(x)$ of (2) and the rescaling (4) below, we will show that the left hand side of (3) converges to 0 as $\omega \to \infty$. For $\phi_\omega(x) \in \mathcal{G}_\omega$, we define $\tilde{\phi}_\omega(x)$ by

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x), \quad \omega \in (\omega_0, \infty). \tag{4}$$

Then, $\tilde{\phi}_\omega(x)$ satisfies

$$-\Delta\phi + \phi + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n.$$

Recall that $\psi_1(x)$ is the unique positive radial solution of (2) with $V(x) \equiv 0$ and $\omega = 1$ in $H^1(\mathbb{R}^n)$, and we put

$$\tilde{I}_\omega(v) := \|\nabla v\|_2^2 + \|v\|_2^2 + \omega^{-1}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|v(x)|^2dx - \|v\|_{p+1}^{p+1},$$
$$I_1^0(v) := \|\nabla v\|_2^2 + \|v\|_2^2 - \|v\|_{p+1}^{p+1}.$$

Lemma 3. Assume (A2) and (V1). Then, we have

(i) $\lim_{\omega\to\infty}\|\tilde{\phi}_\omega\|_{p+1}^{p+1}=\|\psi_1\|_{p+1}^{p+1}$, (ii) $\lim_{\omega\to\infty}I_1^0(\tilde{\phi}_\omega)=0$, (iii) $\lim_{\omega\to\infty}\|\tilde{\phi}_\omega\|_{H^1}^2=\|\psi_1\|_{H^1}^2$.

Proof. First of all, we note that $\tilde{\phi}_\omega(x)$ is a minimizer of

$$\inf\{\|v\|_{p+1}^{p+1}:\ v\in X\setminus\{0\},\ \tilde{I}_\omega(v)\le 0\},$$

and $\psi_1(x)$ is a minimizer of

$$\inf\{\|v\|_{p+1}^{p+1}:\ v\in H^1(\mathbb{R}^n)\setminus\{0\},\ I_1^0(v)\le 0\}.$$

First, we show (i). Since $\tilde{I}_\omega(\tilde{\phi}_\omega)=0$, we have $I_1^0(\tilde{\phi}_\omega)\le 0$. Thus, we have $\|\psi_1\|_{p+1}^{p+1}\le\|\tilde{\phi}_\omega\|_{p+1}^{p+1}$ for any $\omega\in(\omega_0,\infty)$. Moreover, for any $\mu>1$, it follows from $I_1^0(\psi_1)=0$ that

$$\mu^{-2}\tilde{I}_\omega(\mu\psi_1)=-(\mu^{p-1}-1)\|\psi_1\|_{p+1}^{p+1}+\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx.$$

Here, from the assumption (V1), we have

$$\left|\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx\right|\le C\int_{\mathbb{R}^n}(1+\omega^{-m/2}|x|^m)|\psi_1(x)|^2dx.$$

Since $\psi_1(x)$ has exponential decay at infinity, we have $(1+|x|^m)|\psi_1(x)|^2\in L^1(\mathbb{R}^n)$ and

$$\lim_{\omega\to\infty}\omega^{-1}\int_{\mathbb{R}^n}V\left(\frac{x}{\sqrt{\omega}}\right)|\psi_1(x)|^2dx=0.$$

Thus, there exists $\omega(\mu)\in(\omega_0,\infty)$ such that $\tilde{I}_\omega(\mu\psi_1)<0$ for any $\omega\in(\omega(\mu),\infty)$, so we have

$$(\|\psi_1\|_{p+1}^{p+1}\le)\ \|\tilde{\phi}_\omega\|_{p+1}^{p+1}\le\|\mu\psi_1\|_{p+1}^{p+1}=\mu^{p+1}\|\psi_1\|_{p+1}^{p+1}$$

for any $\omega\in(\omega(\mu),\infty)$. Since $\mu>1$ is arbitrary, we conclude (i).

Next, we show (ii). Since $I_1^0(\tilde{\phi}_\omega)\le 0$, for any $\omega\in(\omega_0,\infty)$ there exists $\mu(\omega)\in(0,1]$ such that $I_1^0(\mu(\omega)\tilde{\phi}_\omega)=0$. Thus, we have

$$\|\psi_1\|_{p+1}^{p+1}\le\|\mu(\omega)\tilde{\phi}_\omega\|_{p+1}^{p+1}=\mu(\omega)^{p+1}\|\tilde{\phi}_\omega\|_{p+1}^{p+1},$$

which implies $\|\psi_1\|_{p+1}/\|\tilde{\phi}_\omega\|_{p+1}\le\mu(\omega)\le 1$, and from (i) we have $\lim_{\omega\to\infty}\mu(\omega)=1$. Moreover, from $I_1^0(\mu(\omega)\tilde{\phi}_\omega)=0$, we have

$$I_1^0(\tilde{\phi}_\omega)=(\mu(\omega)^{p-1}-1)\|\tilde{\phi}_\omega\|_{p+1}^{p+1}.$$

Hence, again from (i), we conclude (ii).

Since $I_1^0(\psi_1)=0$, (iii) follows from (i) and (ii) immediately. □

Proof of Theorem 1. As stated above, we have only to show that the left hand side of (3) converges to 0 as $\omega \to \infty$. Since we have

$$\frac{\int_{\mathbb{R}^n} V^*(x)|\phi_\omega(x)|^2 dx}{\|\phi_\omega\|_{p+1}^{p+1}} = \frac{\omega^{-1}\int_{\mathbb{R}^n} V^*(x/\sqrt{\omega})|\tilde{\phi_\omega}(x)|^2 dx}{\|\tilde{\phi_\omega}\|_{p+1}^{p+1}},$$

by Lemma 3 (i), it suffices to prove

$$\lim_{\omega\to\infty} \omega^{-1}\int_{\mathbb{R}^n} V^*\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^2 dx = 0. \tag{5}$$

From $\tilde{I_\omega}(\tilde{\phi_\omega}) = 0$ and Lemma 3 (ii), we have

$$\lim_{\omega\to\infty} \omega^{-1}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^2 dx = -\lim_{\omega\to\infty} I_1^0(\tilde{\phi_\omega}) = 0.$$

By the assumption (V2), we have $|V^*(x)| \le C(1+V(x))$ on $\mathbb{R}^n$, and by Lemma 3 (iii) we obtain (5). □

Remark. Let $\phi_\omega(x) \in \mathcal{G}_\omega$ and we assume (without loss of generality) that $\phi_\omega(x)$ is positive in $\mathbb{R}^n$. By Lemma 3 and the concentration compactness principle, we see that there exist a subsequence $\{\tilde{\phi_{\omega_j}}(x)\}$ of $\{\tilde{\phi_\omega}(x)\}$ and a sequence $\{y_j\} \subset \mathbb{R}^n$ such that

$$\lim_{j\to\infty} \|\tilde{\phi_{\omega_j}} - \psi_1(\cdot + y_j)\|_{H^1} = 0 \tag{6}$$

(see Theorem III.1 in [8]). Although (6) may give some information on the asymptotic behavior of $\phi_\omega(x) \in \mathcal{G}_\omega$ as $\omega \to \infty$, we did not use (6) in the proof of Theorem 1 directly. We also note that Lemma 3 holds for any p such that $1 < p < \infty$ if $n = 1, 2$, and $1 < p < 1 + 4/(n-2)$ if $n \ge 3$. Finally, we remark that, in the case $p = 1 + 4/n$, it follows from (6) that

$$\lim_{\omega\to\infty} \|\phi_\omega\|_2^2 = \|\psi_1\|_2^2.$$

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