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Kyoto University
Generalized fractional integrals

1. INTRODUCTION

The fractional integral $I_{\alpha} (0 < \alpha < n)$ is defined by

$$I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$  

It is known that $I_{\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = n/q > 0$ as the Hardy-Littlewood-Sobolev theorem. The fractional integral was studied by many authors (see, for example, Rubin [10] or Chapter 5 in Stein [11]). The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory. We introduce generalized fractional integrals and extend the Hardy-Littlewood-Sobolev theorem to the Orlicz spaces. We show that, for example, a generalized fractional integral is bounded from $L^p$ to $\exp L^q$.

Let $B(a, r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ with center $a$ and of radius $r > 0$, and $B_0 = B(O, 1)$ with center the origin and of radius 1. The modified fractional integral $\tilde{I}_{\alpha} (0 < \alpha < n+1)$ is defined by

$$\tilde{I}_{\alpha}f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1 - \chi_{B_0}(y)}{|y|^{n-\alpha}} \right) dy,$$

where $\chi_{B_0}$ is the characteristic function of $B_0$. It is known that the modified fractional integral $\tilde{I}_{\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = 0$, from $L^p(\mathbb{R}^n)$ to $\text{Lip}_{\beta}(\mathbb{R}^n)$ when $p > 1$ and $-1 < n/p - \alpha = -\beta < 0$, from $\text{BMO}(\mathbb{R}^n)$ to $\text{Lip}_{\alpha}(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\text{Lip}_{\beta}(\mathbb{R}^n)$ to $\text{Lip}_{\gamma}(\mathbb{R}^n)$ when $0 < \alpha + \beta = \gamma < 1$.

We investigate the boundedness of generalized fractional integrals from the Orlicz space $L^\Phi(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$ and from $\text{BMO}_\phi(\mathbb{R}^n)$ to $\text{BMO}_\psi(\mathbb{R}^n)$. If $\phi(r) \equiv 1$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha (0 < \alpha \leq 1)$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{Lip}_{\alpha}(\mathbb{R}^n)$. We also investigate the boundedness of generalized fractional integrals on the Morrey and Campanato spaces.
2. Notations and Definitions

For a function \( \rho : (0, +\infty) \rightarrow (0, +\infty) \), let
\[
I_{\rho}f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy.
\]
We consider the following conditions on \( \rho \):

\[
\int_0^1 \frac{\rho(t)}{t} \, dt < +\infty,
\]

\[
\frac{1}{A_1} \leq \frac{\rho(s)}{\rho(r)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,
\]

\[
\frac{\rho(r)}{r^n} \leq A_2 \frac{\rho(s)}{s^n} \quad \text{for} \quad s \leq r,
\]
where \( A_1, A_2 > 0 \) are independent of \( r, s \).

If \( \rho(r) = r^\alpha \), \( 0 < \alpha < n \), then \( I_{\rho} \) is the fractional integral or the Riesz potential denoted by \( I_\alpha \).

We define the modified version of \( I_{\rho} \) as follows:
\[
\tilde{I}_{\rho}f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) \, dy.
\]
We consider the following conditions on \( \rho \): (2.1), (2.2) and

\[
\frac{\rho(r)}{r^{n+1}} \leq A_2' \frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad s \leq r,
\]

\[
\int_r^{+\infty} \frac{\rho(t)}{t^2} \, dt \leq A_2'' \frac{\rho(r)}{r},
\]

\[
\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \leq A_3 |r-s| \frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,
\]
where \( A_2', A_2'', A_3 > 0 \) are independent of \( r, s \).

A function \( \Phi : [0, +\infty) \rightarrow [0, +\infty] \) is called a Young function if \( \Phi \) is convex, \( \lim_{r \rightarrow 0} \Phi(r) = \Phi(0) = 0 \) and \( \lim_{r \rightarrow +\infty} \Phi(r) = +\infty \). Any Young function is increasing. For a Young function \( \Phi \), the complementary function is defined by
\[
\overline{\Phi}(r) = \sup\{rs - \Phi(s) : s \geq 0\}, \quad r \geq 0.
\]
For example, if \( \Phi(r) = r^p/p, 1 < p < \infty \), then \( \overline{\Phi}(r) = r^{p'}/p', 1/p + 1/p' = 1 \).
If \( \Phi(r) = r \), then \( \overline{\Phi}(r) = 0(0 \leq r \leq 1), = +\infty(r > 1) \).
For a Young function \( \Phi \), let

\[
L^{\Phi}(\mathbb{R}^{n}) = \left\{ f \in L_{1\text{o.c}}^{1}(\mathbb{R}^{n}) : \int_{\mathbb{R}^{n}} \Phi(\epsilon|f(x)|) \, dx < +\infty \text{ for some } \epsilon > 0 \right\},
\]

\[
\|f\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^{n}} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]

\[
L_{\text{weak}}^{\Phi}(\mathbb{R}^{n}) = \left\{ f \in L_{1\text{o.c}}^{1}(\mathbb{R}^{n}) : \sup_{r>0} \Phi(r) m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},
\]

\[
\|f\|_{\Phi,\text{weak}} = \inf \left\{ \lambda > 0 : \sup_{r>0} \Phi \left( \frac{r}{\lambda} \right) \leq 1 \right\},
\]

where \( m(r, f) = |\{x \in \mathbb{R}^{n} : |f(x)| > r\}| \).

Then

\[
L^{\Phi}(\mathbb{R}^{n}) \subset L_{\text{weak}}^{\Phi}(\mathbb{R}^{n}) \quad \text{and} \quad \|f\|_{\Phi,\text{weak}} \leq \|f\|_{\Phi}.
\]

If a Young function \( \Phi \) satisfies

\[
0 < \Phi(r) < +\infty \quad \text{for } 0 < r < +\infty,
\]

then \( \Phi \) is continuous and bijective from \([0, +\infty)\) to itself. The inverse function \( \Phi^{-1} \) is also increasing and continuous.

A function \( \Phi \) is said to satisfy the \( \nabla_{2} \)-condition, denoted \( \Phi \in \nabla_{2} \), if

\[
\Phi(r) \leq \frac{1}{2k} \Phi(kr), \quad r \geq 0,
\]

for some \( k > 1 \).

Let \( Mf(x) \) be the maximal function, i.e.

\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)| \, dy,
\]

where the supremum is taken over all balls \( B \) containing \( x \).

We assume that \( \Phi \) satisfies (2.7). Then \( M \) is bounded from \( L^{\Phi}(\mathbb{R}^{n}) \) to \( L_{\text{weak}}^{\Phi}(\mathbb{R}^{n}) \) and

\[
\|Mf\|_{\Phi,\text{weak}} \leq C_{0}\|f\|_{\Phi}.
\]

If \( \Phi \in \nabla_{2} \), then \( M \) is bounded on \( L^{\Phi}(\mathbb{R}^{n}) \) and

\[
\|Mf\|_{\Phi} \leq C_{0}\|f\|_{\Phi}.
\]

For functions \( \theta, \kappa : (0, +\infty) \to (0, +\infty) \), we denote \( \theta(r) \sim \kappa(r) \) if there exists a constant \( C > 0 \) such that

\[
C^{-1}\theta(r) \leq \kappa(r) \leq C\theta(r) \quad \text{for } r > 0.
\]
A function \( \theta : (0, +\infty) \to (0, +\infty) \) is said to be almost increasing (almost decreasing) if there exists a constant \( C > 0 \) such that
\[
\theta(r) \leq C \theta(s) \quad (\theta(r) \geq C \theta(s)) \quad \text{for} \quad r \leq s.
\]

A function \( \theta : (0, +\infty) \to (0, +\infty) \) is said to satisfy the doubling condition if there exists a constant \( C > 0 \) such that
\[
C^{-1} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.
\]

For \( 1 \leq p < \infty \) and a function \( \phi : (0, +\infty) \to (0, +\infty) \), let
\[
\|f\|_{L_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_{B} |f(x)|^{p} \, dx \right)^{1/p},
\]
\[
L_{p,\phi}(\mathbb{R}^{n}) = \{ f \in L_{1\mathrm{oc}}^{p}(\mathbb{R}^{n}) : \|f\|_{L_{p,\phi}} < +\infty \}.
\]

We assume that \( \phi \) satisfies the doubling condition and that \( \phi(r)r^{n/p} \) is almost increasing. If \( \phi(r) = r^{\lambda-n}/p \) \((0 \leq \lambda \leq n)\), then \( L_{p,\phi}(\mathbb{R}^{n}) = L^{p,\lambda}(\mathbb{R}^{n}) \) which is the classical Morrey space. If \( \lambda = 0 \), then \( L^{p,\lambda}(\mathbb{R}^{n}) = L^{p}(\mathbb{R}^{n}) \). If \( \lambda = n \), then \( L^{p,\lambda}(\mathbb{R}^{n}) = L^{\infty}(\mathbb{R}^{n}) \).

For \( 1 \leq p < \infty \) and a function \( \phi : (0, +\infty) \to (0, +\infty) \), let
\[
\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_{B} |f(x) - f_{B}|^{p} \, dx \right)^{1/p},
\]
\[
\mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = \{ f \in L_{1\mathrm{oc}}^{p}(\mathbb{R}^{n}) : \|f\|_{\mathcal{L}_{p,\phi}} < +\infty \},
\]

where \( f_{B} = \frac{1}{|B|} \int_{B} f(x) \, dx \).

We assume that \( \phi \) satisfies the doubling condition and that \( \phi(r)r^{n/p} \) is almost increasing. If \( \phi(r) = r^{\lambda-n}/p \) \((0 \leq \lambda \leq n+1)\), then \( \mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = \mathcal{L}^{p,\lambda}(\mathbb{R}^{n}) \) which is the classical Companato space.

If \( \phi \) is almost increasing, then \( \mathcal{L}_{p,\phi}(\mathbb{R}^{n}) = \mathcal{L}_{1,\phi}(\mathbb{R}^{n}) \) for all \( p > 1 \). Let \( \mathrm{BMO}_{\phi}(\mathbb{R}^{n}) = \mathcal{L}_{1,\phi}(\mathbb{R}^{n}) \). If \( \phi \equiv 1 \), then \( \mathrm{BMO}_{\phi}(\mathbb{R}^{n}) = \mathrm{BMO}(\mathbb{R}^{n}) \). If \( \phi(r) = r^{\alpha} \), \( 0 < \alpha \leq 1 \), then it is known that \( \mathrm{BMO}_{\phi}(\mathbb{R}^{n}) = \mathrm{Lip}_{\alpha}(\mathbb{R}^{n}) \).

The letter \( C \) shall always denote a constant, not necessarily the same one.

### 3. Main results

Our main results are as follows:
Theorem 3.1. Let $\rho$ satisfy (2.1)--(2.3). Let $\Phi$ and $\Psi$ be Young functions with (2.7). Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,

\begin{align}
(3.1) & \int_{r}^{+\infty} \frac{\rho(t)}{A \int_{0}^{r} (\rho(s)/s) ds \Phi^{-1}(1/r^{n}) t^{n}} t^{n-1} dt \leq A', \\
(3.2) & \int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^{n}} \right) \leq A'' \Psi^{-1} \left( \frac{1}{r^{n}} \right),
\end{align}

where $\Phi$ is the complementary function with respect to $\Phi$. Then, for any $C_{0} > 0$, there exists a constant $C_{1} > 0$ such that, for $f \in L^{\Phi}(\mathbb{R}^{n})$,

\begin{align}
(3.3) & \frac{\Psi}{C_{1} \|f\|_{\Phi}} \left( \frac{|I_{\rho} f(x)|}{C_{1} \|f\|_{\Phi}} \right) \leq \Phi \left( \frac{Mf(x)}{C_{0} \|f\|_{\Phi}} \right).
\end{align}

Therefore $I_{\rho}$ is bounded from $L^{\Phi}(\mathbb{R}^{n})$ to $L^{\Psi}_{weak}(\mathbb{R}^{n})$. Moreover, if $\Phi \in \nabla_{2}$, then $I_{\rho}$ is bounded from $L^{\Phi}(\mathbb{R}^{n})$ to $L^{\Psi}(\mathbb{R}^{n})$.

Remark 3.1. From (2.2) it follows that

\begin{align}
(3.4) & \rho(r) \leq C \int_{0}^{r} \frac{\rho(t)}{t} dt.
\end{align}

If $\rho(r)/r^{\varepsilon}$ is almost increasing for some $\varepsilon > 0$ and $\rho(t)/t^{n}$ is almost decreasing, then $\rho$ satisfies (2.1)--(2.3) and $\int_{0}^{r} (\rho(t)/t) dt \sim \rho(r)$. Let, for example, $\rho(r) = r^{\alpha} (\log(1/r))^{-\beta}$ for small $r$. If $\alpha = 0$ and $\beta > 1$, then $\int_{0}^{r} (\rho(t)/t) dt \sim (\log(1/r))^{-\beta+1}$. If $\alpha > 0$ and $-\infty < \beta < +\infty$, then $\int_{0}^{r} (\rho(t)/t) dt \sim \rho(r)$.

Remark 3.2. In the case $\Phi(r) = r$, (3.1) is equivalent to

\begin{align}
\frac{\rho(t)}{t^{n}} \leq A \int_{0}^{r} (\rho(s)/s) ds \frac{1}{r^{n}}, & \quad 0 < r \leq t.
\end{align}

This inequality follows from (2.3) and (3.4).

The following corollary is stated without the complementary function.

Corollary 3.2. Let $\rho$ satisfy (2.1)--(2.3). Let $\Phi$ and $\Psi$ be Young functions with (2.7). Assume that

\begin{align}
\int_{0}^{t} \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^{n}} \right).
\end{align}
is almost decreasing and that there exist constants $A, A' > 0$ such that, for all $r > 0$,

\[(3.5) \quad \int_{r}^{+\infty} \frac{\rho(t)}{t} \Phi^{-1} \left( \frac{1}{t^n} \right) dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^n} \right),\]

\[(3.6) \quad \int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^n} \right) \leq A' \Phi^{-1} \left( \frac{1}{r^n} \right).\]

Then (3.3) holds. Therefore $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\psi_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\psi(\mathbb{R}^n)$.

Remark 3.3. If $r^\epsilon \rho(r) \Phi^{-1}(1/r^n)$ is almost decreasing for some $\epsilon > 0$, then

\[(3.7) \quad \int_{r}^{+\infty} \frac{\rho(t)}{t} \Phi^{-1} \left( \frac{1}{t^n} \right) dt \leq C \rho(r) \Phi^{-1} \left( \frac{1}{r^n} \right).\]

This inequality and (3.4) yield (3.5).

Remark 3.4. We cannot replace (3.2) or (3.6) by

\[\rho(r) \Phi^{-1} \left( \frac{1}{r^n} \right) \leq A \Phi^{-1} \left( \frac{1}{r^n} \right) \quad \text{for all } r > 0\]

(see Section 5 in [6]).

O'Neil [7] showed the boundedness for convolution operators on the Orlicz spaces. Cianchi [1] gave a necessary and sufficient condition on $\Phi$ and $\Psi$ so that the fractional integral $I_\alpha$ is bounded from $L^\Phi$ to $L^\Psi$.

**Theorem 3.3.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\Phi$ be Young function with (2.7), $\phi$ satisfy the doubling condition and be almost increasing. Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,

\[(3.7) \quad \int_{r}^{+\infty} \tilde{\Phi} \left( \frac{r \rho(t)}{A \int_{0}^{r} (\rho(s)/s) ds \Phi^{-1}(1/r^n) t^{n+1}} \right) t^{n-1} dt \leq A',\]

\[(3.8) \quad \int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^n} \right) \leq A'' \phi(r),\]

where $\tilde{\Phi}$ is the complementary function with respect to $\Phi$. Then $\tilde{I}_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$.

**Theorem 3.4.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition, and $\phi(r)r^n$ and $\psi(r)r^n$ be almost increasing. Assume
that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{equation}
(3.9) \int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} \, dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} \, dt \frac{\phi(r)}{r},
\end{equation}
\begin{equation}
(3.10) \int_{0}^{r} \frac{\rho(t)}{t} \, dt \phi(r) \leq A' \psi(r).
\end{equation}

Then $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

If $\Phi \in \nabla_2$ and $\Phi^{-1}(1/r^n) = \phi(r)$, then we can show
\begin{equation}
(3.11) \int_{f}^{+\infty} \frac{\rho(t)\Phi^{-1}(1/t^n)}{t^2} \, dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} \, dt \frac{\Phi^{-1}(1/r^n)}{r},
\end{equation}
\begin{equation}
(3.12) \int_{0}^{f} \frac{\rho(t)}{t} \, dt \Phi^{-1}\left(\frac{1}{r^n}\right) \leq A' \phi(r).
\end{equation}

Then $\tilde{I}_\rho$ is bounded from $L^\Phi_{\text{weak}}(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$.

**Corollary 3.5.** Let $\rho$ satisfy (2.1), (2.2), (2.4) and (2.6). Let $\Phi$ be Young function with (2.7), $\Phi \in \nabla_2$, $\phi$ satisfy the doubling condition and be almost increasing. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{equation}
(3.13) \int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} \, dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} \, dt \frac{\phi(r)}{r},
\end{equation}
\begin{equation}
(3.14) \int_{0}^{r} \frac{\rho(t)}{t} \, dt \phi(r) \leq A' \psi(r).
\end{equation}

Then $\tilde{I}_\rho$ is bounded from $L^\Phi_{\text{weak}}(\mathbb{R}^n)$ to $\text{BMO}_\phi(\mathbb{R}^n)$.

**Theorem 3.6.** Let $\rho$ satisfy (2.1), (2.2), (2.5) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition, and $\phi(r)r^n$ and $\psi(r)r^n$ are almost increasing. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,
\begin{equation}
(3.15) \int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} \, dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} \, dt \frac{\phi(r)}{r},
\end{equation}
\begin{equation}
(3.16) \int_{0}^{r} \frac{\rho(t)}{t} \, dt \phi(r) \leq A' \psi(r).
\end{equation}

Then $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

**Remark 3.5.** From Lemma 4.3 it follows that $\tilde{I}_\rho$ is a constant. Hence $\tilde{I}_\rho$ is well defined as an operator from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$.

The boundedness of the fractional integral $I_\alpha$ on the Campanato space is known (Peetre [8]).

**Corollary 3.7.** Let $\rho$ satisfy (2.1), (2.2), (2.5) and (2.6). Let $\phi$ and $\psi$ satisfy the doubling condition and be almost increasing. Assume that there exist
constants \( A, A' > 0 \) such that, for all \( r > 0 \),

\[
\int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} \, dt \leq A \int_{0}^{r} \frac{\rho(t)}{t} \frac{\phi(r)}{r},
\]

(3.15)

\[
\int_{0}^{r} \frac{\rho(t)}{t} \, dt \frac{\phi(r)}{r} \leq A' \psi(r).
\]

(3.16)

Then \( \tilde{I}_\rho \) is bounded from \( \text{BMO}_\phi(\mathbb{R}^n) \) to \( \text{BMO}_\psi(\mathbb{R}^n) \).

This Corollary is a generalization of the well-known result that \( \tilde{I}_\alpha \) is bounded from \( \text{BMO}(\mathbb{R}^n) \) to \( \text{Lip}_\alpha(\mathbb{R}^n) \) when \( 0 < \alpha < 1 \), and from \( \text{Lip}_\beta(\mathbb{R}^n) \) to \( \text{Lip}_{\alpha+\beta}(\mathbb{R}^n) \) when \( \alpha > 0, \beta > 0 \) and \( 0 < \alpha + \beta < 1 \).

The results in Figure 1 are known. Our results contain these. Moreover, we have the results in Figure 2.

\[
\begin{array}{c|c|c|c|c|c|c}
I_p & L^q & \text{BMO} & \text{Lip}_\beta & \text{Lip}_\gamma \\
\hline
I_\alpha & -n/p + \alpha = -n/q & & \tilde{I}_\alpha & & \\
\hline
-\tilde{I}_\alpha & -n/p + \alpha = 0 & & & \tilde{I}_\alpha & \\
\hline
\end{array}
\]

\[\beta + \alpha = \gamma\]

\text{FIGURE 1. Boundedness of fractional integrals}

We can also state our results on spaces of homogeneous type with appropriate conditions.

4. PROOFS

Let \( \Phi \) be a Young function. By the convexity and \( \Phi(0) = 0 \), we have

\[
\Phi(r) \leq \frac{r}{s} \Phi(s) \quad \text{for} \quad r \leq s.
\]

(4.1)

Let \( \tilde{\Phi} \) be the complementary function with respect to \( \Phi \). Then

\[
\tilde{\Phi} \left( \frac{\Phi(r)}{r} \right) \leq \Phi(r), \quad r > 0.
\]

(4.2)
Actually,
\[ \frac{\Phi(r)}{r}s - \Phi(s) \leq \Phi(r) \quad \text{for } s < r \]
and
\[ \frac{\Phi(r)}{r}s - \Phi(s) \leq 0 \quad \text{for } s \geq r. \]

We note that
\[ (4.3) \quad \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq 2\|f\|_{\Phi} \|g\|_{\tilde{\Phi}} \]
(see for example [9]).

**Proof of Theorem 3.1.** Let
\[ J_1 = \int_{|x-y|<r} f(y) \rho(|x-y|) \frac{|x-y|^n}{|x-y|^n} \, dy \quad \text{and} \]
\[ J_2 = \int_{|x-y|\geq r} f(y) \rho(|x-y|) \frac{|x-y|^n}{|x-y|^n} \, dy. \]

Let
\[ h(r) = \inf \left\{ \frac{\rho(s)}{s^n} : s \leq r \right\}, \quad r > 0. \]

Then \( h \) is nonincreasing. It follows that
\[ \int_{|x-y|<r} |f(y)|h(|x-y|) \, dy \leq Mf(x) \int_{|x-y|<r} h(|x-y|) \, dy \]
(see Stein[12, p.57]). Since $h(r) \sim \rho(r)/r^n$,

\begin{equation}
|J_1| \leq CMf(x) \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n} dy \leq CMf(x) \int_0^r \frac{\rho(t)}{t} dt.
\end{equation}

Next we estimate $J_2$. By (4.3) we have

\begin{equation}
|J_2| \leq 2 \left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^c}(\cdot) \right\|_{\tilde{\Phi}} \|f\|_{\Phi},
\end{equation}

where $\chi_{B(x,r)^c}$ is the characteristic function of the complement of $B(x,r)$. Let

\begin{equation}
F(r) = \int_0^r \frac{\rho(s)}{s} ds \Phi^{-1} \left( \frac{1}{r^n} \right).
\end{equation}

We show

\begin{equation}
\left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^c}(\cdot) \right\|_{\tilde{\Phi}} \leq CF(r).
\end{equation}

From (2.2) and the increasingness of $\tilde{\Phi}$ it follows that

\begin{equation}
\int_{|x-y| \geq r} \tilde{\Phi} \left( \frac{\rho(|x-y|)}{\lambda|x-y|^n} \right) dy \leq C_2 \int_r^{+\infty} \tilde{\Phi} \left( \frac{\rho(t)}{\lambda t^n} \right) t^{n-1} dt,
\end{equation}

where $C_2$ is independent of $\lambda > 0$, $r > 0$ and $x \in \mathbb{R}^n$. We may assume that $C_2A' \geq 1$. By (4.1) and (3.1) we have

\begin{equation}
\int_r^{+\infty} \tilde{\Phi} \left( \frac{\rho(t)}{C_2AA'F(r)t^n} \right) t^{n-1} dt \leq \frac{1}{C_2A'} \int_r^{+\infty} \tilde{\Phi} \left( \frac{\rho(t)}{AF(r)t^n} \right) t^{n-1} dt \leq \frac{1}{C_2}.
\end{equation}

Let $\lambda = C_2AA'F(r)$. Then, by (4.8) and (4.9) we have

\begin{equation}
\int_{|x-y| \geq r} \tilde{\Phi} \left( \frac{\rho(|x-y|)}{\lambda|x-y|^n} \right) dy \leq 1,
\end{equation}

and so (4.7). By (4.4), (4.5) and (4.7) we have

\begin{equation}
|I_{\rho}f(x)| = |J_1 + J_2| \leq C \left( Mf(x) + \|f\|_{\Phi} \Phi^{-1} \left( \frac{1}{r^n} \right) \right) \int_0^r \frac{\rho(t)}{t} dt.
\end{equation}

Choose $r > 0$ so that

\begin{equation}
\Phi^{-1} \left( \frac{1}{r^n} \right) = \frac{Mf(x)}{C_0\|f\|_{\Phi}}.
\end{equation}

Then

\begin{equation}
\int_0^r \frac{\rho(t)}{t} dt \leq A'' \Psi^{-1} \left( \frac{1}{r^n} \right) = A'' \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0\|f\|_{\Phi}} \right).
\end{equation}
By (4.10), (4.11) and (4.12) we have

$$|I_\rho f(x)| \leq C_1 ||f||_\Phi \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 ||f||_\Phi} \right).$$

Therefore we have (3.3).

Let $C_0$ be as in (2.8). Then

$$\sup_{r > 0} \Psi(r) m \left( r, \frac{|I_\rho f(x)|}{C_1 ||f||_\Phi} \right) = \sup_{r > 0} r m \left( r, \Psi \left( \frac{|I_\rho f(x)|}{C_1 ||f||_\Phi} \right) \right)$$

$$\leq \sup_{r > 0} r m \left( r, \Phi \left( \frac{Mf(x)}{C_0 ||f||_\Phi} \right) \right) = \sup_{r > 0} \Phi(r) m \left( r, \frac{Mf(x)}{C_0 ||f||_\Phi} \right) \leq 1,$$

i.e.

$$||I_\rho f||_{\Psi,weak} \leq C_1 ||f||_\Phi.$$

Let $C_0$ be as in (2.9). Then

$$\int_{\mathbb{R}^n} \Psi \left( \frac{|I_\rho f(x)|}{C_1 ||f||_\Phi} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{C_0 ||f||_\Phi} \right) dx \leq 1,$$

i.e.

$$||I_\rho f||_{\Psi} \leq C_1 ||f||_\Phi. \quad \Box$$

**Proof of Corollary 3.2.** Let $F(r)$ be as (4.6). By the almost decreasingness of $F(r)$ we have

$$F(t) \leq CF(r) \quad \text{for } 0 < r \leq t < +\infty.$$ 

By (3.4) we have

$$\frac{1}{t^n} \geq \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds} t^n.$$
From (4.1) and (4.2) it follows that

\[
\tilde{\Phi} \left( \frac{\rho(t)}{CF(r)t^n} \right) \leq \frac{F(t)}{CF(r)} \tilde{\Phi} \left( \frac{\rho(t)}{C'F(t)t^n} \right)
\]

\[
= \frac{F(t)}{CF(r)} \tilde{\Phi} \left( \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds} \Phi^{-1}(1/t^n) t^n \right)
\]

\[
\leq \frac{F(t)}{CF(r)} \tilde{\Phi} \left( \frac{\rho(t)}{C'} \int_0^t (\rho(s)/s) ds t^n \right)
\]

\[
\leq \frac{F(t)}{CF(r)} \frac{\rho(t)}{C' \int_0^t (\rho(s)/s) ds t^n} = \frac{1}{CC'F(r)} \frac{\rho(t)}{t^n} \Phi^{-1} \left( \frac{1}{t^n} \right)
\]

By (3.5) we have (3.1). Therefore this corollary follows from Theorem 3.1. \(\square\)

**Lemma 4.1.** Let \(\Phi\) be a Young function with (2.7) and \(\tilde{\Phi}\) be the complementary function with respect to \(\Phi\). Then there exists a constant \(C > 0\) such that, for all \(a \in \mathbb{R}^n\) and \(r > 0\),

\[
\| \chi_{B(a,r)} \|_{\tilde{\Phi}} \leq C \Phi^{-1} \left( \frac{1}{r^n} \right) r^n.
\]

**Proof.** Let \(\lambda = \Phi^{-1} \left( 1/|B(a,r)| \right) |B(a,r)|\). Then we have by (4.2)

\[
\int_{\mathbb{R}^n} \tilde{\Phi} \left( \frac{\chi_{B(a,r)}(x)}{\lambda} \right) dx = \int_{B(a,r)} \tilde{\Phi} \left( \frac{1}{\lambda} \right) dx
\]

\[
= |B(a,r)| \tilde{\Phi} \left( \frac{1}{\Phi^{-1} \left( \frac{1}{|B(a,r)|} \right)} \right) \leq 1. \quad \square
\]

**Proof of Theorem 3.3.** First we note that there exists a constant \(C > 0\) such that, for all \(a \in \mathbb{R}^n\) and \(r > 0\),

\[
(4.13) \quad \left\| \frac{\rho(|a - \cdot|)}{|a - \cdot|^{n+1}} \chi_{B(a,r)c}(\cdot) \right\|_{\tilde{\Phi}} \leq C \frac{1}{r} \int_0^r \frac{\rho(t)}{t} dt \Phi^{-1} \left( \frac{1}{r^n} \right).
\]

We have this inequality (4.13) by (3.7) in a way similar to the proof of (4.7),
For any ball \( B = B(a, r) \), let \( \tilde{B} = B(a, 2r) \) and 

\[
E_B(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} \right) dy,
\]

\[
C_B = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,
\]

\[
E_B^1(x) = \int_{\tilde{B}} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy,
\]

\[
E_B^2(x) = \int_{\tilde{B}^C} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy.
\]

Then 

\[
\int_{\tilde{B}} |\rho(|x-y|)(1 - \chi_{\tilde{B}}(y))| \leq \begin{cases} C, & |y| \leq 2|a|, \\ C|a| \frac{\rho(|y|)}{|y|^{n+1}}, & |y| \geq 2|a|. \end{cases}
\]

By (2.6) we have 

\[
|\frac{\rho(|a-y|)(1 - \chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n}| \leq \begin{cases} C, & |y| \leq 2|a|, \\ C|a| \frac{\rho(|y|)}{|y|^{n+1}}, & |y| \geq 2|a|. \end{cases}
\]

From (4.3) and (4.13) it follows that \( C_B \) is well defined. By (4.3), Lemma 4.1 and (3.8) we have 

\[
\int_{\tilde{B}} \left( \int_{B} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy \leq \int_{\tilde{B}} |f(y)| \left( \int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy 
\]

\[
\leq \int_{\tilde{B}} |f(y)| dy \int_{0}^{3r} \frac{\rho(t)}{t} dt 
\]

\[
\leq C \|f\|_\Phi \|\chi_{\tilde{B}}\| \Phi \int_{0}^{3r} \frac{\rho(t)}{t} dt 
\]

\[
\leq C \|f\|_\Phi \Phi^{-1} \left( \frac{1}{r^n} \right) r^n \int_{0}^{r} \frac{\rho(t)}{t} dt 
\]

\[
\leq C \phi(r) r^n \|f\|_\Phi.
\]

From Fubini’s theorem it follows that \( E_B^1 \) is well defined and that 

\[
(4.14) \quad \int_{B} |E_B^1(x)| dx \leq C \phi(r) r^n \|f\|_\Phi.
\]

By (2.6) we have 

\[
|\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}| \leq C \frac{|a-x| \rho(|a-y|)}{|a-y|^{n+1}}, \quad x \in B \text{ and } y \in \tilde{B}^C.
\]

From (4.3), (4.13) and (3.8) it follows that \( E_B^2 \) is well defined and 

\[
(4.15) \quad |E_B^2(x)| \leq C \phi(r) \|f\|_\Phi.
\]
By (4.14) and (4.14) we have
\[
\frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - C_B| \, dx \leq C\phi(r)\|f\|_{\Phi},
\]
and
\[
\|\tilde{I}_\rho f\|_{\text{BMO}_\phi} \leq C\|f\|_{\Phi}. \quad \square
\]

**Lemma 4.2.** Under the assumption in Theorem 3.4, there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,
\[
\int_{B(a,r)^{\mathcal{C}}} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy \leq C \frac{\psi(r)}{r} \|f\|_{L_{1,\Phi}}.
\]

**Proof.** By (3.9) and (3.10), we have
\[
\int_{B(a,r)^{\mathcal{C}}} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy = \sum_{j=1}^{+\infty} \int_{J^{-1}} 2r \leq |a-y| \leq 2^j r \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y)| \, dy
\]
\[
\leq C \sum_{j=1}^{+\infty} \frac{\rho(2^j r)}{(2^j r)^{n+1}} \int_{B(a,2^j r)} |f(y)| \, dy \leq C \sum_{j=1}^{+\infty} \frac{\rho(2^j r)\phi(2^j r)}{2^j r} \|f\|_{L_{1,\Phi}}
\]
\[
\sim \int_{r}^{+\infty} \frac{\rho(t)\phi(t)}{t^2} \|f\|_{L_{1,\Phi}} \leq C \int_{0}^{r} \frac{\rho(t)}{t} dt \frac{\phi(r)}{r} \|f\|_{L_{1,\Phi}}
\]
\[
\leq C \frac{\psi(r)}{r} \|f\|_{L_{1,\Phi}}. \quad \square
\]

**Proof of Theorem 3.4.** For any ball $B = B(a, r)$, let $\tilde{B} = B(a, 2r)$ and
\[
E_B(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^{n}} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^{n}} \right) \, dy,
\]
\[
C_B = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^{n}} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^{n}} \right) \, dy,
\]
\[
E_B^1(x) = \int_{\tilde{B}} f(y) \frac{\rho(|x-y|)}{|x-y|^{n}} \, dy,
\]
\[
E_B^2(x) = \int_{\tilde{B}^{\mathcal{C}}} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^{n}} - \frac{\rho(|a-y|)}{|a-y|^{n}} \right) \, dy.
\]

Then
\[
\tilde{I}_\rho f(x) - C_B = E_B(x) = E_B^1(x) + E_B^2(x) \quad \text{for } x \in B.
\]
By (2.6) we have
\[
\left| \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right| \leq \begin{cases} 
C, & |a-y| \leq \max(2|a|, 2r) \\
C|a| \frac{\rho(|y|)}{|a-y|^n+1}, & |a-y| \geq \max(2|a|, 2r). 
\end{cases}
\]

From Lemma 4.2 it follows that $C_B$ is well defined. By (3.10) we have
\[
\int_{\tilde{B}} \left( \int_{B} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy 
\leq \int_{\tilde{B}} |f(y)| \left( \int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy 
\leq \int_{\tilde{B}} |f(y)| \int_{0}^{3r} \frac{\rho(t)}{t} \, dt \leq C \|f\|_{L_{1,\phi}} r^n \int_{0}^{\psi(r)} \frac{\rho(t)}{t} \, dt 
\leq C \|f\|_{L_{1,\phi}} r^n \psi(r).
\]

From Fubini's theorem it follows that $E_B^1$ is well defined and that
\[
(4.16) \quad \int_{B} |E_B^1(x)| \, dx \leq C \psi(r) r^n \|f\|_{L_{1,\phi}}.
\]

From (2.6) and Lemma 4.2 it follows that $E_B^2$ is well defined and
\[
(4.17) \quad |E_B^2(x)| \leq C \psi(r) \|f\|_{L_{1,\phi}}.
\]

By (4.16) and (4.17) we have
\[
\frac{1}{|B|} \int_{B} |\tilde{I}_\rho f(x) - C_B| \, dx \leq C \psi(r) \|f\|_{L_{1,\phi}},
\]
and
\[
\|\tilde{I}_\rho f\|_{L_{1,\phi}} \leq C \|f\|_{L_{1,\phi}}. \quad \square
\]

**Lemma 4.3.** If $\rho$ satisfies (2.1), (2.2), (2.5) and (2.6), then
\[
(4.18) \quad \frac{\rho(|x_1-y|)}{|x_1-y|^n} - \frac{\rho(|x_2-y|)}{|x_2-y|^n}
\]
is integrable on $\mathbb{R}^n$ as a function of $y$ and the value is equal to 0 for every choice of $x_1$ and $x_2$. 
Proof. Let $r = |x_1 - x_2|$. For large $R > 0$, let

\[ J_1 = \int_{B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} \, dy - \int_{B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \, dy, \]

\[ J_2 = \int_{B(x_1, R+r) \setminus B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} \, dy - \int_{B(x_1, R+r) \setminus B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \, dy, \]

\[ J_3 = \int_{B(x_1, R+r)^C} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) \, dy. \]

Then

\[ J_1 + J_2 + J_3 = \int_{\mathbb{R}^n} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) \, dy. \]

From (2.1) it follows that \( \frac{\rho(|x_i - y|)}{|x_i - y|^n} \) (\( i = 1, 2 \)) are in \( L_{1\text{oc}}^1(\mathbb{R}^n) \) and that \( J_1 = 0 \). By (2.6) we have

\[ \int_{B(x_1, R+r)^C} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \, dy \leq \int_{B(x_1, R+r)^C} A_3r \frac{\rho(|x_1 - y|)}{|x_1 - y|^{n+1}} \, dy = Cr \int_{R+r}^{\infty} \frac{\rho(t)}{t^2} \, dt. \]

From (2.5) it follows that (4.18) is integrable and that \( |J_3| \to 0 \) as \( R \to +\infty \). By (2.2) and (2.5) we have

\[ |J_2| \leq \int_{B(x_1, R+r) \setminus B(x_1, R-r)} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} + \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) \, dy \]

\[ \sim ((R + r)^n - (R - r)^n) \frac{\rho(R)}{R^n} \leq C \frac{\rho(R)}{R} \to 0 \quad \text{as} \quad R \to +\infty. \]

\[ \square \]

Lemma 4.4. Under the assumption in Theorem 3.6, there exists a constant \( C > 0 \) such that, for all \( a \in \mathbb{R}^n \) and \( r > 0 \),

\[ \int_{B(a, r)^C} \frac{\rho(|a - y|)}{|a - y|^{n+1}} \left| f(y) - f_{B(a, r)} \right| \, dy \leq C \frac{\psi(r)}{r} \| f \|_{\mathcal{L}_{1, \phi}}. \]

Proof. By the doubling condition of \( \phi \) we have

\[ |f_{B(a, 2^k r)} - f_{B(a, 2^{k+1} r)}| \leq \frac{1}{|B(a, 2^k r)|} \int_{B(a, 2^k r)} \left| f(y) - f_{B(a, 2^{k+1} r)} \right| \, dy \]

\[ \leq \frac{1}{|B(a, 2^k r)|} \int_{B(a, 2^{k+1} r)} \left| f(y) - f_{B(a, 2^{k+1} r)} \right| \, dy \]

\[ \leq 2^n \phi(2^{k+1} r) \| f \|_{\mathcal{L}_{1, \phi}} \leq C \int_{2^k r}^{2^{k+1} r} \frac{\phi(s)}{s} \, ds \| f \|_{\mathcal{L}_{1, \phi}}, \]
for \( k = 0, 1, \cdots, j - 1 \), and so
\[
\frac{1}{|B(a, 2^j r)|} \int_{B(a,2^j r)} |f(y) - f_{B(a,r)}| \, dy \leq \frac{1}{|B(a, 2^j r)|} \int_{B(a,2^j r)} |f(y) - f_{B(a,2^j r)}| \, dy + |f_{B(a,r)} - f_{B(a,2^j r)}| \leq C \int_{r}^{2^j r} \frac{\phi(s)}{s} \, ds \|f\|_{L_1, \phi}.
\]
Hence, using (2.5) and (3.15), we have
\[
\int_{B(a,r)^c} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy = \sum_{j=1}^{\infty} \int_{2^{j-1} r \leq |a-y| \leq 2^{j} r} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy \leq C \sum_{j=1}^{\infty} \frac{\rho(2^j r)}{(2^j r)^{n+1}} \int_{B(a,2^j r)} |f(y) - f_{B(a,r)}| \, dy \leq C \sum_{j=1}^{\infty} \frac{\rho(2^j r)}{2^j r} \int_{r}^{2^j r} \frac{\phi(s)}{s} \, ds \|f\|_{L_1, \phi} \leq \int_{r}^{\infty} \frac{\rho(t)}{t^2} \left( \int_{r}^{t} \frac{\phi(s)}{s} \, ds \right) dt \|f\|_{L_1, \phi} \leq C \int_{0}^{r} \frac{\rho(t)}{t} \frac{\phi(r)}{r} \, dt \|f\|_{L_1, \phi} \leq C \frac{\psi(r)}{r} \|f\|_{L_1, \phi}.
\]

**Proof of Theorem 3.6.** For any ball \( B = B(a, r) \), let \( \tilde{B} = B(a, 2r) \) and
\[
E_B(x) = \int_{R^n} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} (1 - \chi_{\tilde{B}}(y)) \right) \, dy,
\]
\[
C_B^1 = \int_{R^n} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|a-y|)}{|a-y|^n} (1 - \chi_{\tilde{B}}(y)) - \frac{\rho(|y|)}{|y|^n} (1 - \chi_{B_0}(y)) \right) \, dy,
\]
\[
C_B^2 = \int_{R^n} f_{\tilde{B}} \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)}{|y|^n} (1 - \chi_{B_0}(y)) \right) \, dy,
\]
\[
E_B^1(x) = \int_{\tilde{B}} (f(y) - f_{\tilde{B}}) \frac{\rho(|x-y|)}{|x-y|^n} \, dy,
\]
\[
E_B^2(x) = \int_{\tilde{B}^{c}} (f(y) - f_{\tilde{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) \, dy.
\]
\[ \tilde{I}_\rho f(x) - (C_B^1 + C_B^2) = E_B(x) = E_B^1(x) + E_B^2(x) \quad \text{for } x \in B. \]

By (2.6) we have
\[
\frac{\rho(|a - y|)(1 - \chi_B(y))}{|a - y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \leq \begin{cases} 
C, & |a - y| \leq \max(2|a|, 2r) \\
C|a| \frac{\rho(|a - y|)}{|a - y|^{n+1}}, & |a - y| \geq \max(2|a|, 2r).
\end{cases}
\]

From Lemma 4.4 it follows that $C_B^1$ is well defined. By Lemma 4.3 and (2.1) we have
\[
\int_{\mathbb{R}^n} \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} dy + \int_{B_0} \frac{\rho(|y|)}{|y|^n} dy = C.
\]

By (3.16) we have
\[
\int_B \left( \int_B |f(y) - f_B| \frac{\rho(|x - y|)}{|x - y|^n} dx \right) dy 
\leq \int_B |f(y) - f_B| \left( \int_{B_{y,3r}} \frac{\rho(|x - y|)}{|x - y|^n} dx \right) dy
\leq \int_B |f(y) - f_B| dy \int_0^{3r} \frac{\rho(t)}{t} dt \leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \phi(r) \int_0^r \frac{\rho(t)}{t} dt
\leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \psi(r).
\]

From Fubini’s theorem it follows that $E_B^1$ is well defined and that
\[ (4.19) \quad \int_B |E_B^1(x)| dx \leq C\psi(r) r^n \|f\|_{\mathcal{L}_{1,\phi}}. \]

From (2.6), Lemma 4.4 and (3.16) it follows that $E_B^2$ is well defined and
\[ (4.20) \quad |E_B^2(x)| \leq C\psi(r) \|f\|_{\mathcal{L}_{1,\phi}}. \]

By (4.19) and (4.20) we have
\[
\frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - (C_B^1 + C_B^2)| dx \leq C\psi(r) \|f\|_{\mathcal{L}_{1,\phi}},
\]
and
\[ \|\tilde{I}_\rho f\|_{\mathcal{L}_{1,\phi}} \leq C \|f\|_{\mathcal{L}_{1,\phi}}. \quad \square \]
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