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'ardinal invariants of the continuum — A survey

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Abstract

These are expanded notes of a series of two lectures given at the meeting on axiomatic set theory (公理的集合論) at Kyōto University (京都大学数理解析研究所) in November 2000. The lectures were intended to survey the state of the art of the theory of cardinal invariants of the continuum, and focused on the interplay between iterated forcing theory and cardinal invariants, as well as on important open problems. To round off the present written account of this survey, we also include sections on ZFC–inequalities between cardinal invariants, and on applications outside of set theory. However, due to the sheer size of the area, proofs had to be mostly left out.

While being more comprehensive than the original talks, the personal flavor of the latter is preserved in the notes. Some of the material included was presented in talks at other conferences.

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1 What are cardinal invariants?

We plan to look at certain basic features of the real line $\mathbb{R}$ from the point of view of combinatorial set theory. For our purposes, it is convenient to work with the *Cantor space* 2^ω or the *Baire space* ω^ω instead of $\mathbb{R}$ itself. Here we put, as usual,

$$\begin{aligned} 2 &= \{0,1\} \\ \omega &= \mathrm{N} = \{0,1,2,3,...\} = \text{the natural numbers} \\ 2^\omega &= \text{the set of functions from } \omega \text{ to } 2 \\ \omega^\omega &= \text{the set of functions from } \omega \text{ to } \omega. \end{aligned}$$

Both 2^ω and ω^ω can be turned into topological spaces in a natural way: 2 and ω carry the discrete topology, and 2^ω and ω^ω are equipped with the product topology. This means that basic open sets in, say, 2^ω are of the form

$$[\sigma] := \{f \in 2^\omega;\ \sigma \subseteq f\}$$

where $\sigma \in 2^{<\omega}$, that is, σ is a finite sequence of 0's and 1's. Whereas open sets have an easy description in any of these spaces (countable unions of open intervals in case of $\mathbb{R}$, countable unions of sets of the form $[\sigma]$ in case of 2^ω and ω^ω), in 2^ω (and ω^ω) closed sets are nicely characterized as well, namely as sets of branches

$$[T] := \{f \in 2^\omega;\ f\restriction n \in T \text{ for all } n\}$$

through trees $T \subseteq 2^{<\omega}$, and this is one of the main reasons for using the Cantor space or the Baire space in combinatorial set theory.

Incidentally, there is an alternative description of this topology, as follows. Given $f, g \in 2^\omega$, define

$$d(f,g) = \begin{cases} 0 & \text{if } f = g \\ \frac{1}{2^{\min\{n;\ f(n) \neq g(n)\}}} & \text{if } f \neq g. \end{cases}$$

d is easily seen to be a metric which turns 2^ω into a *Polish* (= separable complete metric) space. The topology induced by the metric is identical to the product topology outlined in the previous paragraph.

While 2^ω, ω^ω and $\mathbb{R}$ are different objects from the topological point of view (e.g., 2^ω is compact, $\mathbb{R}$ is σ–compact and not compact, whereas ω^ω is not even σ–compact), they are still close enough to each other so that all notions we are interested in carry over from one space to the other. In fact, they are "homeomorphic modulo a countable set". For example, the mapping F sending $f \in 2^\omega$ to the real $F(f) \in [0,1]$ whose binary expansion is

$$F(f) = 0.f(0)f(1)f(2)f(3)...$$

is continuous and onto $[0,1]$, as well as injective on a co–countable subset C of 2^ω, with the inverse mapping of the restriction $F\restriction C$ being continuous too. Similarly, it's well–known that the Baire space is homeomorphic to the irrational numbers.

Recall that a subset A of a topological space X is called *nowhere dense* if its closure has empty interior. $B \subseteq X$ is *meager* (or: *of first category*) if it's a union of countably many nowhere dense sets. Let $\mathcal{M}$ denote the σ–ideal of meager subsets of ω^ω. (For our purposes, it doesn't matter on which space we consider $\mathcal{M}$. The point is the maps almost identifying 2^ω, ω^ω and $\mathbb{R}$ which we briefly discussed above send meager sets to meager sets and vice–versa.) By the Baire category theorem, $\mathcal{M}$ is non–trivial.

As with topology, we can equip 2^ω and ω^ω with a measure in a natural way: 2 carries the measure giving both $\{0\}$ and $\{1\}$ measure $\frac{1}{2}$; ω carries the measure giving $\{n\}$ measure $\frac{1}{2^{n+1}}$; and 2^ω as well as ω^ω both get the product measure. Since this is analogous to the usual construction of Lebesgue measure on the reals $\mathbb{R}$, we may refer to this measure as "Lebesgue measure" as well. Let $\mathcal{N}$ stand for the σ–ideal of null subsets of ω^ω. (Again, the exact nature of the space is not relevant for subsequent considerations.)

These ideals provide us with our first examples of *cardinal invariants of the continuum*, that is, cardinals sitting between $\aleph_1$ and $\mathfrak{c}$ (so that, in particular, they become trivial if the continuum hypothesis CH, that is, the statement $\mathfrak{c} = \aleph_1$, holds) and reflecting part of the combinatorial structure of the real line. Here we put, as usual,

$$\begin{aligned} \aleph_1 &= \text{the first uncountable cardinal} \\ \mathfrak{c} &= |\mathbb{R}| = \text{the cardinality of the continuum.} \end{aligned}$$

Namely let us define

$$\begin{aligned} \mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} \notin \mathcal{I}\} = \text{the } \textit{additivity of } \mathcal{I} \\ \mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} = \omega^\omega\} = \text{the } \textit{covering number of } \mathcal{I} \\ \mathsf{non}(\mathcal{I}) &= \min\{|X|;\ X \subseteq \omega^\omega \text{ and } X \notin \mathcal{I}\} = \text{the } \textit{uniformity of } \mathcal{I} \\ \mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall A \in \mathcal{I}\ \exists B \in \mathcal{F}\ (A \subseteq B)\} = \text{the } \textit{cofinality of } \mathcal{I} \end{aligned}$$

where $\mathcal{I}$ is either $\mathcal{M}$ or $\mathcal{N}$. Note that one has $\mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ as well as $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. To see, e.g., the last inequality, let $\mathcal{F} \subseteq \mathcal{I}$ be a witness for $\mathsf{cof}(\mathcal{I})$, choose for each $A \in \mathcal{F}$ a real $x_A \in \omega^\omega \setminus A$ and put $X = \{x_A;\ A \in \mathcal{F}\}$. $X \notin \mathcal{I}$ is straightforward. Furthermore, the fact that both $\mathcal{M}$ and $\mathcal{N}$ are σ–ideals is rephrased as $\mathsf{add}(\mathcal{M}) \geq \aleph_1$ ($\mathsf{add}(\mathcal{N}) \geq \aleph_1$, respectively) in this language. Since both $\mathcal{M}$ and $\mathcal{N}$ have a basis consisting of Borel sets, $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{c}$ ($\mathsf{cof}(\mathcal{N}) \leq \mathfrak{c}$, resp.) is immediate as well. Here $\mathcal{F} \subseteq \mathcal{I}$ is called a *basis* if it satisfies the defining clause of $\mathsf{cof}(\mathcal{I})$, that is, if for all $A \in \mathcal{I}$ there is $B \in \mathcal{F}$ containing A. Finally notice that $\mathsf{add}(\mathcal{I})$ is always a regular cardinal.

The *eventual dominance order* $\leq^*$ on ω^ω is given by: $f \leq^* g$ iff $f(n) \leq g(n)$ holds for all but finitely many n. In this case we say g *eventually dominates* f. (This relation is transitive and reflexive, but not antisymmetric. It can be turned into a partial order by looking at equivalence classes of functions modulo the relation *almost equal*. However, we have no use for doing this.) Let us say $\mathcal{F} \subseteq \omega^\omega$ is *unbounded* if it's not bounded in this order or, equivalently, if given any $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $f(n) > g(n)$ for infinitely many n. The *unbounding number* $\mathfrak{b}$ is the size of the smallest unbounded family. An easy diagonal argument shows that countable families are bounded so that $\aleph_1 \leq \mathfrak{b}$. (More explicitly, given functions $f_n \in \omega^\omega$, $n \in \omega$, define $g(k) = \max\{f_n(k);\ n \leq k\}$. Then $f_n \leq^* g$ for all n.) Call $\mathcal{F} \subseteq \omega^\omega$ *dominating* if it is cofinal in the eventual dominance order, that is, if, given any $g \in \omega^\omega$, there is $f \in \mathcal{F}$ with $g \leq^* f$. The *dominating number* $\mathfrak{d}$ is the size of the least dominating family. Note that for the definition of $\mathfrak{d}$ it would not matter if we considered *domination everywhere*, i.e., if we replaced $\leq^*$ by $\leq$ where $f \leq g$ iff $f(n) \leq g(n)$ for all n. (However, such a change would affect $\mathfrak{b}$; for indeed, the corresponding cardinal would be $\aleph_0$, and thus uninteresting.) Clearly $\mathfrak{d} \leq \mathfrak{c}$. A little while's thought lets us convince ourselves that $\mathfrak{b}$ is regular and less than $cf(\mathfrak{d})$, the cofinality of $\mathfrak{d}$.

The relationship between these numbers is best illustrated by the following diagram, called *Cichoń's diagram*, where cardinals grow as one moves up and/or to the right.

$$\begin{tikzcd}
 & \mathsf{cov}(\mathcal{N}) \arrow[r, dash] & \mathsf{non}(\mathcal{M}) \arrow[r, dash] & \mathsf{cof}(\mathcal{M}) \arrow[r, dash] & \mathsf{cof}(\mathcal{N}) \arrow[r, dash] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u, dash] \arrow[r, dash] & \mathfrak{d} \arrow[u, dash] & & \\
\aleph_1 \arrow[r, dash] & \mathsf{add}(\mathcal{N}) \arrow[uu, dash] \arrow[r, dash] & \mathsf{add}(\mathcal{M}) \arrow[u, dash] \arrow[r, dash] & \mathsf{cov}(\mathcal{M}) \arrow[u, dash] \arrow[r, dash] & \mathsf{non}(\mathcal{N}) \arrow[uu, dash] &
\end{tikzcd}$$

The inequalities between cardinals exhibited in this diagram (all theorems of ZFC, Zermelo–Fraenkel set theory with choice) were mostly proved in the 70's and 80's (see [BaJ]). The deepest and most important result is perhaps

Theorem 1.1 (Bartoszyński–Raisonnier–Stern Theorem, classical version [BaJ]) $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$.

None of the inequalities above is reversible (see Section 4). However, there are two results each of which relates three of the cardinals.

Theorem 1.2 (Miller–Truss Theorem [BaJ]) $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ *and* $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$.

Except for this restriction, it has been shown [BaJ] that all assignments of the values $\aleph_1$ and $\aleph_2$ to the above cardinals which do not contradict the diagram are consistent with ZFC. (However, if the continuum is larger than $\aleph_2$, a number of deep questions remain open, see below, Sections 4 and 5.)

There are quite a few more interesting and important cardinal invariants which we shall introduce as need arises. At this point, let us rather ask: **Why are cardinal invariants interesting?** Why do we study them?

(1) They are interesting objects of study in their own right. For they describe the underlying combinatorial structure of the real line.

(2) There are lots of connections to other areas of mathematics. In particular, to general topology. But also to algebra (mainly group theory) and real analysis... We shall illustrate this in Section 3.

(3) They provide us with "test problems" for forcing theory, in particular in light of the search for new iteration techniques. We shall come back to this in Section 4.

For a more detailed account of everything we discussed so far see either [BaJ] or [Bl2].

2 Proving inequalities in ZFC

In the preceding section we saw some easy examples of how to prove inequalities between cardinal invariants in ZFC. But how does one proceed in general? Usually, such proofs are first done in an ad hoc fashion, but then a deeper analysis may yield either sharper results or a simplified argument or the very essence of what's going on. There are two important, and not unrelated, ways to look at such arguments, namely

(1) working over models for fragments of ZFC,

(2) reformulating the argument in the language of so–called Galois–Tukey connections.

We shall take a closer look at both.

FIRST METHOD. MODELS OF FRAGMENTS OF ZFC. The Bartoszyński–Raisonnier–Stern Theorem 1.1 is sometimes stated as

Theorem 2.1 (Bartoszyński–Raisonnier–Stern Theorem, model version [BaJ]) *Assume M is a model of a large enough finite fragment of ZFC. Also suppose that there is a null set A which contains all Borel null sets coded in M. Then there is a meager set B which contains all Borel meager sets coded in M.*

Let us first check this indeed implies $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$. For assume $\kappa < \mathsf{add}(\mathcal{N})$, and choose a family $\mathcal{F}$ of κ many meager sets. We can then find a model M of size κ containing Borel codes for all members of $\mathcal{F}$ as well as satisfying a large enough finite fragment of ZFC. Since $\kappa < \mathsf{add}(\mathcal{N})$, the union of the Borel null sets coded in M is null so that there is A satisfying the assumption of the above theorem. Therefore, we obtain a meager set B containing all Borel meager sets coded in M. A fortiori, $\bigcup \mathcal{F} \subseteq B$, as required.

In principle, the proof of 2.1 goes like a "model–free" proof of 1.1. There is one slight simplification, however. The point is to prove a statement like $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$, one has to proceed in the following, rather obvious, manner: fix $\mathcal{F} \subseteq \mathcal{M}$ of size less than $\mathsf{add}(\mathcal{N})$; associate a family $\mathcal{G} \subseteq \mathcal{N}$, still of size less than $\mathsf{add}(\mathcal{N})$, with $\mathcal{F}$ in such a way that the fact that $\bigcup \mathcal{G}$ is null (i.e. contained in a Borel null set A) entails $\bigcup \mathcal{F}$ is meager (i.e. contained in a Borel meager set B). Now, in this as well as in most interesting cases (see below for some counterexamples), $\mathcal{G}$ is constructed from $\mathcal{F}$ in such a way that any model M containing $\mathcal{F}$ will also contain $\mathcal{G}$. Therefore the approach via models allows us *not to care a priori about the concrete nature of* $\mathcal{G}$, and just take any Borel null set A which contains all Borel null sets coded in M. Then use A to construct the appropriate Borel meager set B. Only when doing the last step of the proof do we associate with each meager set X from $\mathcal{F}$ (or from M, for that matter) a null set Y from M such that $Y \subseteq A$ entails $X \subseteq B$. (To appreciate what we mean by "not caring a priori" we recommend the reader to have a look at the proof of Theorem 2.12 below.)

There are, however, more important reasons for formulating statements like 2.1:

(1) They are readily adapted to a descriptive set–theoretic context. For example, 2.1 in particular entails that if there is a null set containing all Borel null sets

coded in $L[x]$, then there is a meager set containing all Borel meager sets coded in $L[x]$. Now, the former statement is well-known to be equivalent to (lightface) $\Sigma^1_2(x)$-measurability, while the latter is the same as the (lightface) $\Sigma^1_2(x)$-Baire property. Therefore, we immediately get the (effective) descriptive set-theoretic version of the Bartoszsyński-Raisonnier-Stern Theorem, namely the statement *$\Sigma^1_2(x)$-measurability implies $\Sigma^1_2(x)$-Baire property.*

(2) They are particularly useful when dealing with cardinal invariants which can be reformulated in *forcing language* (Martin axiom language).

Let me expand somewhat on the second point. Given a p.o. $\mathbb{P}$, denote by $\mathfrak{m}(\mathbb{P})$ the least κ such that Martin's axiom MA_κ fails for $\mathbb{P}$, that is, there are κ many dense sets in $\mathbb{P}$ such that no filter meets all of them. For a class $\mathcal{P}$ of p.o.'s, let $\mathfrak{m}(\mathcal{P}) = \min\{\mathfrak{m}(\mathbb{P});\ \mathbb{P} \in \mathcal{P}\}$. (So $\mathfrak{m}(\{\mathbb{P}\}) = \mathfrak{m}(\mathbb{P})$.) $\mathfrak{m}$ denotes $\mathfrak{m}(ccc)$, i.e., the least cardinal κ such that the standard Martin axiom MA_κ fails.

Fact 2.2 (i) $\mathfrak{m}(\mathbb{C}) = \mathfrak{m}(\text{countable}) = \mathsf{cov}(\mathcal{M})$.

(ii) $\mathfrak{m}(\mathbb{B}) = \mathsf{cov}(\mathcal{N})$.

Proof. (i) The first equality is an immediate consequence of the fact that every countable forcing notion is equivalent to Cohen forcing.

Next we argue that $\mathfrak{m}(\mathbb{C}) \leq \mathsf{cov}(\mathcal{M})$. For indeed if $\mathcal{F} \subseteq \mathcal{M}$ is a family of Borel sets of size less than $\mathfrak{m}(\mathbb{C})$, then there is a model M containing the Borel codes of the members of $\mathcal{F}$ and still of size less than $\mathfrak{m}(\mathbb{C})$. Hence there is a filter F which meets all dense sets of $\mathbb{C}$ of M, that is F is $\mathbb{C}$-generic over M. From F we can decode a real c which is Cohen-generic over M. By Solovay's characterization of being Cohen-generic (that is, a real is Cohen-generic over M iff it avoids all Borel meager sets coded in M), we see that $c \notin \bigcup \mathcal{F}$, so that $\mathcal{F}$ was not a covering family.

To show $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{m}(\mathbb{C})$, we simply trace our way back in the argument of the preceding paragraph.

(ii) is analogous. □

Now, what is the use of this? Suppose for example, we want to prove the inequality $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$ in Cichoń's diagram. Of course, this can be done directly rather easily, but it becomes even more trivial in the "model language". One simply needs to recall that a real c Cohen over M is *unbounded* over M (in the sense that c is not (eventually) dominated by any member from M). Now, if $\mathcal{F} \subseteq \omega^\omega$ is of size less than $\mathsf{cov}(\mathcal{M})$, we can find a model M, still of size less than $\mathsf{cov}(\mathcal{M})$ and containing $\mathcal{F}$, and then a real c Cohen over M. Since c is unbounded over M, we see immediately that $\mathcal{F}$ was not a dominating family. Hence $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$.

In a similar fashion, using that a real r random over M canonically defines a meager set containing $M \cap 2^\omega$, one can prove $\mathsf{cov}(\mathcal{N}) \leq \mathsf{non}(\mathcal{M})$. And $\mathsf{cov}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$ is analogous!

The additivity numbers can also be formulated in forcing language.

Proposition 2.3 (see [BaJ] and [B4])

(i) $\mathsf{add}(\mathcal{M}) = \mathfrak{m}(\mathbb{D})$ *where* $\mathbb{D}$ *denotes* Hechler forcing *(the standard σ-centered forcing for adding a dominating real).*

(ii) $\mathsf{add}(\mathcal{N}) = \mathfrak{m}(\mathbb{A}) = \mathfrak{m}(\mathbb{LOC})$ *where* $\mathbb{A}$ *denotes* amoeba forcing *(the standard forcing for adding a measure one set of random reals), and* $\mathbb{LOC}$ *is* localization forcing *(the standard forcing for adding a function ϕ with domain ω and $|\phi(n)| \leq n$ for all n such that for all $f \in \omega^\omega$ in the ground model, $f(n) \in \phi(n)$ holds for all but finitely many n.)*

Here, we say a forcing notion $\mathbb{P}$ is *σ-centered* if it can be written as $\mathbb{P} = \bigcup_n P_n$ with each P_n being centered, that is, each finite $F \subseteq P_n$ has a lower bound $p \in \mathbb{P}$. Similarly, $\mathbb{P}$ is *σ-linked* if it is of the form $\mathbb{P} = \bigcup_n P_n$ with each P_n being linked, i.e. any two members of P_n are compatible. Clearly any σ–centered forcing is σ–linked as well. $\mathbb{A}$ and $\mathbb{LOC}$ above are σ–linked (but not σ–centered).

Let $\mathbb{C}_\kappa$ denote the forcing for adding κ many Cohen reals; and let $\mathbb{B}_\kappa$ stand for the p.o. adjoining κ many random reals (the product measure algebra). Then $\mathbb{C}_\mathfrak{c}$ is σ–centered, and $\mathbb{B}_\mathfrak{c}$ is σ–linked (but not σ–centered). So we see

Observation 2.4 (i) $\mathfrak{m}(\mathbb{C}) \geq \mathfrak{m}(\mathbb{C}_\mathfrak{c}) \geq \mathfrak{m}(\sigma - \text{centered})$.

(ii) $\mathfrak{m}(\mathbb{B}) \geq \mathfrak{m}(\mathbb{B}_\mathfrak{c}) \geq \mathfrak{m}(\sigma - \text{linked})$.

In fact one can prove

Theorem 2.5 [B4] $\mathbb{C}_\mathfrak{c}$ *completely embeds into* $\mathbb{LOC}$.

In view of 2.3 this entails

Corollary 2.6 [B4] $\mathfrak{m}(\mathbb{C}_\mathfrak{c}) \geq \mathsf{add}(\mathcal{N})$.

SECOND METHOD. GALOIS–TUKEY CONNECTIONS. Most cardinal invariants of the continuum come in dual pairs. For example add is dual to cof, and non is dual to cov. Similarly, $\mathfrak{b}$ and $\mathfrak{d}$ are dual. Now, it has been realized early on that dual inequalities between cardinal invariants have similar proofs. This is reflected rather badly by the "model language" above, and some other approach is needed to economize on proofs. Consider

Theorem 2.7 (Bartoszyński–Raisonnier–Stern Theorem, Galois–Tukey version, see [BaJ], [Ba] or [PR]) *There are "definable" functions $X \mapsto Y_X : \mathcal{M} \to \mathcal{N}$ and $A \mapsto B_A : \mathcal{N} \to \mathcal{M}$ such that whenever Y_X is contained in A, X is contained in B_A.*

Let us quickly argue this implies Theorem 2.1 above, and thus $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$, as well as the dual inequality $\mathsf{cof}(\mathcal{N}) \geq \mathsf{cof}(\mathcal{M})$. For the former, let M and A be as in 2.1, and check that $B = B_A$ is as required. Indeed, let X be a meager set coded in M. By definability of the mapping $\mathcal{M} \to \mathcal{N}$, Y_X also belongs to M. Therefore $Y_X \subseteq A$. Hence $X \subseteq B$, which is what we wanted to show. For the latter, let $\mathcal{F}$ be a basis of $\mathcal{N}$. Then $\mathcal{G} = \{B_A;\ A \in \mathcal{F}\}$ is easily seen to be a basis of $\mathcal{M}$, as required.

A comment concerning the word "definable" in the statement of the theorem is in order. Of course, from the argument in the preceding paragraph it is clear that it suffices that the process leading from X to Y_X (from A to B_A, respectively) is constructive enough so that every ZFC–model containing X (A, resp.) will also contain Y_X (B_A, resp.). However, one can usually do much better and get functions which are Borel or even (in most cases) continuous. (Let me briefly dwell on what would be meant by "Borel" or "continuous" in the above theorem. It is relatively easy to see there is a Borel *master meager set* C in the plane $\omega^\omega \times \omega^\omega$, that is, a meager set C such that all vertical sections C_x are meager and for all meager sets X there is a real x such that $X \subseteq C_x$. Similarly there is a Borel master null set D. Now, "continuity" simply means that there are continuous maps $\phi : \omega^\omega \to \omega^\omega$ and $\psi : \omega^\omega \to \omega^\omega$ such that whenever $D_{\phi(x)} \subseteq D_y$, then $C_x \subseteq C_{\psi(y)}$. Then $X \mapsto Y_X$ is simply $C_x \mapsto D_{\phi(x)}$, and $A \mapsto B_A$ is $D_y \mapsto C_{\psi(y)}$. See [Ba] or [PR] for details.)

The language with pairs of functions, as exemplified by Theorem 2.7, is originally due to Fremlin, has been put into a more general framework by Vojtáš [Vo], and reformulated by Blass [Bl2]. The optimal results, using continuous functions, were obtained by Pawlikowski and Recław [PR] (see also [Ba]).

Unfortunately, things are not always as easy as in 2.7, and sometimes one needs either several functions or functions defined on products (several models, resp.) to arrive at a reformulation of some inequality between cardinal invariants. We give some examples.

Theorem 2.8 (Truss' Theorem [BaJ], see also [Ba] or [PR]) *There are functions $(f, x) \mapsto A_{f,x} : 2^\omega \times \omega^\omega \to \mathcal{M}$ and $(f, B) \mapsto y_{f,B} : 2^\omega \times \mathcal{M} \to \omega^\omega$ such that for all x, B, f, if $f \notin B$ and $y_{f,B} \leq^* x$, then $B \subseteq A_{f,x}$.*

In terms of the "model language" this means that given ZFC–models $M_0 \subseteq M_1$ as well as reals $f \in M_1$ Cohen over M_0 and x dominating over M_1, $A_{f,x}$ contains all Borel meager sets coded in M_0. In the language of cardinal invariants this then entails $\mathsf{add}(\mathcal{M}) \geq \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$, one inequality in the Miller–Truss Theorem 1.2 quoted above. However, we used two models, and it is known one is not enough. For Pawlikowski [Pa] has shown that if d is $\mathbb{D}$–generic (Hechler–generic) over M, then the union of the Borel meager sets coded in M is not meager in $M[d]$, yet $M[d]$ contains both a Cohen real and a dominating real over M (d is dominating, and d mod $2 \in 2^\omega$ is Cohen). We leave it to the reader to infer from 2.8 that for the dual inequality, namely $\mathsf{cof}(\mathcal{M}) \leq \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$, one model is indeed enough.

Let $\exists^\infty n$ abbreviate *there are infinitely many n*. Similarly, $\forall^\infty n$ stands for *for all but finitely many n*.

Theorem 2.9 (Bartoszyński–Miller characterization of $\mathsf{non}(\mathcal{M})$ and $\mathsf{cov}(\mathcal{M})$ [BaJ], see also [Ba] or [PR]) *There are functions $A \mapsto x_A : \mathcal{M} \to \omega^\omega$, $(A, y) \mapsto z_{A,y} : \mathcal{M} \times \omega^\omega \to \omega^\omega$, and $(y, v) \mapsto w_{y,v} : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that whenever $\exists^\infty n\ (x_A(n) \leq y(n))$ and $\exists^\infty n\ (z_{A,y}(n) = v(n))$, then $w_{y,v} \notin A$.*

Now, this shows that given ZFC–models $M_0 \subseteq M_1$ as well as reals $y \in M_1$ unbounded over M_0 and v infinitely often equal over M_1, then $w_{y,z}$ is Cohen over M_0. (Check details!) In this case it is not known whether one model is sufficient or not:

Problem 2.10 (half–a–Cohen–real problem) *Assume there is a real $x \in \omega^\omega$ which is infinitely often equal over M, that is, given any $y \in \omega^\omega \cap M$, there are infinitely many n such that $x(n) = y(n)$. Does there exist a Cohen real over M?*

It is easy to see a Cohen real is infinitely often equal in this sense, and by the above, from two infinitely often equal reals, one over the other, we can reconstruct a Cohen real. Therefore an infinitely often equal real is sometimes labeled "half a Cohen real", and 2.10 asks whether half a Cohen real is the same as one Cohen real. In terms of cardinal invariants this gives us

Theorem 2.11 (Bartoszyński–Miller characterization of $\mathsf{non}(\mathcal{M})$ [BaJ])
$\mathsf{cov}(\mathcal{M}) = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \omega^\omega;\ \forall x \in \omega^\omega\ \exists y \in \mathcal{F}\ \forall^\infty n\ (x(n) \neq y(n))\}$

Again we leave it to the reader to verify that for the characterization of $\mathsf{non}(\mathcal{M})$, one model is enough.

We finally discuss a recent result of the author, dealing with a situation which is even more complex. Call a function $\pi : 2^{<\omega} \to 2$ a *predictor*. Say π *k–constantly predicts* a real $x \in 2^\omega$ if for almost all intervals I of length k, there is $i \in I$ such that $x(i) = \pi(x \restriction i)$. In case π k–constantly predicts x for some k, say that π *constantly predicts* x. The *constant prediction number* $\mathfrak{v}_2^{\text{const}}$ is the smallest size of a set of predictors Π such that every $x \in 2^\omega$ is constantly predicted by some $\pi \in \Pi$. The concept of prediction, which we shall encounter again in Section 3 below, is originally due to Blass [Bl1] who also put it into a much more general framework in [Bl2, Section 10]. The notion of constant prediction and the definition of $\mathfrak{v}_2^{\text{const}}$, however, are due to Kamo (see [Ka1] and [Ka2]), and the notation $\mathfrak{v}_2^{\text{const}}$ is due to Kada.

Kamo observed that $\mathfrak{v}_2^{\text{const}} \geq \mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N})$ [Ka1]. He also proved that $\mathfrak{v}_2^{\text{const}}$ may be larger than all cardinal invariants in Cichoń's diagram [Ka1], and smaller than the dominating number $\mathfrak{d}$ [Ka2]. He asked whether it can even be smaller than the unbounding number $\mathfrak{b}$. The following is the main step towards the solution of his problem.

Theorem 2.12 [B3] *Fix $k \in \omega$. Let $\ell = 2^k - 1$. Assume there are ZFC–models $M_0 \subset M_1 \subset \ldots \subset M_\ell$ and reals $f_0, \ldots, f_{\ell-1} \in \omega^\omega$ such that $f_i \in M_{i+1}$ is dominating over M_i. Then there is $x \in 2^\omega \cap M_\ell$ which is not k–constantly predicted by any predictor from M_0.*

Proof. Assume without loss all f_i are strictly increasing, $f_i(0) > 0$ and $f_i(n+1) > f_i(n) + k$. Define $h_i \in \omega^\omega \cap M_{i+1}$ by the recursion $h_i(0) = f_i(0)$ and $h_i(n+1) = f_i(h_i(n))$. Without loss we may assume $\text{ran}(h_{i+1}) \subseteq \text{ran}(h_i)$ for all i. Clearly $h_i \geq f_i$ for all i. List $\{s \in 2^k;\ s \neq 0\}$ (where 0 denotes the sequence with constant value 0) as $\{s_i;\ i < \ell\}$. Define $x \in 2^\omega$ as follows:

$$x(n) = \begin{cases} 0 & \text{if } n \notin \{h_0(m) + j;\ m \in \omega \text{ and } j < k\} \\ s_i(j) & \text{if } n \text{ is of the form } h_i(m) + j,\ i < \ell - 1 \text{ and } j < k, \\ & \text{and } h_i(m) \notin \text{ran}(h_{i+1}) \\ s_{\ell-1}(j) & \text{if } n \text{ is of the form } h_{\ell-1}(m) + j, j < k \end{cases}$$

We also define, for each $t \in 2^{<\omega}$ and each $i \leq \ell$, a real $x_{t,i} \in 2^\omega \cap M_i$:

$$x_{t,0} = t^\frown 0 \text{ (this means } x_{t,0} \text{ is constantly 0 past } |t|) \text{ and}$$

$$x_{t,i}(n) = \begin{cases} t(n) & \text{if } n \in |t| \\ 0 & \text{if } n \notin \{h_0(m)+j;\ m \in \omega \text{ and } j < k\} \cup |t| \\ s_{i'}(j) & \text{if } n \text{ is of the form } h_{i'}(m)+j;\ i' < i-1 \text{ and } j < k, \\ & h_{i'}(m) \notin \mathrm{ran}(h_{i'+1}), \text{ and } n \notin |t| \\ s_{i-1}(j) & \text{if } n \text{ is of the form } h_{i-1}(m)+j, j < k, \text{ and } n \notin |t| \end{cases}$$

for $i > 0$. So $x = x_{\langle\rangle,\ell}$. Moreover, the $x_{t,i}$ can be thought of as approximations to x with initial segment t in the intermediate models M_i.

Fix a predictor $\pi \in M_0$. In M_i, $i < \ell$, define $g_i \in \omega^\omega$ by

$$\begin{aligned} g_0(n) &= \min\{m;\ \text{ for all } t \in 2^n\text{: if there is } m' \geq n \text{ such that} \\ &\qquad \pi(x_{t,0}{\restriction}m'+j) \neq x_{t,0}(m'+j) \text{ for all } j < k, \text{ then } m > m'+k\} \text{ and} \\ g_i(n) &= \min\{m;\ \text{ for all } t \in 2^n\text{: if there is } m' \in \mathrm{ran}(h_{i-1}),\ m' \geq n, \text{ such that} \\ &\qquad \pi(x_{t,i}{\restriction}m'+j) \neq x_{t,i}(m'+j) \text{ for all } j < k, \text{ then } m > m'+k\} \end{aligned}$$

for $i > 0$. We digress briefly on why models are useful in this proof. In a minute we will use that $g_i \in M_i$ and that f_i is dominating over M_i to infer that $g_i \leq^* f_i$. If we had not chosen f_i and M_i in this fashion a priori, we would have had to choose f_0 only after getting g_0, then construct the $x_{t,1}$ and g_1, then choose f_1, etc... This is rather cumbersome, and this is what we meant at the beginning of this section by "not caring a priori".

Now, there is n_0 such that for all $i < \ell$ and all $n \geq n_0$ we have $f_i(n) > g_i(n+k)$. The following is clear from the way things were set up.

Claim 2.13 *For all $i < \ell$, all $n, n' > n_0$, all $t \in 2^{n+k}$ such that n and n' are consecutive members of $\mathrm{ran}(h_i)$: if there is no $m' \in \mathrm{ran}(h_{i-1}) \cap [n+k, n'-k]$ ($m' \in [n+k, n'-k]$ in case $i = 0$) such that $\pi(x_{t,i}{\restriction}m'+j) \neq x_{t,i}(m'+j)$ for all j, then it's not true that $\pi(x_{t,i}{\restriction}n'+j) \neq x_{t,i}(n'+j)$ for all j.*

Proof. If n, n' are consecutive members of $\mathrm{ran}(h_i)$, we must have $n' = f_i(n)$. Since $g_i(n+k) < f_i(n)$, the claim follows. □

Put $s_{-1} = 0$ (the sequence in 2^k with constant value 0).

Claim 2.14 *For all i, all $n, n' > n_0$, all t as in Claim 2.13: if there is no $m' \in [n+k, n'-k]$ such that $\pi(x_{t,i}{\restriction}m'+j) \neq x_{t,i}(m'+j)$ for all j, then for all $i' < i$, it's not true that $\pi(x_{t,i}{\restriction}n'^\frown s_{i'}{\restriction}j) \neq (x_{t,i}{\restriction}n'^\frown s_{i'})(n'+j)$ for all j.*

Proof. We make induction on i: the case $i = 0$ is clear from Claim 2.13.

$i \longrightarrow i+1$. n and n' are consecutive members of $\mathrm{ran}(h_{i+1})$. So there is $n^* \geq n$ such that n^* and n' are consecutive members of $\mathrm{ran}(h_i)$. Let $t^* := x_{t,i+1}{\restriction}n^*+k \in 2^{n^*+k}$. Note that $x_{t^*,i}{\restriction}n' = x_{t,i+1}{\restriction}n'$. So we may apply the induction hypothesis to get the conclusion of the claim for all $i' < i$. The case $i' = i$, however, follows from Claim 2.13 (for $i+1$).

Applying Claim 2.14 to $i = \ell - 1$, we see that if $n, n' > n_0$ are consecutive members of $\mathrm{ran}(h_{\ell-1})$ and $t \in 2^{n+k}$, then there is $m' \in [n+k, n']$ such that $\pi(x_{t,\ell}\restriction m'+j) \neq x_{t,\ell}(m'+j)$ for all j. (Using that $x_{t,\ell}\restriction n' = x_{t,\ell-1}\restriction n'$, we see that if there is no $m' \in [n+k, n'-k]$ with this property, then, by the claim, $\pi(x_{t,\ell-1}\restriction n'^\frown s_{\ell-1}\restriction j) \neq (x_{t,\ell-1}\restriction n'^\frown s_{\ell-1})(n'+j)$ for all j. However, $x_{t,\ell-1}\restriction n'^\frown s_{\ell-1}\restriction k = x_{t,\ell}\restriction n'+k$.) This completes the proof of the theorem.
□

Before proceeding we mention that this result is optimal.

Theorem 2.15 [B3] *Fix $\ell \in \omega$. Denote by $\mathbb{L}_\ell$ the ℓ–stage iteration of Laver forcing $\mathbb{L}$. Let G_ℓ be $\mathbb{L}_\ell$–generic over V, and let $x \in 2^\omega \cap V[G_\ell]$. Given k with $\ell < 2^k - 1$, there is a predictor $\pi : 2^{<\omega} \to 2$ in V which k–constantly predicts x.*

It is thus clear that if we want to solve Kamo's question by showing $\mathfrak{b} \leq \mathfrak{v}_2^{\mathrm{const}}$ in ZFC, we need at least ω many dominating reals corresponding to an ω–sequence of models. In fact, the following holds.

Lemma 2.16 [B3] *Assume there are ZFC–models $M_0 \subset M_1 \subset \ldots \subset M_i \subset \ldots$ and reals $f_0, \ldots, f_i, \ldots \in \omega^\omega$ such that $f_i \in M_{i+1}$ is dominating over M_i. Also assume $N_0 \subset N_1$ are ZFC–models containing $\langle M_i;\ i \in \omega\rangle, \langle f_i;\ i \in \omega\rangle$ and $f \in N_1$ is dominating over N_0. Then there is $x \in 2^\omega \cap N_1$ which is not constantly predicted by any predictor from M_0.*

Thus we infer

Theorem 2.17 [B3] $\mathfrak{b} \leq \mathfrak{v}_2^{\mathrm{const}}$.

Proof. For indeed, if we had $\mathfrak{v}_2^{\mathrm{const}} < \mathfrak{b}$, we could find first a model M_0 of size $\mathfrak{v}_2^{\mathrm{const}}$, and then M_i $(i > 0)$, f_i, N_0, N_1, and f which satisfy the hypotheses of the previous lemma. Thus we reach a contradiction. □

So we get the following "local" diagram for $\mathfrak{v}_2^{\mathrm{const}}$.

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[r, dash] & \max\{\mathfrak{b}, \mathsf{cov}(\mathcal{N})\} \arrow[r, dash] \arrow[d, dash] & \mathfrak{v}_2^{\mathrm{const}} \arrow[r, dash] \arrow[d, dash] & \mathfrak{c} \\
 & \mathfrak{b} & \mathsf{cov}(\mathcal{M}) &
\end{tikzcd}$$

One more comment concerning $\mathfrak{b}$ and $\mathfrak{v}_2^{\mathrm{const}}$ is in order. Shortly before we obtained our result outlined above, Kamo (unpublished) proved that an ω–stage iteration of Laver forcing adjoins $x \in 2^\omega$ which is not constantly predicted by any predictor from the ground model. This shows that $\mathfrak{v}_2^{\mathrm{const}} = \aleph_2$ after adding ω_2 Laver reals with countable support over a model for CH. This was strong evidence, and also an incentive, for our 2.16 and 2.17. For Zapletal [Z] has proved, assuming a proper class of measurable Woodin cardinals, that the iterated Laver model is a minimal model for $\mathfrak{b}$ in the sense that whenever a cardinal invariant $\mathfrak{i}$ with a reasonably easy definition has value $\aleph_2$ in that model, then $\mathfrak{b} \leq \mathfrak{i}$ is provable. Now, $\mathfrak{v}_2^{\mathrm{const}}$ indeed falls into Zapletal's framework. However, our result does not follow from Kamo's and Zapletal's work because the latter uses a large cardinal assumption while ours is in ZFC alone. Moreover, it turns out our proof of 2.17 is much simpler than Kamo's argument referred to above.

As further reading which contains most of the material of this section with proofs we recommend [Bl2] and [Ba], two survey articles written for the *Handbook on set theory* which is scheduled to appear on Sankt–Nimmerleins–Tag.

3 An example from algebra

Much of the original motivation for studying cardinal invariants came from set–theoretic topology. In fact, the two standard survey articles on cardinal invariants which have been published so far appeared in topology books. Even though somewhat outdated, they are still good introductions to the subject, see [vD] and [Va]. In the meantime, cardinal invariants have found many more applications, in particular in topology and algebra.

We shall give one example the choice of which is determined by our wish to illustrate two quite distinct phenomena at the same time. The first of these is fairly common and can be roughly described as follows.

> *If a certain cardinal invariant is small ($\aleph_1$)/large ($\mathfrak{c}$), then a topological/algebraic object of a certain kind exists/does not exist.*

Of course, one ideally looks for *iff* statements (that is, characterizations of the (non)existence of certain objects in terms of certain cardinal invariants assuming certain values), but this is often difficult, if not impossible, to achieve.

Let $\mathbb{K}$ be an at most countable field. Assume E is an uncountable–dimensional vector space over $\mathbb{K}$, and $\Phi : E^2 \to \mathbb{K}$ is a symmetric bilinear form (i.e. $\Phi(a,b) = \Phi(b,a)$ for all $a, b \in E$ and Φ is linear in both coordinates). (E, Φ) is called a *Gross space* if for any infinite–dimensional subspace $U \leq E$, the orthogonal complement $U^\perp = \{x;\ \Phi(x,y) = 0$ for all $x \in U\}$ has dimension less than $\dim(E)$. In case one even has that $\dim(U^\perp)$ is at most countable for all such U, (E, Φ) is called a *strong Gross space.* (Of course, the two notions agree in case $\dim(E) = \aleph_1$.) So a Gross space is rather different from Hilbert space.

Let us introduce the cardinal invariant which plays the main role in the study of Gross spaces. Let D be an infinite subset of ω. A function $\varphi : D \to [\omega]^{<\omega}$ with $|\varphi(n)| \leq n$ for all $n \in D$ is called a *slalom.* Given $f \in \omega^\omega$ and a slalom φ with domain D, we say that φ *localizes* f if $f(n) \in \varphi(n)$ holds for all but finitely many $n \in D$. The *linear evasion number* $\mathfrak{e}_\ell$ is the size of the least $\mathcal{F} \subseteq \omega^\omega$ such that no slalom localizes all members of $\mathcal{F}$. By Bartoszyński's characterization of $\mathsf{add}(\mathcal{N})$ [BaJ], the size of the least $\mathcal{F} \subseteq \omega^\omega$ such that no slalom *with domain* $D = \omega$ localizes all members of $\mathcal{F}$ is equal to $\mathsf{add}(\mathcal{N})$. Moreover, the standard proof of the Bartoszyński–Raisonnier–Stern Theorem 1.1 uses this combinatorial characterization, and it turns out the same proof still workes with the present concept of slalom (with variable D). Therefore $\mathfrak{e}_\ell$ sits between the two additivities [Bl1], i.e. we get the following "local" diagram for $\mathfrak{e}_\ell$.

$$\begin{tikzcd}[arrows=-] \mathsf{add}(\mathcal{M}) \arrow[d] \\ \mathfrak{e}_\ell \arrow[d] \\ \mathsf{add}(\mathcal{N}) \arrow[d] \\ \aleph_1 \end{tikzcd}$$

We are ready to characterize existence of strong Gross spaces over countable fields.

Theorem 3.1 [B4] *The following are equivalent.*

(i) *There is a strong Gross space over some countable field.*

(ii) *There is a strong Gross space over every countable field.*

(iii) $\mathfrak{e}_\ell = \aleph_1$.

Sketch of proof. To give a rough idea of the proof we need some "intermediate combinatorics". Given an at most countable field $\mathbb{K}$ and $D \subseteq \omega$ infinite, call $\pi = \langle \pi_n;\ n \in D \rangle$ a *predictor* if $\pi_n : \mathbb{K}^n \to \mathbb{K}$ is $\mathbb{K}$–linear for all $n \in D$. (Note the connection to the concept of "prediction" discussed in Section 2 above.) Such a π *predicts* $f \in \mathbb{K}^\omega$ if $f(n) = \pi_n(f{\restriction}n)$ holds for all but finitely many $n \in D$. If π does not predict f we say that f *evades* π. The $\mathbb{K}$*–linear evasion number* $\mathfrak{e}_{\mathbb{K}}$ is the cardinality of the smallest $\mathcal{F} \subseteq \mathbb{K}^\omega$ such that every predictor is evaded by some member of $\mathcal{F}$. Then $\mathfrak{e}_{\mathbb{K}} = \mathfrak{e}_\ell$ holds for any countable field $\mathbb{K}$ [BS]. In fact, for $\mathbb{K} = \mathbb{Q}$, this was the original definition of $\mathfrak{e}_\ell$ [Bl1]. This is the reason for its name, "linear evasion number". $\mathfrak{e}_{\mathbb{K}} \geq \mathfrak{e}_\ell$ is also true for finite fields. Yet, for finite fields, $\mathfrak{e}_{\mathbb{K}}$ may be strictly larger than $\mathfrak{e}_\ell$ [B1]. In fact, it is consistently even larger than the dominating number $\mathfrak{d}$.

Say an uncountable–dimensional subspace $E \leq \mathbb{K}^\omega$ is a $\mathbb{K}$*–Luzin space* if every predictor π predicts at most countably many members of E. Now, for countable $\mathbb{K}$, $\mathfrak{e}_{\mathbb{K}} = \aleph_1$ is equivalent to the existence of a $\mathbb{K}$–Luzin space. Of course, the reverse direction is trivial and holds for finite fields as well. The forward direction, however, is the main point of the proof of 3.1, see [B4]. As shown in [B1], it is consistently false for finite fields. This should be seen as saying that there is no reasonable characterization of the existence of strong Gross spaces over finite fields in terms of cardinal invariants.

Finally, for any at most countable field $\mathbb{K}$, the existence of a $\mathbb{K}$–Luzin space is equivalent to the existence of a strong Gross space over $\mathbb{K}$ [B1]. Again, the reverse direction is rather straightforward. In fact, a similar argument works to derive the existence of a generalized $\mathbb{K}$–Luzin space from a Gross space. The forward direction, however, uses the combinatorics of ω_1 in a crucial way. □

Let us now turn to the second phenomenon which can be described roughly as follows.

Cardinal invariants may be a useful tool to prove something which doesn't depend on the values of cardinal invariants at all. E.g. they may be used to split the proof of an "absolute" result into cases, according to the value of a certain cardinal invariant.

As an illustration we address

Question 3.2 *Do Gross spaces exist?*

An old result of Gross says that they exist under the continuum hypothesis CH (of course, this also follows from 3.1 above for countable fields). On the negative side one has

Theorem 3.3 (Shelah–Spinas [SSp]) *It is consistent that $\mathfrak{c} = \aleph_2$ and there are no Gross spaces over any finite field.*

This is proved by a countable support iteration of proper forcing (see below, Section 4, for more on forcing). With a very complicated new iteration technique involving "mixed support", Shelah got

Theorem 3.4 (Shelah [S2]) *It is consistent that $\mathfrak{c} = \aleph_3$ (or larger) and there are no Gross spaces over any (at most countable) field.*

See below (Theorem 4.12) for a more explicit formulation of this result.

But what about $\mathfrak{c} = \aleph_2$? It should be remarked here that, apart from trivial exceptions, almost all combinatorial problems on the reals which are independent of ZFC are also independent of $ZFC + \mathfrak{c} = \aleph_2$. However, this is not true in our situation.

Theorem 3.5 [B4] $\mathfrak{c} = \aleph_2$ *implies the existence of a Gross space over every countable field.*

Sketch of proof. The proof splits into two cases according to the pattern sketched above.

Case 1. $\mathfrak{e}_\ell = \aleph_1$. Then use Theorem 3.1: there are strong Gross spaces (in particular, Gross spaces of dimension $\aleph_1$) over every countable field.

Case 2. $\mathfrak{e}_\ell = \mathfrak{c}$. One argues in two steps that this implies that Martin's axiom MA holds for any Cohen algebra: first show that $\mathbb{C}_\mathfrak{c}$, the forcing for adding $\mathfrak{c}$ many Cohen reals, completely embeds into $\mathbb{P}$ [B4], the standard σ–centered forcing for making $\mathfrak{e}_\ell$ large (this is like the proof of Theorem 2.5 mentioned above); then show that $\mathfrak{e}_\ell = \mathfrak{m}(\mathbb{P})$ [B4]. (Of course, $\mathfrak{e}_\ell \geq \mathfrak{m}(\mathbb{P})$ is obvious. What we need here, however, is $\mathfrak{e}_\ell \leq \mathfrak{m}(\mathbb{P})$. This is proved by showing that, given a model M of ZFC of size less than $\mathfrak{e}_\ell$, one can reconstruct a $\mathbb{P}$–generic filter over M using a slalom localizing all members of M, as well as a Cohen real and dominating reals (over M). The latter is OK because $\mathfrak{e}_\ell \leq \mathfrak{b}, \mathsf{cov}(\mathcal{M})$ in ZFC.) Taken together, the two results imply that $\mathfrak{e}_\ell \leq \mathfrak{m}(\mathbb{C}_\mathfrak{c})$. By $\mathfrak{e}_\ell = \mathfrak{c}$, MA holds for $\mathbb{C}_\mathfrak{c}$.

By an old result of Baumgartner and Spinas [BSp], saying that MA for $\mathbb{C}_\mathfrak{c}$ implies the existence of Gross spaces of dimension $\mathfrak{c}$ over any at most countable field, we're done.

As mentioned already, unlike for the first phenomenon, there seem to be few other examples for the second phenomenon so far. More specifically, there seem to be few non–trivial problems which are decided by $\mathfrak{c} = \aleph_2$ but independent of $\mathfrak{c} \geq \aleph_3$. See [Ko] for one example. Let us briefly discuss two cases which are good candidates though independence from $\mathfrak{c} \geq \aleph_3$ has not been shown yet in either case.

Let $A, B \in [\omega]^\omega$. We say that A is *almost included* or *almost contained* in B (and write $A \subseteq^* B$) if $A \setminus B$ is finite. Given a filter base $\mathcal{F} \subseteq [\omega]^\omega$, call $A \in [\omega]^\omega$ a *pseudointersection* of $\mathcal{F}$ if $A \subseteq^* X$ for all $X \in \mathcal{F}$. The *pseudointersection number* $\mathfrak{p}$ is the size of the least filter base $\mathcal{F} \subseteq [\omega]^\omega$ which has no pseudointersection. $\mathfrak{p}$ can be characterized in forcing language (see Section 2).

Theorem 3.6 (Bell's Theorem, see [BaJ] or [Bl2]) $\mathfrak{p} = \mathfrak{m}(\sigma - \text{centered})$.

Let κ be a cardinal. Call a sequence $\mathcal{T} = \langle T_\alpha;\ \alpha < \kappa \rangle$ a *tower of height* κ if it is decreasing modulo almost inclusion and has no pseudointersection. The *tower number* $\mathfrak{t}$ is the least κ such that there is a tower of height κ. $\mathfrak{p} \leq \mathfrak{t}$ is immediate from the definition. The Piotrowski–Szymański Theorem [BaJ] says that $\mathfrak{t} \leq \mathsf{add}(\mathcal{M})$. In fact $\mathfrak{t} \leq \mathfrak{e}_\ell$ [La]. So we get the following "local diagram" for $\mathfrak{p}$ and $\mathfrak{t}$.

$$\begin{tikzcd} \mathsf{add}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathfrak{e}_\ell \arrow[r, dash] \arrow[d, dash] & \mathsf{add}(\mathcal{M}) \\ & \mathfrak{t} \arrow[d, dash] & \\ \aleph_1 \arrow[r, dash] & \mathfrak{p} & \end{tikzcd}$$

Theorem 3.7 [Bl2] *Assume* $\mathfrak{c} = \aleph_2$. *Then* $\mathfrak{p} = \mathfrak{t}$.

Sketch of proof. Split into two cases according to the value of $\mathfrak{d}$.

Case 1. $\mathfrak{d} = \aleph_1$. Since $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{d}$ in ZFC, we're done.

Case 2. $\mathfrak{d} = \mathfrak{c}$. There is a well–known combinatorial lemma to the effect that given $\mathcal{G} \subseteq [\omega]^\omega$ of size less than $\mathfrak{d}$ as well as a countable filter base $\mathcal{F}$ such that $X \cap Y$ is infinite for all $X \in \mathcal{F}$ and $Y \in \mathcal{G}$, there is a pseudointersection A of $\mathcal{F}$ which still has infinite intersection with all members of $\mathcal{G}$. Now assume $\mathfrak{t} = \aleph_2$. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a filter base of size $\aleph_1$. Using the lemma one can recursively construct a decreasing sequence $\mathcal{T}$ of length ω_1 below $\mathcal{F}$ (that is, for all $X \in \mathcal{F}$ there is $Y \in \mathcal{G}$ almost contained in X). By $\mathfrak{t} = \aleph_2$, $\mathcal{T}$ is not a tower. Ergo $\mathcal{F}$ has a pseudointersection, and we're done. $\square$

Note the above proof works more generally under the assumption $\mathfrak{p} = \aleph_1$. The consistency of $\mathfrak{p} < \mathfrak{t}$ is still open (see Problem 5.1).

We turn to the second example. Recall that an ultrafilter $\mathcal{U}$ on ω is called a *P–point* if any countable $\mathcal{F} \subseteq \mathcal{U}$ has a pseudointersection belonging to $\mathcal{U}$. $\mathcal{U}$ is a *Q–point* if given any partition $\langle X_n;\ n \in \omega \rangle$ of ω into finite sets, there is a *selector* A belonging to $\mathcal{U}$, i.e. $A \cap X_n$ has at most one element for all n. Shelah proved it is consistent there are no P–points and Miller showed it is consistent there are no Q–points (see [BaJ] for both proofs). In both models $\mathfrak{c} = \aleph_2$ holds. However

Theorem 3.8 [BaJ] *Assume $\mathfrak{c} = \aleph_2$. Then either there is a P–point or there is a Q–point.*

Sketch of proof. Again we split into two cases according to the value of $\mathfrak{d}$.

Case 1. $\mathfrak{d} = \aleph_1$. Then $\mathfrak{d} = \mathsf{cov}(\mathcal{M})$, and we can use Canjar's Theorem [BaJ] saying that under the latter assumption every filter base of size less than $\mathsf{cov}(\mathcal{M})$ can be extended to a Q–point.

Case 2. $\mathfrak{d} = \mathfrak{c}$. Then we use Ketonen's Theorem [BaJ] saying that under this assumption every filter base of size less than $\mathfrak{d}$ can be extended to a P–point. □

Again, the simultaneous consistency of the non–existence of P–points and Q–points is still open (see Problem 5.8).

4 Cardinal invariants and iterated forcing

Forcing, invented by Cohen to prove the independence of CH, was soon transformed into a general and powerful technique to obtain a plethora of independence results. In particular, Solovay and Tennenbaum developed iterated forcing to show the consistency of Martin's axiom MA and, a fortiori, Suslin's hypothesis.

In our context, using this technique, quite a number of statements of the form $\mathfrak{i} < \mathfrak{j}$ have been proved to be consistent where $\mathfrak{i}$ and $\mathfrak{j}$ are cardinal invariants. For example $\mathsf{add}(\mathcal{N})$ is consistently less than $\mathsf{add}(\mathcal{M})$ and, in fact, any of the inequalities between cardinal invariants shown in the diagrams in the preceding sections may be strict except for the restriction imposed by the Miller–Truss Theorem 1.2 and the still open $\mathfrak{p}$–versus–$\mathfrak{t}$ problem 5.1 (see [BaJ] for most of the consistency proofs). However, though many consistency results have been proved concerning combinatorial properties of the reals, this has been done almost exclusively with two very specific iteration techniques which describe only a small fraction of what an iteration is in general, namely

(1) *finite support iteration of ccc forcing*, the technique originally developed by Solovay and Tennenbaum,

(2) *countable support iteration of proper forcing*, created by Shelah [S1].

Both approaches are well–understood by now, the latter being much more complicated as well as more powerful.

In fact, whenever two cardinals have been shown to be consistently different under $\mathfrak{c} = \aleph_2$, this could be proved as well using a countable support iteration of proper forcing of length ω_2 over a model of CH though the latter proof may be more complicated than the original one. There is a general reason for this, see Zapletal's work [Z] which we mentioned at the end of Section 2. To enjoy countable support iteration in its full baroque splendor see [RS].

On the other hand, one of the nice features of finite support iterations is that consistency results usually dualize in a fairly straightforward manner. For example, as mentioned in Section 2, $\mathsf{cof}(\mathcal{N}) \geq \mathsf{cof}(\mathcal{M})$ is dual to $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$, but likewise,

$CON(\mathsf{cof}(\mathcal{N}) > \mathsf{cof}(\mathcal{M}))$ is dual to $CON(\mathsf{add}(\mathcal{N}) < \mathsf{add}(\mathcal{M}))$. The latter proof proceeds by a long finite support iteration of Hechler forcing over a model of CH while the former uses a short finite support iteration of the same forcing over a model of MA.

Both methods have rather obvious flaws. An iteration of the first kind (of length at least the value of the continuum in the extension) must force Martin's axiom MA for any Cohen algebra (that is, $\mathfrak{m}(\mathbb{C}_\mathfrak{c}) = \mathfrak{c}$), and thus in particular, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. (A shorter iteration, say of length $\kappa \geq \aleph_1$, still forces $\mathfrak{m}(\mathbb{C}_\kappa) \geq \kappa$, and thus $\mathsf{cov}(\mathcal{M}) \geq \kappa$, yet $\mathsf{non}(\mathcal{M}) \leq \kappa$.) The second technique necessarily forces $\mathfrak{c} \leq \aleph_2$. So there are a number of natural problems for which none of the techniques can work.

Of course, there are alternative constructions. One may use a large measure algebra (adding many random reals) or force with a large (countable support) product. The latter approach has been used quite successfully to obtain models where many cardinal invariants assume different values, see e.g. [GS]. Yet, this works only for a rather small class of invariants, namely those which can be increased by forcing with compact infinitely often branching trees, a fortiori only for cardinals which are consistently larger than $\mathfrak{d}$. Sometimes one may also use an

(3) *iterated forcing construction with mixed support*

but, unlike the first two methods, this one is not very well understood yet. It consists merely of a number of consistency results, due mainly to Shelah, in part in joint work (see for example [FSS], [DS1], and [S2]), and scattered over the literature. Below, we shall give an example illustrating both how this method works and why it is needed.

However, let us first mention a relatively recent technique, namely

(4) *non–well–founded iteration (of reasonably definable ccc forcing)*

developed by Shelah to solve a long–standing open problem on cardinal invariants by showing

Theorem 4.1 (Shelah [S4]) *It is consistent that* $\mathfrak{a} < \mathfrak{d}$.

Here, $\mathfrak{a}$ denotes the *almost disjointness number.* A family $\mathcal{A} \subseteq [\omega]^\omega$ is called an *almost disjoint family* (a.d. family) if the intersection of any two members of $\mathcal{A}$ is finite. $\mathcal{A}$ is a *mad family* (maximal almost disjoint family) if it is almost disjoint and cannot be extended to a strictly larger a.d. family. $\mathfrak{a}$ is the size of the smallest mad family. $\mathfrak{a}$ is well–known to be larger or equal than $\mathfrak{b}$, and it has been known for a while there were no other lower bounds, $\mathfrak{a}$ being consistently smaller than any cardinal which is not provably below $\mathfrak{b}$. However, concerning upper bounds, not much had been known, and Shelah's result, and its variations for other cardinals, can be construed as saying essentially there are none except $\mathfrak{c}$. So we get the following "local diagram" for $\mathfrak{a}$.

$$\begin{array}{c} \mathfrak{c} \\ | \\ \mathfrak{a} \\ | \\ \mathfrak{b} \end{array}$$

More specifically, Shelah proved

Theorem 4.2 (Shelah [S4])

(i) *Assume κ is measurable, and $\kappa < \mu < \lambda$ are regular. Then there is a ccc p.o. forcing that $\mu = \mathfrak{b} = \mathfrak{d}$ and $\mathfrak{a} = \mathfrak{c} = \lambda$.*

(ii) *The same conclusion is true, assuming only the consistency of ZFC, for any triple of uncountable regular cardinals $\kappa < \mu < \lambda$. For example, $\mathfrak{b} = \mathfrak{d} = \aleph_2$ and $\mathfrak{a} = \mathfrak{c} = \aleph_3$ is consistent (assuming the consistency of ZFC).*

Sketch of proof. First add μ Hechler reals in a finite support iteration of length μ. Call this forcing $\mathbb{P}$. It adds a μ–scale (that is, it forces $\mathfrak{b} = \mathfrak{d} = \mu$). Now consider $\mathbb{P}^\kappa/\mathcal{U}$ where $\mathcal{U}$ is an ultrafilter on the measurable cardinal κ (in case (i)). This is still a ccc forcing. $\mathbb{P}$ completely embeds into $\mathbb{P}^\kappa/\mathcal{U}$ so that we may think of $\mathbb{P}^\kappa/\mathcal{U}$ as a two–step iteration. An analysis of $\mathbb{P}^\kappa/\mathcal{U}$–names shows that the scale added by $\mathbb{P}$ is preserved, while every a.d. family of the intermediate extension is not maximal in the extension via $\mathbb{P}^\kappa/\mathcal{U}$. (The latter uses that $\mathfrak{a} > \kappa$ in the intermediate model).

One now tries to iterate this procedure of taking ultrapowers for λ many steps. The main problem occurs in limits of of countable cofinality, and this is where the new method, non–well–founded iteration comes in.

In case (ii), taking ultrapowers is replaced by repeatedly inserting new coordinates in the forcing $\mathbb{P}$ and then using an isomorphism–of–names argument.

In fact, the proof shows more than just the consistency of $\mathfrak{a} > \mathfrak{d}$. By a straightforward modification one gets e.g. $CON(\mathfrak{a} > \mathsf{cof}(\mathcal{N}))$. Furthermore, case (i) also works for other invariants like the ultrafilter number $\mathfrak{u}$ and the independence number $\mathfrak{i}$. See [S4] for details. $\square$

Using the same method Shelah also got

Theorem 4.3 (Shelah [S4]) *$\mathfrak{a}$ is consistently singular (of uncountable cofinality).*

As promised we now turn to mixed support iterations. Recall the combinatorial principle $\clubsuit$ ("club") says there is a sequence $\langle A_\alpha;\ A_\alpha \subseteq \alpha \text{ is cofinal}, \alpha < \omega_1 \text{ is a limit}\rangle$ such that for all uncountable $A \subseteq \omega_1$ there is α with $A_\alpha \subseteq A$. If we require only that $A_\alpha \subseteq \alpha$ be countable we get the weaker principle $\dagger$ ("stick"). Of course, this can be reformulated as saying there is a family $\mathcal{F} \subseteq [\omega_1]^\omega$ such that for all $A \in [\omega_1]^{\omega_1}$ there is $B \in \mathcal{F}$ contained in A. It is well–known that $\diamondsuit$ is equivalent to $\clubsuit + CH$ while $CH + \neg\clubsuit$ (Jensen) and $\clubsuit + \neg CH$ (Shelah) are both consistent. Furthermore, $\dagger$ is a trivial consequence of both CH and $\clubsuit$, while $\dagger + \neg CH + \neg\clubsuit$ [DS1] is consistent. We want to address questions of the following pattern.

Question 4.4 *Let $\mathfrak{i}$ be a cardinal invariant. Is $\clubsuit + \mathfrak{i} > \aleph_1$ consistent?*

To this end first note

Fact 4.5 (i) *$\mathfrak{m}(\mathbb{C}_{\omega_1}) \geq \aleph_2$ implies $\neg\dagger$, a fortiori $\neg\clubsuit$. Similarly for $\mathbb{B}_{\omega_1}$.*

(ii) *$\mathfrak{e}_\ell \geq \aleph_2$ implies $\neg\dagger$, a fortiori $\neg\clubsuit$. Similarly for $\mathsf{add}(\mathcal{N})$, $\mathfrak{p}$, and even $\mathfrak{t}$.*

Proof. (i) Assume $\mathcal{F}$ is a witness for stick. So $|\mathcal{F}| = \aleph_1$. Let M be a model of a large enough fragment of ZFC, still of size $\aleph_1$ and containing $\mathcal{F}$. By $\mathfrak{m}(\mathbb{C}_{\omega_1}) \geq \aleph_2$, there is a $\mathbb{C}_{\omega_1}$–generic filter over M. From this filter, we can reconstruct an uncountable subset of ω_1 which contains no countable set of M, contradicting $M \cap [\omega_1]^\omega$ is a witness for $\dagger$.

Use the same argument for the measure algebra.

(ii) For $\mathfrak{e}_\ell$, use $\mathfrak{e}_\ell \leq \mathfrak{m}(\mathbb{C}_\mathfrak{c})$, see the proof of 3.5. Then use $\mathsf{add}(\mathcal{N}), \mathfrak{p}, \mathfrak{t} \leq \mathfrak{e}_\ell$. Or see Theorem 2.6 for $\mathsf{add}(\mathcal{N})$, and use Observation 2.4 and Bell's Theorem 3.6 for $\mathfrak{p}$. For $\mathsf{add}(\mathcal{N})$, there is an alternative argument: Truss [T] proved $\min\{\mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N})\} \geq \aleph_2$ implies the failure of $\dagger$. $\square$

In the other direction we have

Theorem 4.6 (Fuchino, Shelah and Soukup [FSS]) *Assume CH. Let $\kappa = \kappa^\omega$. Then there is a cardinal–preserving extension satisfying* $\mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \kappa$ *and* $\clubsuit$.

This is proved by adding many Cohen reals with a mixed support product. Why are other constructions not possible? Let us note first that finite support products and iterations add generics for large Cohen algebras, and are thus excluded by the argument above (Fact 4.5). A countable support product of Cohen forcing collapses the continuum. The case of countable support iterations is less clear, that is, we do not know whether they can preserve $\clubsuit$, but in any case, as mentioned above, they yield only models with $\mathfrak{c} \leq \aleph_2$.

Now, how does "mixed support" work? As mentioned above, we subsume under this name several constructions which do not fit well into a single framework (yet??), but all work roughly as follows: conditions have two components, we use finite support on one and countable support on the other. The order is σ–closed on the countable pieces so that a pressing–down argument shows the p.o. is proper. Moreover, if we assume CH and work either with a product (as e.g. in [FSS]) or restrict the countable piece in the iteration to something from the ground model (as e.g. in [S2] or [B2]) or consider only iterations of length ω_2 (as e.g. in [DS2]), we will have the $\aleph_2$–cc. So cardinals are preserved. Finally an elaboration of the pressing–down argument showing properness usually gives combinatorial principles like $\clubsuit$. As an illustration consider

Theorem 4.7 [B4] *Assume CH. Let $\kappa = \kappa^\omega$. Then there is a cardinal–preserving extension satisfying* $\mathsf{cov}(\mathcal{N}) = \mathfrak{c} = \kappa$ *and* $\clubsuit$.

Sketch of proof. Let $\kappa > \aleph_1$ be a cardinal and let $C \subseteq \kappa$ be non–empty. Given a partial function $s : C \to \omega$, let $a_s = \{(\alpha, n);\ \alpha \in \mathrm{dom}(s) \text{ and } n < s(\alpha)\}$ and $b_s = (\mathrm{dom}(s) \times \omega) \setminus a_s$.

The p.o. $\mathbb{P}_C$ consists of all triples (s, f, A) such that $s : C \to \omega$ is an at most countable partial function, $f : a_s \to 2$, and $A \subseteq 2^{\mathrm{dom}(s)\times\omega}$ is Borel with $f \subseteq x$ for all $x \in A$ and such that A has positive measure as a subset of the space 2^{b_s} (which is equipped with the product measure as usual). We let $(s, f, A) \leq (t, g, B)$ iff $\mathrm{dom}(s) \supseteq \mathrm{dom}(t)$, $s(\alpha) \geq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t)$, $s(\alpha) = t(\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(t)$, $f \supseteq g$ (that is $f \restriction a_t = g$) and $x \restriction (\mathrm{dom}(t) \times \omega) \in B$ for all $x \in A$. (In particular, $\{x \restriction (\mathrm{dom}(t) \times \omega);\ x \in A\} \subseteq B$ still has positive measure inside 2^{b_t}.) The following is easy to show.

Fact 4.8 (i) *$\mathbb{P}_C$ is $\aleph_2$–cc.*

(ii) *$\mathbb{P}_\kappa$ forces* $\mathsf{cov}(\mathcal{N}) = \mathfrak{c} = \kappa$.

The main point for proving cardinals are preserved is

Lemma 4.9 (i) *Let $p = (s^p, f^p, A^p) \in \mathbb{P}_C$, and let $\dot\gamma$ be a $\mathbb{P}_C$–name for an ordinal. Then there is $q = (s^q, f^q, A^q) \le p$ such that $\{r \le q;\ \mathrm{dom}(s^r) = \mathrm{dom}(s^q)$ and r decides $\dot\gamma\}$ is predense below q.*

(ii) *$\mathbb{P}_C$ is proper.*

Proof. (i) Assume not, and construct recursively $\langle p_\zeta;\ \zeta < \omega_1\rangle$ (a sequence of conditions) and $\langle s_\zeta;\ \zeta < \omega_1\rangle$ (a sequence of countable partial functions $C \to \omega$) such that

(i) $p_\zeta \le p$, $s_\zeta \supseteq s^p$

(ii) $\zeta \le \xi$ implies $s_\zeta \subseteq s_\xi$

(iii) $\mathrm{dom}(s_\zeta) = \mathrm{dom}(s^{p_\zeta})$ and we have:
$s_\zeta(\alpha) \le s^{p_\zeta}(\alpha)$ for all $\alpha \in \mathrm{dom}(s_\zeta)$
$s_\zeta(\alpha) = s^{p_\zeta}(\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(s_\zeta)$
$s_\zeta(\alpha) = s^{p_\zeta}(\alpha)$ for all $\alpha \in \mathrm{dom}(s_\zeta) \setminus \bigcup_{\xi<\zeta} \mathrm{dom}(s_\xi)$

(iv) $\zeta \le \xi$ implies $f^{p_\zeta} \restriction a_{s_\zeta} = f^{p_\xi} \restriction a_{s_\zeta}$

(v) p_ζ decides $\dot\gamma$

(vi) all p_ζ are pairwise incompatible

This can be done easily: assume we are at step ζ of the construction. Put $t_\zeta = \bigcup_{\xi<\zeta} s_\xi$ and $g_\zeta = \bigcup_{\xi<\zeta} (f^{p_\xi} \restriction a_{s_\xi})$. Then let $q_\zeta = (t_\zeta, g_\zeta, B_\zeta)$ where $B_\zeta = \{x \in 2^{\mathrm{dom}(t_\zeta)\times\omega};\ g_\zeta \subseteq x$ and $x \restriction (\mathrm{dom}(s^p) \times \omega) \in A^p\}$. So $q_\zeta \le p$. By assumption, q_ζ does not satisfy the conclusion of the lemma. Therefore there is $p_\zeta \le q_\zeta$ deciding $\dot\gamma$ and incompatible with all p_ξ, $\xi < \zeta$. Now let $s_\zeta = t_\zeta \cup (s^{p_\zeta} \restriction \mathrm{dom}(s^{p_\zeta}) \setminus \mathrm{dom}(t_\zeta))$ and check clauses (i) to (vi) are satisfied.

A standard pressing–down argument (i.e. an application of Fodor's Lemma) now gives us a stationary $S \subseteq \omega_1$ and a partial function $h : C \to \omega$ with $\mathrm{dom}(h)$ of size at most $\aleph_1$ such that $s^{p_\zeta} \subseteq h$ for any $\zeta \in S$ (by (ii) and (iii)) and such that $f^{p_\zeta} \subseteq f^{p_\xi}$ for $\zeta \le \xi$ from S (by (iv)). The standard proof that $\mathbb{B}_{\omega_1}$ is ccc shows there are $\zeta \neq \xi$ both from S such that p_ζ and p_ξ are compatible, a contradiction.

(ii) is an easy consequence of (i). $\square$

The following lemma completes the proof of Theorem 4.7.

Main Lemma 4.10 (i) *If CH holds in the ground model V, then $\Vdash_{\mathbb{P}_\kappa} \dagger$.*

(ii) *If $\diamondsuit$ holds in V, then $\Vdash_{\mathbb{P}_\kappa} \clubsuit$.*

The proof of this is similar to the proof of Lemma 4.9 above but technically more involved and longer. □

Next, by retrieving as much as possible from the flawed [DS2], one can prove

Theorem 4.11 (Džamonja and Shelah [DS2]) *Assume CH. Then there is a cardinal–preserving extension satisfying* $\mathsf{add}(\mathcal{M}) = \mathfrak{c} = \aleph_2$ *and* $\clubsuit$.

Finally, let us give a rough idea of what Shelah did to prove the consistency of the non–existence of Gross spaces (Theorem 3.4 above). Let $S^2_0 = \{\alpha < \omega_2;\ cf(\alpha) = \omega\}$. $\clubsuit_{S^2_0}$ means there is a sequence $\langle A_\alpha;\ A_\alpha \subseteq \alpha$ cofinal, $\alpha \in S^2_0\rangle$ such that for every $A \subseteq \omega_2$ of size $\aleph_2$ there is α with $A_\alpha \subseteq A$.

Theorem 4.12 (Shelah [S2]) *It is consistent that*

(a) $\mathfrak{c} = \aleph_3$ *(or larger)*

(b) $\mathfrak{m}(\sigma-\text{centered}) = \aleph_2$

(c) *Given* $\mathcal{F} \subseteq \omega^\omega$ *of size* $\aleph_2$*, there is* $\mathcal{G} \subseteq \mathcal{F}$*, still of size* $\aleph_2$*, which is localized by a single slalom*

(d) $\clubsuit_{S^2_0}$

This is proved using a complicated mixed support iteration. In fact, this may arguably be the most sophisticated proof produced so far in iterated forcing theory, the definition of the iteration alone taking roughly seven pages.

To derive 3.4 from 4.12, note (b) implies $\mathfrak{e}_\ell \geq \aleph_2$, so by 3.1 there are no Gross spaces of dimension $\aleph_1$. (c) is a weakening of $\mathfrak{e}_\ell > \aleph_2$ still strong enough to get rid of Gross spaces of dimension $\aleph_2$ (by an argument analogous to the one in the proof of 3.1). In fact, by the argument in 3.5, $\mathfrak{e}_\ell = \aleph_2$ must hold in Shelah's model. Finally, $\clubsuit_{S^2_0}$ entails there are no Gross spaces of dimension $\mathfrak{c} = \aleph_3$.

The preceding discussion shows there is a two–way interplay between iterated forcing and cardinal invariants.

> *On the one hand, we need forcing to prove consistency results on the possible values of, and possible inequalities between, cardinal invariants. On the other hand, cardinal invariants provide us with test cases which might help us to either develop new iteration techniques or prove ZFC–results saying this is impossible.*

5 Open problems

We close our considerations with a selection of open problems on cardinal invariants, this selection being guided mainly by personal interest, but also by choosing problems which seem to require further development of new iteration techniques so that their solution may shed new light on the interplay between forcing and cardinal invariants. For more comprehensive problem lists, not only on cardinal invariants, see either [M2] (recent update available on www) or [S3].

(1) PROBLEMS FOR FSI. Though the method is well-understood, there are still a number of important problems which may be solvable by finite support iterations of ccc forcing, for it is sometimes very difficult to find the right iterands.

Problem 5.1 (van Douwen's problem [M2]) *Is* $\mathfrak{p} < \mathfrak{t}$ *consistent?*

See the end of Section 3 for a discussion of this.

Let $A, X \in [\omega]^\omega$. Say X *splits* (or: *reaps*) A if $X \cap A$ and $A \setminus X$ are both infinite. $\mathcal{F} \subseteq [\omega]^\omega$ is a *splitting family* if every $A \in [\omega]^\omega$ is split by a member of $\mathcal{F}$. The *splitting number* $\mathfrak{s}$ is the size of the least splitting family. It is well-known that $\mathfrak{s} \geq \mathfrak{t}$ and $\mathfrak{s} \leq \mathfrak{d}, \mathsf{non}(\mathcal{M}), \mathsf{non}(\mathcal{N})$ [B12], and there are a number of consistency results saying that $\mathfrak{s}$ may be smaller and/or larger than any other cardinal invariant unless this is forbidden by the aforementioned restrictions. So we get the following "local diagram" for $\mathfrak{s}$.

$$\begin{tikzcd}
\mathsf{non}(\mathcal{M}) & \mathfrak{d} & \\
\mathfrak{s} \arrow[u, dash] \arrow[r, dash] & \min\{\mathfrak{d}, \mathsf{non}(\mathcal{N})\} \arrow[u, dash] \arrow[r, dash] & \mathsf{non}(\mathcal{N}) \\
\mathfrak{t} \arrow[u, dash] & &
\end{tikzcd}$$

Furthermore, $\mathfrak{s}$ must have uncountable cofinality.

Problem 5.2 (Vaughan's problem [Va]) *Can* $\mathfrak{s}$ *be singular?*

(2) PROBLEMS FOR MIXED SUPPORT. Theorem 4.11 suggests the following problem.

Problem 5.3 *Is it consistent that* $\mathsf{add}(\mathcal{M}) > \aleph_2$ *yet* $\clubsuit$ *holds.*

For a failed attempt see [B2].

The following also seems to be a good candidate for "mixed support".

Problem 5.4 (Judah [M2]) *Let* V *be any model of set theory. Can one force* $\mathfrak{b} > \aleph_2$ *without adding Cohen reals over* V*?*

Note that this is easy for some models: add κ Hechler reals d_β ($\kappa > \omega_2$) with a finite support iteration over W. Then add ω_1 random reals r_α over $W[d_\beta;\ \beta < \kappa]$ with the measure algebra. Let $V = W[r_\alpha;\ \alpha < \omega_1]$ and note there are no reals Cohen over V in $V[d_\beta;\ \beta < \kappa]$. So the problem asks this for arbitrary models, e.g. for L.

(3) PROBLEMS AROUND $\mathfrak{a}$. It is conceivable that some of the problems below can be solved by the method of [S4], namely, non-well-founded iteration. First note that since Shelah only proved the consistency of $\mathfrak{d} < \mathfrak{a}$ with $\mathfrak{d} \geq \aleph_2$ (see 4.2), the following is still

Problem 5.5 (Roitman's problem [M2]) *Assume* $\mathfrak{d} = \aleph_1$. *Does* $\mathfrak{a} = \aleph_1$*?*

Recall that a *Luzin set* is an uncountable set of reals $X \subseteq \omega^\omega$ such that $X \cap A$ is at most countable for every meager set A. The existence of a Luzin set obviously implies $\mathsf{non}(\mathcal{M}) = \aleph_1$ (the converse is false in general, see [JS]). The consistency of $\mathfrak{a} < \mathsf{non}(\mathcal{M})$ is well-known (this is true, e.g., in the random real model), and $\mathfrak{a} > \mathsf{non}(\mathcal{M})$ holds in Shelah's model [S4]. However, since $\mathsf{non}(\mathcal{M}) \geq \aleph_2$ in that model, there cannot be a Luzin set. So the following is yet unknown.

Problem 5.6 (Fleissner's problem [M2]) *Assume there is a Luzin set. Does* $\mathfrak{a} = \aleph_1$*?*

In view of Theorem 4.3 we may ask

Problem 5.7 *Can* $\mathfrak{a}$ *be singular of countable cofinality?*

There are a number of problems concerning relatives of $\mathfrak{a}$ which are still open, see the discussion at the end of [BSZ].

(4) MISCELLANEA.

Problem 5.8 (van Mill's problem [M2]) *Is it consistent there are neither* P*-points nor* Q*-points?*

See the end of Section 3 for a discussion of this. Since the continuum must be at least $\aleph_3$ and $\mathsf{cov}(\mathcal{M})$ must be small, neither finite nor countable support iteration will work.

$\mathcal{F} \subseteq [\omega]^\omega$ is an *unreaped family* if it is not reaped (=split) by a single real, that is for all $X \in [\omega]^\omega$ there is $A \in \mathcal{F}$ such that either $A \subseteq^* X$ or $A \subseteq^* \omega \setminus X$. The *reaping number* $\mathfrak{r}$ is the size of the least unreaped family. Note that $\mathfrak{r}$ is dual to $\mathfrak{s}$. So the *ZFC*–results about $\mathfrak{s}$ mentioned above dualize to $\mathfrak{r} \geq \mathfrak{b}, \mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N})$. Also let $\mathfrak{r}_\sigma$ denote the size of the least $\mathcal{F} \subseteq [\omega]^\omega$ which is not reaped by a countable set $\mathcal{B} \subseteq [\omega]^\omega$, that is for all countable $\mathcal{B} \subseteq [\omega]^\omega$ there is $A \in \mathcal{F}$ such that for all $X \in \mathcal{B}$, either $A \subseteq^* X$ or $A \subseteq^* \omega \setminus X$. Clearly $\mathfrak{r} \leq \mathfrak{r}_\sigma$, and we get the following "local diagram" for $\mathfrak{r}$ and $\mathfrak{r}_\sigma$.

$$
\begin{tikzcd}
& & \mathfrak{r}_\sigma \arrow[d, dash] \\
\mathsf{cov}(\mathcal{N}) \arrow[r, dash] & \max\{\mathfrak{b}, \mathsf{cov}(\mathcal{N})\} \arrow[r, dash] \arrow[d, dash] & \mathfrak{r} \arrow[d, dash] \\
& \mathfrak{b} & \mathsf{cov}(\mathcal{M})
\end{tikzcd}
$$

As explained in [BJL], neither a finite nor a countable support iteration can be used to solve the following

Problem 5.9 (Vojtáš' problem) *Is it consistent that* $\mathfrak{r} < \mathfrak{r}_\sigma$*?*

Both $\mathfrak{r}$ and $\mathfrak{r}_\sigma$ may be singular. However, $cf(\mathfrak{r}_\sigma) > \omega$. In view of this, a positive answer to the following problem would also yield a positive answer to 5.9 above.

Problem 5.10 (Miller [M2]) *Is it consistent that $cf(\mathfrak{r}) = \omega$?*

Say that $\mathcal{F} \subseteq [\omega]^\omega$ is an ω–splitting family if given any countable $\mathcal{A} \subseteq [\omega]^\omega$ there is $X \in \mathcal{F}$ which splits all members of $\mathcal{A}$. $\mathfrak{s}_\omega$ is the size of the least ω–splitting family. Clearly $\mathfrak{s} \leq \mathfrak{s}_\omega$.

Problem 5.11 (Steprāns [St]) *Is it consistent that $\mathfrak{s} < \mathfrak{s}_\omega$?*

Let us close with two problems on cardinal invariants in Cichoń's diagram.

Problem 5.12 (Miller [M1]) *Is $\mathsf{non}(\mathcal{M}) = \aleph_3$ and $\mathsf{cov}(\mathcal{M}) = \aleph_2$ (simultaneously) consistent?*

In fact, in all known models which satisfy $\mathsf{non}(\mathcal{M}) > \mathsf{cov}(\mathcal{M})$, one has $\mathsf{cov}(\mathcal{M}) = \aleph_1$. Such models are gotten either by a countable support iteration of proper forcing, or by adding many random reals, or by adding a single Laver real to a model of MA.

Problem 5.13 (Bartoszyński–Judah [M2]) *Is $cf(\mathsf{cov}(\mathcal{M})) < \mathsf{add}(\mathcal{M})$ consistent?*

We formulated almost all problems as consistency questions and, indeed, this is our bias in most cases. However there may well be ZFC–results lurking behind, too.

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