Title: Generalized fractional integrals (Harmonic Analysis and Nonlinear P.D.E.)

Author(s): Nakai, Eiichi

Citation: 数理解析研究所講究録 (2001), 1201: 56-74

Issue Date: 2001-04

URL: http://hdl.handle.net/2433/40948

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
1. INTRODUCTION

The fractional integral $I_{\alpha} (0 < \alpha < n)$ is defined by

$$I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$ 

It is known that $I_{\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{q}(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = n/q > 0$ as the Hardy-Littlewood-Sobolev theorem. The fractional integral was studied by many authors (see, for example, Rubin [10] or Chapter 5 in Stein [11]). The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory. We introduce generalized fractional integrals and extend the Hardy-Littlewood-Sobolev theorem to the Orlicz spaces. We show that, for example, a generalized fractional integral is bounded from $\exp L^p$ to $\exp L^q$.

Let $B(a, r)$ be the ball $\{x \in \mathbb{R}^n : |x - a| < r\}$ with center $a$ and of radius $r > 0$, and $B_0 = B(O, 1)$ with center the origin and of radius 1. The modified fractional integral $\tilde{I}_{\alpha} (0 < \alpha < n + 1)$ is defined by

$$\tilde{I}_{\alpha}f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1 - \chi_{B_0}(y)}{|y|^{n-\alpha}} \right) dy,$$

where $\chi_{B_0}$ is the characteristic function of $B_0$. It is known that the modified fractional integral $\tilde{I}_{\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = 0$, from $L^p(\mathbb{R}^n)$ to $\text{Lip}_\beta(\mathbb{R}^n)$ when $p > 1$ and $-1 < n/p - \alpha = -\beta < 0$, from $\text{BMO}(\mathbb{R}^n)$ to $\text{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\text{Lip}_\beta(\mathbb{R}^n)$ to $\text{Lip}_\gamma(\mathbb{R}^n)$ when $0 < \alpha + \beta = \gamma < 1$.

We investigate the boundedness of generalized fractional integrals from the Orlicz space $L^\phi(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$ and from $\text{BMO}_\phi(\mathbb{R}^n)$ to $\text{BMO}_\psi(\mathbb{R}^n)$. If $\phi(r) \equiv 1$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha (0 < \alpha \leq 1)$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n)$. We also investigate the boundedness of generalized fractional integrals on the Morrey and Campanato spaces.
2. Notations and Definitions

For a function $\rho : (0, +\infty) \rightarrow (0, +\infty)$, let

$$ I_{\rho}f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy. $$

We consider the following conditions on $\rho$:

1. \[ \int_{0}^{1} \frac{\rho(t)}{t} dt < +\infty, \]
2. \[ \frac{1}{A_1} \leq \frac{\rho(s)}{\rho(r)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2, \]
3. \[ \frac{\rho(r)}{r^n} \leq A_2 \frac{\rho(s)}{s^n} \quad \text{for} \quad s \leq r, \]

where $A_1, A_2 > 0$ are independent of $r, s > 0$. If $\rho(r) = r^\alpha, 0 < \alpha < n$, then $I_{\rho}$ is the fractional integral or the Riesz potential denoted by $I_{\alpha}$.

We define the modified version of $I_{\rho}$ as follows:

$$ \tilde{I}_{\rho}f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)}{|y|^n}(1 - \chi_{B_0}(y)) \right) dy. $$

We consider the following conditions on $\rho$: (2.1), (2.2) and

1. \[ \frac{\rho(r)}{r^{n+1}} \leq A_2' \frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad s \leq r, \]
2. \[ \int_{r}^{+\infty} \frac{\rho(t)}{t^2} dt \leq A_2' \frac{\rho(r)}{r}, \]
3. \[ \left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \leq A_3 |r-s| \frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2, \]

where $A_2', A_2'', A_3 > 0$ are independent of $r, s > 0$. If $\rho(r)^{r^\alpha}$ is increasing for some $\alpha \geq 0$ and $\rho(r)/r^{\beta}$ is decreasing for some $\beta \geq 0$, then $\rho$ satisfies (2.2) and (2.6). If $\rho(r) = r^\alpha, 0 < \alpha < n+1$, then $\tilde{I}_{\rho} = \tilde{I}_{\alpha}$. If $\tilde{I}_{\rho}f$ and $I_{\rho}f$ are well defined, then $\tilde{I}_{\rho}f - I_{\rho}f$ is a constant.

A function $\Phi : [0, +\infty) \rightarrow [0, +\infty]$ is called a Young function if $\Phi$ is convex, $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \rightarrow +\infty} \Phi(r) = +\infty$. Any Young function is increasing. For a Young function $\Phi$, the complementary function is defined by

$$ \Phi^-(r) = \sup\{rs - \Phi(s) : s \geq 0\}, \quad r \geq 0. $$

For example, if $\Phi(r) = r^p/p, 1 < p < \infty$, then $\Phi^-(r) = r^{p'}/p', 1/p + 1/p' = 1$. If $\Phi(r) = r$, then $\Phi^-(r) = 0(0 \leq r \leq 1), = +\infty(r > 1)$. 

For a Young function $\Phi$, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon |f(x)|) \, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\},$$

$$L^\Phi_{\text{weak}}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r) \, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi,\text{weak}} = \inf \left\{ \lambda > 0 : \sup_{r>0} \Phi(r) \, m \left( r, \frac{f}{\lambda} \right) \leq 1 \right\},$$

where $m(r, f) = \{|x \in \mathbb{R}^n : |f(x)| > r\}|$.

Then

$$L^\Phi(\mathbb{R}^n) \subset L^\Phi_{\text{weak}}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{\Phi,\text{weak}} \leq \|f\|_\Phi.$$
A function \( \theta : (0, +\infty) \rightarrow (0, +\infty) \) is said to be almost increasing (almost decreasing) if there exists a constant \( C > 0 \) such that
\[
\theta(r) \leq C\theta(s) \quad (\theta(r) \geq C\theta(s)) \quad \text{for} \quad r \leq s.
\]

A function \( \theta : (0, +\infty) \rightarrow (0, +\infty) \) is said to satisfy the doubling condition if there exists a constant \( C > 0 \) such that
\[
C^{-1} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.
\]

For \( 1 \leq p < \infty \) and a function \( \phi : (0, +\infty) \rightarrow (0, +\infty) \), let
\[
\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_{B} |f(x)|^{p} \, dx \right)^{1/p},
\]
\[
\mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = \{ f \in L_{\text{loc}}^{p}(\mathbb{R}^{n}) : \|f\|_{\mathcal{L}_{p,\phi}} < +\infty \}.
\]

We assume that \( \phi \) satisfies the doubling condition and that \( \phi(r)r^{n/p} \) is almost increasing. If \( \phi(r) = r^{(\lambda-n)/p} \) \( (0 \leq \lambda \leq n) \), then \( \mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = L^{p,\lambda}(\mathbb{R}^{n}) \) which is the classical Morrey space. If \( \lambda = 0 \), then \( L^{p,\lambda}(\mathbb{R}^{n}) = L^{p}(\mathbb{R}^{n}) \). If \( \lambda = n \), then \( L^{p,\lambda}(\mathbb{R}^{n}) = L^{\infty}(\mathbb{R}^{n}) \).

For \( 1 \leq p < \infty \) and a function \( \phi : (0, +\infty) \rightarrow (0, +\infty) \), let
\[
\|f\|_{L_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_{B} |f(x)|^{p} \, dx \right)^{1/p},
\]
\[
L_{p,\phi}(\mathbb{R}^{n}) = \{ f \in L_{\text{loc}}^{p}(\mathbb{R}^{n}) : \|f\|_{L_{p,\phi}} < +\infty \}.
\]

We assume that \( \phi \) satisfies the doubling condition and that \( \phi(r)r^{n/p} \) is almost increasing. If \( \phi(r) = r^{(\lambda-n)/p} \) \( (0 \leq \lambda \leq n+1) \), then \( \mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = L^{p,\lambda}(\mathbb{R}^{n}) \) which is the classical Companato space.

If \( \phi \) is almost increasing, then \( L_{p,\phi}(\mathbb{R}^{n}) = L^{1,\phi}(\mathbb{R}^{n}) \) for all \( p > 1 \). Let \( \text{BMO}_{\phi}(\mathbb{R}^{n}) = L^{1,\phi}(\mathbb{R}^{n}) \). If \( \phi \equiv 1 \), then \( \text{BMO}_{\phi}(\mathbb{R}^{n}) = \text{BMO}(\mathbb{R}^{n}) \). If \( \phi(r) = r^{\alpha} \), \( 0 < \alpha \leq 1 \), then it is known that \( \text{BMO}_{\phi}(\mathbb{R}^{n}) = \text{Lip}_{\alpha}(\mathbb{R}^{n}) \).

The letter \( C \) shall always denote a constant, not necessarily the same one.

3. **Main results**

Our main results are as follows:
**Theorem 3.1.** Let $\rho$ satisfy (2.1)--(2.3). Let $\Phi$ and $\Psi$ be Young functions with (2.7). Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,

$$(3.1) \quad \int_{r}^{+\infty} \tilde{\Phi} \left( \frac{\rho(t)}{A \int_{0}^{r} (\rho(s)/s) \, ds \, \Phi^{-1}(1/r^n) t^n} \right) t^{n-1} \, dt \leq A',$$

$$(3.2) \quad \int_{0}^{r} \frac{\rho(t)}{t} \, dt \, \Phi^{-1} \left( \frac{1}{r^n} \right) \leq A'' \Psi^{-1} \left( \frac{1}{r^n} \right),$$

where $\tilde{\Phi}$ is the complementary function with respect to $\Phi$. Then, for any $C_{0} > 0$, there exists a constant $C_{1} > 0$ such that, for $f \in L^{\Phi}(\mathbb{R}^n)$,

$$(3.3) \quad \Psi \left( \frac{|I_{\rho}f(x)|}{C_{1} ||f||_{\Phi}} \right) \leq \Phi \left( \frac{Mf(x)}{C_{0} ||f||_{\Phi}} \right).$$

Therefore $I_{\rho}$ is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}_{\text{weak}}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_{2}$, then $I_{\rho}$ is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

**Remark 3.1.** From (2.2) it follows that

$$(3.4) \quad \rho(r) \leq C \int_{0}^{r} \frac{\rho(t)}{t} \, dt.$$  

If $\rho(r)/r^\varepsilon$ is almost increasing for some $\varepsilon > 0$ and $\rho(t)/t^n$ is almost decreasing, then $\rho$ satisfies (2.1)--(2.3) and $\int_{0}^{r} (\rho(t)/t) \, dt \sim \rho(r)$. Let, for example, $\rho(r) = r^\alpha (\log(1/r))^{-\beta}$ for small $r$. If $\alpha = 0$ and $\beta > 1$, then $\int_{0}^{r} (\rho(t)/t) \, dt \sim (\log(1/r))^{-\beta+1}$. If $\alpha > 0$ and $-\infty < \beta < +\infty$, then $\int_{0}^{r} (\rho(t)/t) \, dt \sim \rho(r)$.

**Remark 3.2.** In the case $\Phi(r) = r$, (3.1) is equivalent to

$$\frac{\rho(t)}{t^n} \leq \frac{A \int_{0}^{r} (\rho(s)/s) \, ds}{r^n}, \quad 0 < r \leq t.$$  

This inequality follows from (2.3) and (3.4).

The following corollary is stated without the complementary function.

**Corollary 3.2.** Let $\rho$ satisfy (2.1)--(2.3). Let $\Phi$ and $\Psi$ be Young functions with (2.7). Assume that

$$\int_{0}^{r} \frac{\rho(t)}{t} \, dt \, \Phi^{-1} \left( \frac{1}{r^n} \right).$$
is almost decreasing and that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{align}
(3.5) \quad & \int_r^{+\infty} \frac{\rho(t)}{t} \Phi^{-1}\left(\frac{1}{t^n}\right) \, dt \leq A \int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}\left(\frac{1}{r^n}\right), \\
(3.6) \quad & \int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}\left(\frac{1}{r^n}\right) \leq A' \Psi^{-1}\left(\frac{1}{r^n}\right).
\end{align}

Then (3.3) holds. Therefore $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi_{\text{weak}}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

**Remark 3.3.** If $r^\epsilon \rho(r) \Phi^{-1}(1/r^n)$ is almost decreasing for some $\epsilon > 0$, then
\[ \int_r^{+\infty} \frac{\rho(t)}{t} \Phi^{-1}\left(\frac{1}{t^n}\right) \, dt \leq C r^\epsilon \rho(r) \Phi^{-1}\left(\frac{1}{r^n}\right). \]

This inequality and (3.4) yield (3.5).

**Remark 3.4.** We cannot replace (3.2) or (3.6) by
\[ \rho(r) \Phi^{-1}\left(\frac{1}{r^n}\right) \leq A \Psi^{-1}\left(\frac{1}{r^n}\right) \quad \text{for all } r > 0 \]
(see Section 5 in [6]).

O'Neil [7] showed the boundedness for convolution operators on the Orlicz spaces. Cianchi [1] gave a necessary and sufficient condition on $\Phi$ and $\Psi$ so that the fractional integral $I_\alpha$ is bounded from $L^\Phi$ to $L^\Psi$.

**Theorem 3.3.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\Phi$ be Young function with (2.7), $\phi$ satisfy the doubling condition and be almost increasing. Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,
\begin{align}
(3.7) \quad & \int_r^{+\infty} \tilde{\Phi}\left(\frac{r \rho(t)}{A \int_0^r (\rho(s)/s) \, ds \Phi^{-1}(1/r^n) t^{n+1}}\right) t^{n-1} \, dt \leq A', \\
(3.8) \quad & \int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}\left(\frac{1}{r^n}\right) \leq A'' \phi(r),
\end{align}

where $\tilde{\Phi}$ is the complementary function with respect to $\Phi$. Then $\tilde{I}_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$.

**Theorem 3.4.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition, and $\phi(r)r^n$ and $\psi(r)r^n$ be almost increasing. Assume
that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{align}
\int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} dt &\leq A \int_{0}^{r} \frac{\rho(t)}{t} dt \frac{\phi(r)}{r}, \\
\int_{0}^{r} \frac{\rho(t)}{t} dt \phi(r) &\leq A' \psi(r).
\end{align}

Then $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

If $\Phi \in \nabla_2$ and $\Phi^{-1}(1/r^n) = \phi(r)$, then we can show

$$L^\Phi_{weak}(\mathbb{R}^n) \subset L^{1,\phi}(\mathbb{R}^n) \text{ and } \|f\|_{L^{1,\phi}} \leq C \|f\|_{\Phi,weak}.$$ 

Then we have the following.

**Corollary 3.5.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\Phi$ be Young function with (2.7), $\Phi \in \nabla_2$, $\phi$ satisfy the doubling condition and be almost increasing. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{align}
\int_{r}^{+\infty} \frac{\rho(t)\Phi^{-1}(1/t^n)}{t^2} dt &\leq A \int_{0}^{r} \frac{\rho(t)}{t} dt \frac{\Phi^{-1}(1/r^n)}{r}, \\
\int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1}(\frac{1}{r^n}) &\leq A' \phi(r).
\end{align}

Then $\tilde{I}_\rho$ is bounded from $L^\Phi_{weak}(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$.

**Theorem 3.6.** Let $\rho$ satisfy (2.1), (2.2), (2.5) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition, and $\phi(r)r^n$ and $\psi(r)r^n$ are almost increasing. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{align}
\int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} dt &\leq A \int_{0}^{r} \frac{\rho(t)}{t} dt \frac{\phi(r)}{r}, \\
\int_{0}^{r} \frac{\rho(t)}{t} dt \phi(r) &\leq A' \psi(r).
\end{align}

Then $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

**Remark 3.5.** From Lemma 4.3 it follows that $\tilde{I}_\rho 1$ is a constant. Hence $\tilde{I}_\rho$ is well defined as an operator from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

The boundedness of the fractional integral $I_\alpha$ on the Campanato space is known (Peetre [8]).

**Corollary 3.7.** Let $\rho$ satisfy (2.1), (2.2), (2.5) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition and be almost increasing. Assume that there exist
constants $A, A' > 0$ such that, for all $r > 0$,
\begin{align}
(3.15) & \quad \int_r^{+\infty} \rho(t)\phi(t) \frac{dt}{t^2} \leq A \int_0^r \rho(t) \frac{\phi(r)}{t} \frac{dt}{r}, \\
(3.16) & \quad \int_0^r \rho(t) \frac{dt}{t} \phi(r) \leq A' \psi(r).
\end{align}

Then $\tilde{I}_\rho$ is bounded from $\text{BMO}_\phi(\mathbb{R}^n)$ to $\text{BMO}_\psi(\mathbb{R}^n)$.

This Corollary is a generalization of the well-known result that $\tilde{I}_\alpha$ is bounded from $\text{BMO}(\mathbb{R}^n)$ to $\text{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\text{Lip}_\beta(\mathbb{R}^n)$ to $\text{Lip}_{\alpha+\beta}(\mathbb{R}^n)$ when $\alpha > 0$, $\beta > 0$ and $0 < \alpha + \beta < 1$.

The results in Figure 1 are known. Our results contain these. Moreover, we have the results in Figure 2.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\matrix (m) [matrix of math nodes, row sep=1.5em, column sep=2.5em, text height=1.5ex, text depth=0.25ex]
{ (1 < p < q < \infty) & (0 < \beta < \gamma < 1) \\
 L^p & L^q & \text{BMO} & \text{Lip}_\beta & \text{Lip}_\gamma \\
 I_\alpha & & & & \\
 -n/p + \alpha = -n/q & & & & \\
 \tilde{I}_\alpha & & & & \\
 -n/p + \alpha = 0 & & & & \\
 \tilde{I}_\alpha & & & & \\
 -n/p + \alpha = \beta & & & & \\
};
\end{tikzpicture}
\caption{Boundedness of fractional integrals}
\end{figure}

We can also state our results on spaces of homogeneous type with appropriate conditions.

4. PROOFS

Let $\Phi$ be a Young function. By the convexity and $\Phi(0) = 0$, we have
\begin{equation}
(4.1) \quad \Phi(r) \leq \frac{r}{s} \Phi(s) \quad \text{for } r \leq s.
\end{equation}

Let $\tilde{\Phi}$ be the complementary function with respect to $\Phi$. Then
\begin{equation}
(4.2) \quad \tilde{\Phi} \left( \frac{\Phi(r)}{r} \right) \leq \Phi(r), \quad r > 0.
\end{equation}
Actually,
\[ \frac{\Phi(r)}{r}s - \Phi(s) \leq \Phi(r) \quad \text{for } s < r \]
and
\[ \frac{\Phi(r)}{r}s - \Phi(s) \leq 0 \quad \text{for } s \geq r. \]

We note that
\[ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq 2\|f\|_{\Phi}\|g\|_{\tilde{\Phi}} \quad \text{(see for example [9]).} \]

**Proof of Theorem 3.1.** Let
\[ J_1 = \int_{|x-y|<r} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy \quad \text{and} \]
\[ J_2 = \int_{|x-y|\geq r} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy. \]

Let
\[ h(r) = \inf \left\{ \frac{\rho(s)}{s^n} : s \leq r \right\}, \quad r > 0. \]

Then \( h \) is nonincreasing. It follows that
\[ \int_{|x-y|<r} |f(y)| h(|x-y|) \, dy \leq Mf(x) \int_{|x-y|<r} h(|x-y|) \, dy \]
(see Stein[12, p.57]). Since $h(r) \sim \rho(r)/r^n$,

\[ |J_1| \leq CMf(x) \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n} \, dy \leq CMf(x) \int_0^r \frac{\rho(t)}{t} \, dt. \tag{4.4} \]

Next we estimate $J_2$. By (4.3) we have

\[ |J_2| \leq 2 \left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^c}(\cdot) \right\|_\Phi \|f\|_\Phi, \tag{4.5} \]

where $\chi_{B(x,r)^c}$ is the characteristic function of the complement of $B(x,r)$. Let

\[ F(r) = \int_0^r \frac{\rho(s)}{s} ds \Phi^{-1} \left( \frac{1}{r^n} \right). \tag{4.6} \]

We show

\[ \left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^c}(\cdot) \right\|_\Phi \leq CF(r). \tag{4.7} \]

From (2.2) and the increasingness of $\Phi$ it follows that

\[ \int_{|x-y|\geq r} \Phi \left( \frac{\rho(|x-y|)}{\lambda|x-y|^n} \right) \, dy \leq C_2 \int_r^{+\infty} \Phi \left( \frac{\rho(t)}{\lambda t^n} \right) t^{n-1} \, dt, \tag{4.8} \]

where $C_2$ is independent of $\lambda > 0$, $r > 0$ and $x \in \mathbb{R}^n$. We may assume that $C_2A' \geq 1$. By (4.1) and (3.1) we have

\[ \int_r^{+\infty} \Phi \left( \frac{\rho(t)}{C_2AA'F(r)t^n} \right) t^{n-1} \, dt \leq \frac{1}{C_2A'} \int_r^{+\infty} \Phi \left( \frac{\rho(t)}{AF(r)t^n} \right) t^{n-1} \, dt \leq \frac{1}{C_2}. \tag{4.9} \]

Let $\lambda = C_2AA'F(r)$. Then, by (4.8) and (4.9) we have

\[ \int_{|x-y|\geq r} \Phi \left( \frac{\rho(|x-y|)}{\lambda|x-y|^n} \right) \, dy \leq 1, \]

and so (4.7). By (4.4), (4.5) and (4.7) we have

\[ |J \rho f(x)| = |J_1 + J_2| \leq C \left( Mf(x) + \|f\|_\Phi \Phi^{-1} \left( \frac{1}{r^n} \right) \right) \int_0^r \frac{\rho(t)}{t} \, dt. \tag{4.10} \]

Choose $r > 0$ so that

\[ \Phi^{-1} \left( \frac{1}{r^n} \right) = \frac{Mf(x)}{C_0\|f\|_\Phi}. \tag{4.11} \]

Then

\[ \int_0^r \frac{\rho(t)}{t} \, dt \leq A'' \frac{\Psi^{-1} \left( \frac{1}{r^n} \right)}{\Phi^{-1} \left( \frac{1}{r^n} \right)} = A'' \frac{\Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0\|f\|_\Phi} \right)}{\frac{Mf(x)}{C_0\|f\|_\Phi}}. \tag{4.12} \]
By (4.10), (4.11) and (4.12) we have

\[ |I_{\rho}f(x)| \leq C_{1}||f||_{\Phi} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_{0}||f||_{\Phi}} \right). \]

Therefore we have (3.3).

Let \( C_{0} \) be as in (2.8). Then

\[
\sup_{r>0} \Psi(r) m \left( r, \frac{|I_{\rho}f(x)|}{C_{1}||f||_{\Phi}} \right) = \sup_{r>0} r m \left( r, \Psi \left( \frac{|I_{\rho}f(x)|}{C_{1}||f||_{\Phi}} \right) \right) 
\leq \sup_{r>0} r \Phi \left( \frac{Mf(x)}{C_{0}||f||_{\Phi}} \right) = \sup_{r>0} \Phi(r) m \left( r, \frac{Mf(x)}{C_{0}||f||_{\Phi}} \right) \leq 1,
\]

i.e.

\[ ||I_{\rho}f||_{\Psi,\text{weak}} \leq C_{1}||f||_{\Phi}. \]

Let \( C_{0} \) be as in (2.9). Then

\[
\int_{\mathbb{R}^{n}} \Psi \left( \frac{|I_{\rho}f(x)|}{C_{1}||f||_{\Phi}} \right) dx \leq \int_{\mathbb{R}^{n}} \Phi \left( \frac{Mf(x)}{C_{0}||f||_{\Phi}} \right) dx \leq 1,
\]

i.e.

\[ ||I_{\rho}f||_{\Psi} \leq C_{1}||f||_{\Phi}. \]

Proof of Corollary 3.2. Let \( F(r) \) be as (4.6). By the almost decreasingness of \( F(r) \) we have

\[ F(t) \leq CF(r) \quad \text{for } 0 < r \leq t < +\infty. \]

By (3.4) we have

\[ \frac{1}{t^{n}} \geq \frac{\rho(t)}{C' \int_{0}^{t} (\rho(s)/s) ds t^{n}}. \]
From (4.1) and (4.2) it follows that

$$\tilde{\Phi} \left( \frac{\rho(t)}{CC'F(r)t^n} \right) \leq \frac{F(t)}{CF(r)} \tilde{\Phi} \left( \frac{\rho(t)}{C'F(t)t^n} \right)$$

$$= \frac{F(t)}{CF(r)} \tilde{\Phi} \left( \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds \Phi^{-1}(1/t^n) t^n} \right)$$

$$\leq \frac{F(t)}{CF(r)} \Phi^{-1} \left( \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds t^n} \right)$$

$$\leq \frac{F(t)}{CF(r)} \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds t^n} = \frac{1}{CC'F(r)} \frac{\rho(t)}{t^n} \Phi^{-1} \left( \frac{1}{t^n} \right).$$

By (3.5) we have (3.1). Therefore this corollary follows from Theorem 3.1. $\square$

**Lemma 4.1.** Let $\Phi$ be a Young function with (2.7) and $\tilde{\Phi}$ be the complementary function with respect to $\Phi$. Then there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,

$$\|\chi_{B(a,r)}\|_{\tilde{\Phi}} \leq C \Phi^{-1} \left( \frac{1}{r^n} \right) r^n.$$

**Proof.** Let $\lambda = \Phi^{-1}(1/|B(a, r)|)|B(a, r)|$. Then we have by (4.2)

$$\int_{\mathbb{R}^n} \tilde{\Phi} \left( \frac{\chi_{B(a,r)}(x)}{\lambda} \right) dx = \int_{B(a,r)} \tilde{\Phi} \left( \frac{1}{\lambda} \right) dx$$

$$= |B(a, r)| \tilde{\Phi} \left( \frac{1}{\Phi^{-1} \left( \frac{1}{|B(a,r)|} \right)} \right) \leq 1. \quad \square$$

**Proof of Theorem 3.3.** First we note that there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,

(4.13) $$\left\| \frac{\rho(|a - \cdot|)}{|a - \cdot|^{n+1}} \chi_{B(a,r)^C} (\cdot) \right\|_{\tilde{\Phi}} \leq \frac{1}{r} \int_0^r \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^n} \right).$$

We have this inequality (4.13) by (3.7) in a way similar to the proof of (4.7),
For any ball $B = B(a, r)$, let $\tilde{B} = B(a, 2r)$ and

\[
E_B(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} \right) dy,
\]

\[
C_B = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,
\]

\[
E_B^1(x) = \int_{\tilde{B}} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy.
\]

Then

\[
\tilde{I}_\rho f(x) - C_B = E_B(x) = E_B^1(x) + E_B^2(x) \quad \text{for } x \in B.
\]

By (2.6) we have

\[
\left| \frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right| \leq \begin{cases} C, & |y| \leq 2|a|, \\
C|a| \frac{\rho(|y|)}{|y|^{n+1}}, & |y| \geq 2|a|. \end{cases}
\]

From (4.3) and (4.13) it follows that $C_B$ is well defined. By (4.3), Lemma 4.1 and (3.8) we have

\[
\int_{B} \left( \int_{B} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy \leq \int_{B} |f(y)| \left( \int_{B(y, 3r)} \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy \\
\leq \int_{B} |f(y)| dy \int_{0}^{3r} \frac{\rho(t)}{t} dt \leq C \|f\|_{\Phi} \|\chi_{\tilde{B}}\|_{B} \int_{0}^{r} \frac{\rho(t)}{t} dt \\
\leq C \|f\|_{\Phi} \Phi^{-1} \left( \frac{1}{r^n} \right) r^n \int_{0}^{r} \frac{\rho(t)}{t} dt \leq C\phi(r) r^n \|f\|_{\Phi}.
\]

From Fubini's theorem it follows that $E_B^1$ is well defined and that

\[
(4.14) \quad \int_{B} |E_B^1(x)| dx \leq C\phi(r) r^n \|f\|_{\Phi}.
\]

By (2.6) we have

\[
\left| \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right| \leq C \frac{|a-x| \rho(|a-y|)}{|a-y|^{n+1}}, \quad x \in B \text{ and } y \in \tilde{B}^C.
\]

From (4.3), (4.13) and (3.8) it follows that $E_B^2$ is well defined and

\[
(4.15) \quad |E_B^2(x)| \leq C\phi(r) \|f\|_{\Phi}.
\]
By (4.14) and (4.14) we have
\[ \frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - C_B| \, dx \leq C\phi(r)\|f\|_\Phi, \]
and
\[ \|\tilde{I}_\rho f\|_{\text{BMO}_\phi} \leq C\|f\|_\Phi. \]

Lemma 4.2. Under the assumption in Theorem 3.4, there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,
\[ \int_{B(a, r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy \leq C\frac{\psi(r)}{r} \|f\|_{L_{1,\Phi}}. \]

Proof. By (3.9) and (3.10), we have
\[ \int_{B(a, r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy = \sum_{j=1}^{+\infty} \int_{B(a, 2^{j}r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy \leq C \sum_{j=1}^{+\infty} \frac{\rho(2^{j}r)}{(2^{j}r)^{n+1}} \int_{B(a, 2^{j}r)} |f(y)| \, dy \leq C \sum_{j=1}^{+\infty} \frac{\rho(2^{j}r)\phi(2^{j}r)}{2^{j}r} \|f\|_{L_{1,\Phi}} \sim \int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^{2}} \|f\|_{L_{1,\Phi}} \leq C \int_{0}^{r} \frac{\rho(t)}{t} \frac{\phi(r)}{r} \|f\|_{L_{1,\Phi}}. \]

Proof of Theorem 3.4. For any ball $B = B(a, r)$, let $\bar{B} = B(a, 2r)$ and
\begin{align*}
E_B(x) &= \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1 - \chi_{\bar{B}}(y))}{|a-y|^n} \right) \, dy, \\
C_B &= \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a-y|)(1 - \chi_{\bar{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) \, dy, \\
E_B^1(x) &= \int_{\bar{B}} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy, \\
E_B^2(x) &= \int_{\bar{B}^C} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) \, dy.
\end{align*}

Then
\[ \tilde{I}_\rho f(x) - C_B = E_B(x) = E_B^1(x) + E_B^2(x) \quad \text{for} \ x \in B. \]
By (2.6) we have

\[
\left| \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right| \leq \begin{cases} 
C, & |a-y| \leq \max(2|a|, 2r) \\
C|a| \frac{\rho(|a-y|)}{|a-y|^{n+1}}, & |a-y| \geq \max(2|a|, 2r). 
\end{cases}
\]

From Lemma 4.2 it follows that $C_B$ is well defined. By (3.10) we have

\[
\int_{\tilde{B}} \left( \int_{B} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy 
\leq \int_{\tilde{B}} |f(y)| \left( \int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy 
\leq \int_{\tilde{B}} |f(y)| \int_{0}^{3r} \frac{\rho(t)}{t} \, dt \leq C\|f\|_{L_{1,\psi}} r^n \phi(r) \int_{0}^{f} \frac{\rho(t)}{t} \, dt 
\leq C\|f\|_{L_{1,\phi}} r^n \psi(r).
\]

From Fubini's theorem it follows that $E_B^1$ is well defined and that

(4.16) \[ \int_{B} |E_B^1(x)| \, dx \leq C\psi(r) r^n \|f\|_{L_{1,\phi}}. \]

From (2.6) and Lemma 4.2 it follows that $E_B^2$ is well defined and

(4.17) \[ |E_B^2(x)| \leq C\psi(r) \|f\|_{L_{1,\phi}}. \]

By (4.16) and (4.17) we have

\[
\frac{1}{|B|} \int_{B} \left| \tilde{I}_\rho f(x) - C_B \right| \, dx \leq C\psi(r) \|f\|_{L_{1,\phi}},
\]

and

\[
\|\tilde{I}_\rho f\|_{L_{1,\psi}} \leq C\|f\|_{L_{1,\phi}}. \quad \square
\]

**Lemma 4.3.** If $\rho$ satisfies (2.1), (2.2), (2.5) and (2.6), then

(4.18) \[ \frac{\rho(|x_1-y|)}{|x_1-y|^n} - \frac{\rho(|x_2-y|)}{|x_2-y|^n} \]

is integrable on $\mathbb{R}^n$ as a function of $y$ and the value is equal to 0 for every choice of $x_1$ and $x_2$. 


Proof. Let \( r = |x_1 - x_2| \). For large \( R > 0 \), let

\[
J_1 = \int_{B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} dy - \int_{B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} dy,
\]

\[
J_2 = \int_{B(x_1, R+r) \setminus B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} dy - \int_{B(x_1, R+r) \setminus B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} dy,
\]

\[
J_3 = \int_{B(x_1, R+r)^c} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.
\]

Then

\[
J_1 + J_2 + J_3 = \int_{\mathbb{R}^n} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.
\]

From (2.1) it follows that \( \frac{\rho(|x_i - y|)}{|x_i - y|^n} \) \((i = 1, 2)\) are in \( L_{1\text{loc}}(\mathbb{R}^n) \) and that \( J_1 = 0 \).

By (2.6) we have

\[
\int_{B(x_1, R+r)^c} \left| \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right| dy.
\]

\[
\leq \int_{B(x_1, R+r)^c} A_3 r \frac{\rho(|x_1 - y|)}{|x_1 - y|^{n+1}} dy = C r \int_{R+r}^{+\infty} \frac{\rho(t)}{t^2} dt.
\]

From (2.5) it follows that (4.18) is integrable and that \( |J_3| \to 0 \) as \( R \to +\infty \).

By (2.2) and (2.5) we have

\[
|J_2| \leq \int_{B(x_1, R+r) \setminus B(x_1, R-r)} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} + \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy
\]

\[
\sim ((R + r)^n - (R - r)^n) \frac{\rho(R)}{R^n} \leq C r \frac{\rho(R)}{R} \to 0 \text{ as } R \to +\infty. \]

Lemma 4.4. Under the assumption in Theorem 3.6, there exists a constant \( C > 0 \) such that, for all \( a \in \mathbb{R}^n \) and \( r > 0 \),

\[
\int_{B(a, r)^c} \frac{\rho(|a - y|)}{|a - y|^{n+1}} |f(y) - f_{B(a, r)}| dy \leq C \frac{\psi(r)}{r} \|f\|_{\mathcal{L}_{1, \phi}}.
\]

Proof. By the doubling condition of \( \phi \) we have

\[
|f_{B(a, 2^k r)} - f_{B(a, 2^{k+1} r)}| \leq \frac{1}{|B(a, 2^{k} r)|} \int_{B(a, 2^k r)} |f(y) - f_{B(a, 2^{k+1} r)}| dy
\]

\[
\leq \frac{1}{|B(a, 2^{k} r)|} \int_{B(a, 2^{k+1} r)} |f(y) - f_{B(a, 2^{k+1} r)}| dy
\]

\[
\leq 2^n \phi(2^{k+1} r) \|f\|_{\mathcal{L}_{1, \phi}} \leq C \int_{2^k r}^{2^{k+1} r} \frac{\phi(s)}{s} ds \|f\|_{\mathcal{L}_{1, \phi}}.
\]
for $k=0,1,\ldots,j-1$, and so
\[
\frac{1}{|B(a,2^{j}r)|} \int_{B(a,2^{j}r)} |f(y) - f_{B(a,r)}| \, dy
\leq \frac{1}{|B(a,2^{j}r)|} \int_{B(a,2^{j}r)} |f(y) - f_{B(a,2^{j}r})| \, dy + |f_{B(a,r)} - f_{B(a,2^{j}r})|
\leq C \int_{r}^{2^{j}r} \frac{\phi(s)}{s} \, ds \|f\|_{\mathcal{L}_{1,\phi}}.
\]

Hence, using (2.5) and (3.15), we have
\[
\int_{B(a,r)^{C}} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
= \sum_{j=1}^{+\infty} \int_{2^{j-1}r \leq |a-y| \leq 2^{j}r} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
\leq C \sum_{j=1}^{+\infty} \frac{\rho(2^{j}r)}{(2^{j}r)^{n+1}} \int_{B(a,2^{j}r)} |f(y) - f_{B(a,r)}| \, dy
\leq C \sum_{j=1}^{+\infty} \frac{\rho(2^{j}r)}{2^{j}r} \int_{r}^{2^{j}r} \frac{\phi(s)}{s} \, ds \|f\|_{\mathcal{L}_{1,\phi}}
\leq C \int_{r}^{2^{j}r} \frac{\phi(s)}{s} \, ds \|f\|_{\mathcal{L}_{1,\phi}}
= C \int_{r}^{\psi(r)} \frac{\phi(s)}{s} \, ds \|f\|_{\mathcal{L}_{1,\phi}}
\leq C \int_{0}^{\psi(r)} \frac{\rho(t)}{t^{2}} \phi(t) \, dt \|f\|_{\mathcal{L}_{1,\phi}}
\leq C \int_{0}^{\psi(r)} \frac{\rho(t)}{t} dt \frac{\phi(r)}{r} \|f\|_{\mathcal{L}_{1,\phi}}
\leq C \frac{\psi(r)}{r} \|f\|_{\mathcal{L}_{1,\phi}}.
\]

**Proof of Theorem 3.6.** For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

\[
E_{B}(x) = \int_{\mathbb{R}^{n}} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^{n}} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^{n}} \right) \, dy,
\]
\[
C_{B}^{1} = \int_{\mathbb{R}^{n}} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^{n}} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^{n}} \right) \, dy,
\]
\[
C_{B}^{2} = \int_{\mathbb{R}^{n}} f_{\tilde{B}} \left( \frac{\rho(|x-y|)}{|x-y|^{n}} - \frac{\rho(|a-y|)(1-\chi_{B_0}(y))}{|a-y|^{n}} \right) \, dy,
\]
\[
E_{B}^{1}(x) = \int_{\tilde{B}} (f(y) - f_{\tilde{B}}) \frac{\rho(|x-y|)}{|x-y|^{n}} \, dy,
\]
\[
E_{B}^{2}(x) = \int_{\tilde{B}^{C}} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^{n}} - \frac{\rho(|a-y|)}{|a-y|^{n}} \right) \, dy.
\]
\[ \tilde{I}_\rho f(x) - (C_B^1 + C_B^2) = E_B(x) = E_B^1(x) + E_B^2(x) \quad \text{for } x \in B. \]

By (2.6) we have
\[
\frac{\rho(|a-y|)(1 - \chi_B(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n}
\leq \begin{cases}
C, & |a-y| \leq \max(2|a|, 2r) \\
C|a|\frac{\rho(|a-y|)}{|a-y|^{n+1}}, & |a-y| \geq \max(2|a|, 2r).
\end{cases}
\]

From Lemma 4.4 it follows that $C_B^1$ is well defined. By Lemma 4.3 and (2.1) we have
\[
\int_{\mathbb{R}^n} \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy
= \int_{\mathbb{R}^n} \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)}{|y|^n} \right) dy + \int_{B_0} \frac{\rho(|y|)}{|y|^n} dy = C.
\]

By (3.16) we have
\[
\int_B \left( \int_B |f(y) - f_B| \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy
\leq \int_B |f(y) - f_B| \left( \int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy
\leq \int_B |f(y) - f_B| dy \int_0^{3r} \frac{\rho(t)}{t} dt \leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \phi(r) \int_0^r \frac{\rho(t)}{t} dt
\leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \psi(r).
\]

From Fubini's theorem it follows that $E_B^1$ is well defined and that
\[
(4.19) \int_B |E_B^1(x)| dx \leq C\psi(r)r^n\|f\|_{\mathcal{L}_{1,\phi}}.
\]

From (2.6), Lemma 4.4 and (3.16) it follows that $E_B^2$ is well defined and
\[
(4.20) |E_B^2(x)| \leq C\psi(r)\|f\|_{\mathcal{L}_{1,\phi}}.
\]

By (4.19) and (4.20) we have
\[
\frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - (C_B^1 + C_B^2)| dx \leq C\psi(r)\|f\|_{\mathcal{L}_{1,\phi}},
\]
and
\[\|\tilde{I}_\rho f\|_{\mathcal{L}_{1,\phi}} \leq C\|f\|_{\mathcal{L}_{1,\phi}}. \square\]
REFERENCES


Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

E-mail address: enakai@cc.osaka-kyoiku.ac.jp