数理解析研究所講究録 1202 巻 2001 年 1-6

Normal ultrafilters without the partition property

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1 Introduction

Let κ be a measurable cardinal and $\kappa \leq \lambda$. Concerning the partition property of a normal ultrafilter on $\mathcal{P}_\kappa\lambda$, Solovay (see Menas [6]) proved the existence of a normal ultrafilter without the partition property under the assumption of that the existence of a certain large cardinal greater than κ. After Solovay established this result, Kunen (see Kunen-Pelletier [3]) improved his results, and proved that the existence of a normal ultrafilter without the partition property implies the existence of a certain large cardinal above κ. On the other hand, Menas [6] proved that there exist $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters with the partition property, if κ is $2^{\lambda^{<\kappa}}$ supercompact. In the talk, we prove

Theorem 1 *Suppose that U is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ without the partition property. Define θ by*

$\mathrm{Ult}_U(\mathbf{V}) \models$ "*θ is the first Mahlo cardinal greater than λ*".

Then, it holds that

$\mathrm{Ult}_U(\mathbf{V}) \models$ "*κ is γ-supercompact for all $\gamma < \theta$*".

As a corollary, we have the following which has been proved in [1].

Corollary 2 *If κ is λ-supercompact, then there exists a normal ultarfilter on $\mathcal{P}_\kappa\lambda$ with the partition property.*

2 Notations and definitions

We use standard $\mathcal{P}_\kappa\lambda$-combinatorial terminologies (e.g., see [2]). Throughout this paper, κ denotes a regular uncountable cardinal. Let A be a set such that $\kappa \subset A$. $\mathcal{P}_\kappa A$ denotes the set $\{ x \subset A \mid |x| < \kappa \}$.

Let $Y \subset \mathcal{P}_\kappa A$. $[Y]^2$ denotes the set $\{ (x, y) \in Y \times Y \mid x \subset y \text{ and } x \neq y \}$. For any function $f : [Y]^2 \to 2$, a subset H of Y is said to be homogeneous for f, if $|f``[H]^2| = 1$.

For each $x \in \mathcal{P}_\kappa A$, $\widehat{x}$ denotes the set $\{ y \in \mathcal{P}_\kappa A \mid x \subset y \text{ and } x \neq y \}$.

Let U be a κ-complete ultrafilter on $\mathcal{P}_\kappa A$. The ultrapower of the universe $\mathbf{V}$ modular U is denoted by $\mathrm{Ult}_U(\mathbf{V})$. We say that U is fine, if $\widehat{x} \in u$ for all $x \in \mathcal{P}_\kappa A$. A fine ultrafilter U is said to be normal, if it is closed under the diagonal intersection. U has the partition property, if for any $X \in U$ and any $f : [X]^2 \to 2$, there exists $Y \in U$ such that $Y \subset X$ and Y is homogeneous for f.

3 Preparations for a proof of Theorem 1

In this section, we prove a lemma which will be used to prove the theorem.

Define $X_0 \subset \mathcal{P}_\kappa \lambda$ by:

$x \in X_0$ if and only if $x \in \mathcal{P}_\kappa \lambda$ and the following (1) and (2) hold.

(1) $x \cap \kappa$ is a Mahlo cardinal.

(2) ξ is inaccessible iff $\mathrm{ot}(x \cap \xi)$ is inaccessible , for all $\xi \in x \cup \{ \lambda \}$.

Since $\langle \mathrm{ot}(x \cap \xi) \mid x \in \mathcal{P}_\kappa \lambda \rangle$ represents ξ in $\mathrm{Ult}_U(\mathbf{V})$ for every $\xi \leq \lambda$, $X_0 \in U$ for every normal ultrafilter U on $\mathcal{P}_\kappa \lambda$. Now we can prove the lemma.

Lemma 3 *Let U be a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ and $\kappa \leq \gamma \leq \lambda$. Suppose that*

$$\forall X \in U \, \exists (x, y) \in [X]^2 \, (x \cap \gamma = y \cap \gamma).$$

Let σ be the least ordinal $\delta \leq \lambda$ which satisfies

$$\forall X \in U \, \exists (x, y) \in [X]^2 \, (x \cap \gamma = y \cap \gamma \text{ and } x \cap \delta \neq y \cap \delta).$$

Then, σ is a Mahlo cardinal.

Proof For each $\xi \in [\gamma, \sigma)$, take a $Y_\xi \in U$ such that

$$\forall (x, y) \in [Y_\xi]^2 \, (\text{if } x \cap \gamma = y \cap \gamma \text{ then } x \cap \xi = y \cap \xi).$$

Set $X_1 = X_0 \cap \Delta_{\gamma \leq \xi < \sigma} Y_\xi$. Note that, for any $(x, y) \in [X_1]^2$, if $x \cap \gamma = y \cap \gamma$

and $x \cap \sigma \neq y \cap \sigma$ then $y \cap \sigma$ is an end extension of $x \cap \sigma$.

We first show that σ is a strong limit cardinal. To get a contradiction, assume that there is a $\delta < \sigma$ such that $\sigma \leq 2^\delta$. Put

$$Y_0 = \{\, x \in X_1 \mid \delta \in x \text{ and } \mathrm{ot}(x \cap \sigma) \leq 2^{\mathrm{ot}(x \cap \delta)} \,\}.$$

Since $\sigma \leq 2^\delta$ also holds in $\mathrm{Ult}_U(\mathbf{V})$ and $\langle\, \mathrm{ot}(x \cap \delta) \mid x \in \mathcal{P}_\kappa\lambda \,\rangle$ represents δ, we have that $Y_0 \in U$. For each $\alpha < \kappa$, take an injection $f_\alpha : 2^\alpha + 1 \to \mathcal{P}(\alpha)$. For each $x \in Y_0$, let $\pi_x : \mathrm{ot}(x \cap \delta) \to x \cap \delta$ be the order isomorphism, and put $a_x = \pi_x'' f_{\mathrm{ot}(x \cap \delta)}(\mathrm{ot}(x \cap \sigma))$. Since $a_x \subset x \cap \delta$ for all $x \in Y_0$, there is an $A \subset \delta$ such that

$$Y_1 = \{\, x \in Y_0 \mid a_x = A \cap x \,\} \in U.$$

Take a pair $(x, y) \in [Y_1]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Since $\delta \in x \subset y$, it holds that $x \cap \delta = y \cap \delta$. By this, we have $\pi_x = \pi_y$ and $a_x = A \cap x \cap \delta = A \cap y \cap \delta = a_y$. So, $\mathrm{ot}(x \cap \sigma) = \mathrm{ot}(y \cap \sigma)$. This contradicts that $y \cap \sigma$ is an end extension of $x \cap \sigma$.

Next, we show that σ is a regular cardinal. To get a contradiction, assume that $\delta = \mathrm{cof}(\sigma) < \sigma$. Take a normal cofinal function $f : \delta \to \sigma$. Put

$$Y_2 = \{\, x \in X_1 \mid \delta \in x \text{ and } x \text{ is } f,\ f^{-1}\text{-closed and } f''x \cap \delta \text{ is cofinal in } x \cap \sigma \,\}.$$

It is easy to check that $Y_2 \in U$. So, there is a pair $(x, y) \in [Y_2]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Since $\delta \in x$ and $x \in X_1$, it holds that $x \cap \delta = y \cap \delta$. So, we have that $\sup(x \cap \sigma) = \sup f''x \cap \delta = \sup f''y \cap \delta = \sup(y \cap \sigma)$. This contradicts that $y \cap \sigma$ is an end extension of $x \cap \sigma$.

Finally we show that σ is a Mahlo cardinal. Note that $\mathrm{ot}(x \cap \sigma)$ is inaccessible for all $x \in X_1$, since $X_0 \subset X_1$ and σ is inaccessible. Put $S = \{\, \alpha < \sigma \mid \alpha \text{ is inaccessible} \,\}$. To get a contradiction, assume that S is non-stationary. Take a closed unbounded subset C of σ such that $\min C > \gamma$ and $S \cap C = \phi$. For each $x \in \mathcal{P}_\kappa\lambda$, let $\rho_x : \mathrm{ot}(x \cap \sigma) \to x \cap \sigma$ be an order isomorphism and put $C_x = \rho^{-1}(x \cap C)$. Since $\langle\, C_x \mid x \in \mathcal{P}_\kappa\lambda \,\rangle$ represents C in $\mathrm{Ult}_U(\mathbf{V})$, it holds that

$$Y_3 = \{\, x \in X_1 \mid C_x \text{ is club in } \mathrm{ot}(x \cap \sigma) \,\} \in U.$$

Take a pair $(x, y) \in [Y_3]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Let η be the least element of $y \cap \sigma \setminus x \cap \sigma$ and $\overline{\eta} = \rho^{-1}(\eta)$. Since $\mathrm{ot}(x \cap \sigma) = \mathrm{ot}(y \cap \eta)$, we have that $\rho_x = \rho_y \restriction \overline{\eta}$ and $\mathrm{ot}(y \cap \eta)$ is inaccessible. So, $\overline{\eta} \in C_y$ and η is inaccessible. Hence $\eta \in C \cap S$. This is a contradiction. □

4 Proofs of Theorem 1 and Corollary 2

In order to prove the theorem, we need the notion of ω-Jonsson functions and some known results. Let S be an infinite set. We denote by ${}^{\omega}S$ the set of functions from ω to S. A function F from ${}^{\omega}S$ to S is called an <u>ω-Jonsson function</u> for S if $F``{}^{\omega}T = S$ for any $T \subset S$ with $|T| = |S|$. Concerning ω-Jonsson functions, Erdös-Hajnal (e.g., see [2, Theorem 23.13]) proved:

Lemma 4 (Erdös-Hajnal) *For any infinite set S, there exists an ω-Jonsson function for S.*

Solovay proved:

Lemma 5 (Solovay [5]) *Let U be a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ and $F : {}^{\omega}\lambda \to \lambda$ an ω-Jonsson function. Then*

$$\{\, x \in \mathcal{P}_\kappa\lambda \mid F \restriction {}^{\omega}x \text{ is an } \omega\text{-Jonsson function for } x \,\} \in U.$$

The next lemma is due to Magidor.

Lemma 6 (Magidor [4]) *If κ is $<\lambda$-supercompact and λ is θ-supercompact, then κ is θ-supercompact.*

The next lemma is due to Menas.

Lemma 7 (Menas [6]) *Let U be a normal ultrafilter on $\mathcal{P}_\kappa\lambda$. Then, the following* (a) *and* (b) *are equivalent.*

(a) *U has the partition property.*

(b) *There exists an $X \in U$ such that $\forall\, (x, y) \in [X]^2$ $(\, |x| < |y \cap \kappa| \,)$.*

Now we can prove the theorem.

Theorem 1 *Suppose that U is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ without the partition property. Define θ by*

$\mathrm{Ult}_U(\mathbf{V}) \models$*"$\theta$ is the first Mahlo cardinal greater than λ".*

Then, it holds that

$\mathrm{Ult}_U(\mathbf{V}) \models$*"$\kappa$ is γ-supercompact for all $\gamma < \theta$".*

Proof To get a contradiction, assume that

$\mathrm{Ult}_U(\mathbf{V}) \models$"$\kappa$ is not γ-supercompact for some $\gamma < \theta$".

Define $f : \mathcal{P}_\kappa\lambda \to \kappa$ by

$f(x) =$ the least Mahlo cardinal greater than $\mathrm{ot}(x)$.

Since f represents θ in $\mathrm{Ult}_U(\mathbf{V})$,

$Y_0 = \{\, x \in X_0 \mid x \cap \kappa$ is not ξ-supercompact for some $\xi < f(x) \,\} \in U$.

Let $\gamma = \sup\{\, \delta \leq \lambda \mid \delta$ is a Mahlo cardinal$\}$. Since γ satisfies the same statement in $\mathrm{Ult}_U(\mathbf{V})$, it holds that

$Y_0 = \{\, x \in X_0 \mid \mathrm{ot}(x \cap \gamma) = \sup\{\, \delta \leq \mathrm{ot}(x) \mid \delta$ is a Mahlo cardinal$\} \in U$.

Futhermore, since

$\mathrm{Ult}_U(\mathbf{V}) \models$" κ is ξ-supercompact for all $\xi < \gamma$",

it holds that

$Y_1 = \{\, x \in Y_0 \mid x \cap \kappa$ is ξ-supercompact for all $\xi < \mathrm{ot}(x \cap \gamma) \,\} \in U$.

By the previous lemma, we can take a $Z \in U$ such that $x \cap \gamma \neq y \cap \gamma$, for all $(x, y) \in Z$. Take an ω-Jonsson function F for γ and put

$Y_3 = \{\, x \in Y_2 \cap Z \mid F \restriction {}^{\omega}(x \cap \gamma)$ is ω-Jonsson function for $x \cap \gamma \,\} \in U$.

Note that $|x \cap \gamma| < |y \cap \gamma|$ for all $(x, y) \in [Y_3]^2$. Since U does not have the partition property, there is a pair $(x, y) \in [Y_3]^2$ such that $y \cap \kappa \leq |x|$. Since $x \in Y_1$ and $y \cap \kappa$ is Mahlo, it holds that $y \cap \kappa \leq \mathrm{ot}(x \cap \gamma)$. So, $x \cap \kappa$ is ξ-supercompact for all $\xi < y \cap \kappa$. By this, since $y \in X_2$, it holds that

$x \cap \kappa$ is ξ-supercompact for all $\xi < \mathrm{ot}(y \cap \gamma)$.

So, $f(x) \leq \mathrm{ot}(y \cap \gamma)$. Hence, it holds that

$x \cap \kappa$ is not ξ-supercompact for some $\xi < \mathrm{ot}(y \cap \gamma)$.

This is a desired contradiction. □

Corollary 2 directly follows from Theorem 1 and the following Menas's result.

Lemma 8 (Menas [5]) *If κ is λ-supercompact, then there exists a normal ultrafilter U on $\mathcal{P}_\kappa\lambda$ such that*

$\mathrm{Ult}_U(\mathbf{V}) \models \kappa$ *is not λ-supercompact.*

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