Normal ultrafilters without the partition property

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1 Introduction

Let $\kappa$ be a measurable cardinal and $\kappa \leq \lambda$. Concerning the partition property of a normal ultrafilter on $\mathcal{P}_{\kappa}\lambda$, Solovay (see Menas [6]) proved the existence of a normal ultrafilter without the partition property under the assumption of that the existence of a certain large cardinal greater than $\kappa$. After Solovay established this result, Kunen (see Kunen-Pelletier [3]) improved his results, and proved that the existence of a normal ultrafilter without the partition property implies the existence of a certain large cardinal above $\kappa$. On the other hand, Menas [6] proved that there exist $2^{2^{\lambda<\kappa}}$ normal ultrafilters with the partition property, if $\kappa$ is $2^{\lambda<\kappa}$ supercompact. In the talk, we prove

**Theorem 1** Suppose that $U$ is a normal ultrafilter on $\mathcal{P}_{\kappa}\lambda$ without the partition property. Define $\theta$ by

$$\text{Ult}_U(V) \models \ " \theta \ is \ the \ first \ Mahlo \ cardinal \ greater \ than \ \lambda \ ".$$  

Then, it holds that

$$\text{Ult}_U(V) \models \ " \kappa \ is \ \gamma \text{-supercompact for all } \gamma < \theta \ ".$$  

As a corollary, we have the following which has been proved in [1].

**Corollary 2** If $\kappa$ is $\lambda$-supercompact, then there exists a normal ultrafilter on $\mathcal{P}_{\kappa}\lambda$ with the partition property.

2 Notations and definitions

We use standard $\mathcal{P}_{\kappa}\lambda$-combinatorial terminologies (e.g., see [2]). Throughout this paper, $\kappa$ denotes a regular uncountable cardinal. Let $A$ be a set such
that $\kappa \subset A$. $\mathcal{P}_\kappa A$ denotes the set $\{ x \subset A \mid |x| < \kappa \}$.

Let $Y \subset \mathcal{P}_\kappa A$. $[Y]^2$ denotes the set $\{ (x, y) \in Y \times Y \mid x \subset y \text{ and } x \neq y \}$.

For any function $f : [Y]^2 \to 2$, a subset $H$ of $Y$ is said to be homogeneous for $f$, if $|f^{-1}[H]^2| = 1$.

For each $x \in \mathcal{P}_\kappa A$, $\hat{x}$ denotes the set $\{ y \in \mathcal{P}_\kappa A \mid x \subset y \text{ and } x \neq y \}$.

Let $U$ be a $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa A$. The ultrapower of the universe $\mathbb{V}$ modular $U$ is denoted by $\text{Ult}_U(\mathbb{V})$. We say that $U$ is fine, if $\hat{x} \in U$ for all $x \in \mathcal{P}_\kappa A$. A fine ultrafilter $U$ is said to be normal, if it is closed under the diagonal intersection. $U$ has the partition property, if for any $X \in U$ and any $f : [X]^2 \to 2$, there exists $Y \in U$ such that $Y \subset X$ and $Y$ is homogeneous for $f$.

3 Preparations for a proof of Theorem 1

In this section, we prove a lemma which will be used to prove the theorem.

Define $X_0 \subset \mathcal{P}_\kappa \lambda$ by:

$x \in X_0$ if and only if $x \in \mathcal{P}_\kappa \lambda$ and the following (1) and (2) hold.

1. $x \cap \kappa$ is a Mahlo cardinal.
2. $\xi$ is inaccessible iff $\text{ot}(x \cap \xi)$ is inaccessible, for all $\xi \in x \cup \{ \lambda \}$.

Since $(\text{ot}(x \cap \xi) \mid x \in \mathcal{P}_\kappa \lambda)$ represents $\xi$ in $\text{Ult}_U(\mathbb{V})$ for every $\xi \leq \lambda$, $X_0 \in U$ for every normal ultrafilter $U$ on $\mathcal{P}_\kappa \lambda$. Now we can prove the lemma.

Lemma 3 Let $U$ be a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ and $\kappa \leq \gamma \leq \lambda$. Suppose that

$\forall X \in U \exists (x, y) \in [X]^2 \ (x \cap \gamma = y \cap \gamma)$.

Let $\sigma$ be the least ordinal $\delta \leq \lambda$ which satisfies

$\forall X \in U \exists (x, y) \in [X]^2 \ (x \cap \gamma = y \cap \gamma \text{ and } x \cap \delta \neq y \cap \delta)$.

Then, $\sigma$ is a Mahlo cardinal.

Proof For each $\xi \in [\gamma, \sigma)$, take a $Y_\xi \in U$ such that

$\forall (x, y) \in [Y_\xi]^2 \ (\text{if } x \cap \gamma = y \cap \gamma \text{ then } x \cap \xi = y \cap \xi)$.

Set $X_1 = X_0 \cap \triangle_{\gamma \leq \xi < \sigma} Y_\xi$. Note that, for any $(x, y) \in [X_1]^2$, if $x \cap \gamma = y \cap \gamma$
and $x \cap \sigma \neq y \cap \sigma$ then $y \cap \sigma$ is an end extension of $x \cap \sigma$. 

We first show that $\sigma$ is a strong limit cardinal. To get a contradiction, assume that there is a $\delta < \sigma$ such that $\sigma \leq 2^\delta$. Put

$$Y_0 = \{ x \in X_1 \mid \delta \in x \text{ and } \text{ot}(x \cap \sigma) \leq 2^{\text{ot}(x \cap \delta)} \}.$$ 

Since $\sigma \leq 2^\delta$ also holds in $\text{Ult}_U(V)$ and $\langle \text{ot}(x \cap \delta) \mid x \in \mathcal{P}_\kappa \lambda \rangle$ represents $\delta$, we have that $Y_0 \in U$. For each $\alpha < \kappa$, take an injection $f_\alpha : 2^\alpha + 1 \rightarrow \mathcal{P}(\alpha)$. For each $x \in Y_0$, let $\pi_x : \text{ot}(x \cap \delta) \rightarrow x \cap \delta$ be the order isomorphism, and put $a_x = \pi_x'' \text{ot}(x \cap \delta)(\text{ot}(x \cap \sigma))$. Since $a_x \subset x \cap \delta$ for all $x \in Y_0$, there is an $A \subset \delta$ such that

$$Y_1 = \{ x \in Y_0 \mid a_x = A \cap x \} \in U.$$ 

Take a pair $(x, y) \in [Y_1]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Since $\delta \in x \subset y$, it holds that $x \cap \delta = y \cap \delta$. By this, we have $\pi_x = \pi_y$ and $a_x = A \cap x \cap \delta = A \cap y \cap \delta = a_y$. So, $\text{ot}(x \cap \sigma) = \text{ot}(y \cap \sigma)$. This contradicts that $y \cap \sigma$ is an end extension of $x \cap \sigma$.

Next, we show that $\sigma$ is a regular cardinal. To get a contradiction, assume that $\delta = \text{cof}(\sigma) < \sigma$. Take a normal cofinal function $f : \delta \rightarrow \sigma$. Put

$$Y_2 = \{ x \in X_1 \mid \delta \in x \text{ and } x \text{ is } f, f^{-1}\text{-closed and } f''x \cap \delta \text{ is cofinal in } x \cap \sigma \}.$$ 

It is easy to check that $Y_2 \in U$. So, there is a pair $(x, y) \in [Y_2]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Since $\delta \in x$ and $x \in X_1$, it holds that $x \cap \delta = y \cap \delta$. So, we have that $\sup(x \cap \sigma) = \sup(f''x \cap \delta) = \sup(f''y \cap \delta) = \sup(y \cap \sigma)$. This contradicts that $y \cap \sigma$ is an end extension of $x \cap \sigma$.

Finally we show that $\sigma$ is a Mahlo cardinal. Note that $\text{ot}(x \cap \sigma)$ is inaccessible for all $x \in X_1$, since $X_0 \subset X_1$ and $\sigma$ is inaccessible. Put $S = \{ \alpha < \sigma \mid \alpha \text{ is inaccessible} \}$. To get a contradiction, assume that $S$ is non-stationary. Take a closed unbounded subset $C$ of $\sigma$ such that $\min C > \gamma$ and $S \cap C = \phi$. For each $x \in \mathcal{P}_\kappa \lambda$, let $\rho_x : \text{ot}(x \cap \sigma) \rightarrow x \cap \sigma$ be an order isomorphism and put $C_x = \rho^{-1}(x \cap C)$. Since $\langle C_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ represents $C$ in $\text{Ult}_U(V)$, it holds that

$$Y_3 = \{ x \in X_1 \mid C_x \text{ is club in } \text{ot}(x \cap \sigma) \} \in U.$$ 

Take a pair $(x, y) \in [Y_3]^2$ such that $x \cap \gamma = y \cap \gamma$ and $x \cap \sigma \neq y \cap \sigma$. Let $\eta$ be
the least element of \( y \cap \sigma \setminus x \cap \sigma \) and \( \overline{\eta} = \rho^{-1}(\eta) \). Since \( \ot(x \cap \sigma) = \ot(y \cap \eta) \), we have that \( \rho_x = \rho_y \upharpoonright \overline{\eta} \) and \( \ot(y \cap \eta) \) is inaccessible. So, \( \overline{\eta} \in C_y \) and \( \eta \) is inaccessible. Hence \( \eta \in C \cap S \). This is a contradiction. \( \square \)

## 4 Proofs of Theorem 1 and Corollary 2

In order to prove the theorem, we need the notion of \( \omega \)-Jonsson functions and some known results. Let \( S \) be an infinite set. We denote by \( \omega \) the set of functions from \( \omega \) to \( S \). A function \( F \) from \( \omega \) to \( S \) is called an \( \omega \)-Jonsson function for \( S \) if \( F^\omega T = S \) for any \( T \subset S \) with \( |T| = |S| \).

Concerning \( \omega \)-Jonsson functions, Erdős-Hajnal (e.g., see [2, Theorem 23.13]) proved:

**Lemma 4 (Erdős-Hajnal)** For any infinite set \( S \), there exists an \( \omega \)-Jonsson function for \( S \).

Solovay proved:

**Lemma 5 (Solovay [5])** Let \( U \) be a normal ultrafilter on \( \mathcal{P}_\kappa \lambda \) and \( F : \omega \lambda \to \lambda \) an \( \omega \)-Jonsson function. Then
\[
\{ x \in \mathcal{P}_\kappa \lambda \mid F \upharpoonright \omega x \text{ is an } \omega \text{-Jonsson function for } x \} \in U.
\]

The next lemma is due to Magidor.

**Lemma 6 (Magidor [4])** If \( \kappa \) is \( <\lambda \)-supercompact and \( \lambda \) is \( \theta \)-supercompact, then \( \kappa \) is \( \theta \)-supercompact.

The next lemma is due to Menas.

**Lemma 7 (Menas [6])** Let \( U \) be a normal ultrafilter on \( \mathcal{P}_\kappa \lambda \). Then, the following (a) and (b) are equivalent.

(a) \( U \) has the partition property.

(b) There exists an \( X \in U \) such that \( \forall (x, y) \in [X]^2 \) \( (|x| < |y \cap \kappa|) \).

Now we can prove the theorem.
Theorem 1. Suppose that $U$ is a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ without the partition property. Define $\theta$ by

$$\text{Ult}_U(V) \models "\theta \text{ is the first Mahlo cardinal greater than } \lambda".$$ 

Then, it holds that

$$\text{Ult}_U(V) \models "\kappa \text{ is } \gamma\text{-supercompact for all } \gamma < \theta".$$ 

Proof. To get a contradiction, assume that

$$\text{Ult}_U(V) \models "\kappa \text{ is not } \gamma\text{-supercompact for some } \gamma < \theta".$$ 

Define $f: \mathcal{P}_\kappa \lambda \to \kappa$ by

$$f(x) = \text{the least Mahlo cardinal greater than } \text{ot}(x).$$ 

Since $f$ represents $\theta$ in $\text{Ult}_U(V)$,

$$Y_0 = \{ x \in X_0 \mid x \cap \kappa \text{ is not } \xi\text{-supercompact for some } \xi < f(x) \} \in U.$$ 

Let $\gamma = \sup\{ \delta \leq \lambda \mid \delta \text{ is a Mahlo cardinal} \}$. Since $\gamma$ satisfies the same statement in $\text{Ult}_U(V)$, it holds that

$$Y_0 = \{ x \in X_0 \mid \text{ot}(x \cap \gamma) = \sup\{ \delta \leq \text{ot}(x) \mid \delta \text{ is a Mahlo cardinal} \} \in U.$$ 

Furthermore, since

$$\text{Ult}_U(V) \models "\kappa \text{ is } \xi\text{-supercompact for all } \xi < \gamma",$$ 

it holds that

$$Y_1 = \{ x \in Y_0 \mid x \cap \kappa \text{ is } \xi\text{-supercompact for all } \xi < \text{ot}(x \cap \gamma) \} \in U.$$ 

By the previous lemma, we can take a $Z \in U$ such that $x \cap \gamma \neq y \cap \gamma$, for all $(x, y) \in Z$. Take an $\omega$-Jonsson function $F$ for $\gamma$ and put

$$Y_3 = \{ x \in Y_2 \cap Z \mid F^{\omega}(x \cap \gamma) \text{ is } \omega\text{-Jonsson function for } x \cap \gamma \} \in U.$$ 

Note that $|x \cap \gamma| < |y \cap \gamma|$ for all $(x, y) \in [Y_3]^2$. Since $U$ does not have the partition property, there is a pair $(x, y) \in [Y_3]^2$ such that $y \cap \kappa \leq |x|$. Since $x \in Y_1$ and $y \cap \kappa$ is Mahlo, it holds that $y \cap \kappa \leq \text{ot}(x \cap \gamma)$. So, $x \cap \kappa$ is $\xi$-supercompact for all $\xi < y \cap \kappa$. By this, since $y \in X_2$, it holds that

$$x \cap \kappa \text{ is } \xi\text{-supercompact for all } \xi < \text{ot}(y \cap \gamma).$$ 

So, $f(x) \leq \text{ot}(y \cap \gamma)$. Hence, it holds that

$$x \cap \kappa \text{ is not } \xi\text{-supercompact for some } \xi < \text{ot}(y \cap \gamma).$$ 

This is a desired contradiction. \qed
Corollary 2 directly follows from Theorem 1 and the following Menas's result.

**Lemma 8** (Menas [5]) \textit{If $\kappa$ is $\lambda$-supercompact, then there exists a normal ultrafilter $U$ on $\mathcal{P}_\kappa \lambda$ such that}

\[ \text{Ult}_U(V) \models \kappa \text{ is not } \lambda \text{-supercompact.} \]

**References**


