WHAT IF $\lambda$ IS A STRONG LIMIT SINGULAR CARDINAL?

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1. BACKGROUND

Let $\kappa$ denote a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa$. Let $P_{\kappa}\lambda$ denote the set $\{x \subseteq \lambda || x | < \kappa\}$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $P_{\kappa}\lambda$.

Suppose $I$ is an ideal over $P_{\kappa}\lambda$. Let $I^+ = \{X \subseteq P_{\kappa}\lambda | X \not\in I\}$. Let $P_I$ denote the p.o. of members of $I^+$ ordered by $X \leq P_I Y \iff X \subseteq Y$.

Definition 1.1.

We say that an ideal $I$ is precipitous if $\models_{P_I} \text{"Ult}(V;G) \text{ is wellfounded"}$.

Let $NS_{\kappa\lambda} = \{ X \subseteq P_{\kappa}\lambda | X \text{ is the non-stationary}\}$. $NS_{\kappa\lambda}$ is known as the non-stationary ideal over $P_{\kappa}\lambda$. For a stationary $X \subseteq P_{\kappa}\lambda$, let $NS_{\kappa\lambda} | X$ denote the ideal over $P_{\kappa}\lambda$ defined by $Y \in NS_{\kappa\lambda} | X \iff Y \cap X \in NS_{\kappa\lambda}$.

Can $NS_{\kappa\lambda}$ or $NS_{\kappa\lambda} | X$ be precipitous?

Answer.: Yes (sometimes assuming ...).

Note The existence of a precipitous ideal has the strength of some large cardinal because it provides us with a "generic" elementary embedding of $V$.

Theorem 1.2 (Foreman, Magidor, Shelah, Goldring) [3][6].

If $\lambda$ is regular and $\delta$ is a Woodin cardinal $> \lambda$, then $\models_{Coll(\lambda, < \delta)} \text{"NS}_{\kappa\lambda}$ is precipitous". (Coll($\lambda$, $< \delta$) is the Levy collapse of $\delta$ to $\lambda^+$.)

Question. What if $\lambda$ is singular?

Burke and Matsubara [1] conjectured that $NS_{\kappa\lambda}$ cannot be precipitous if $\lambda$ is singular.

Definition 1.3. Let $\delta$ be a cardinal. We say that an ideal $I$ is $\delta$-saturated if $P_I$ satisfies the $\delta$ chain condition.

Fact. If $I$ is a $\lambda^+$-saturated $\kappa$-complete normal ideal over $P_{\kappa}\lambda$, then $I$ is precipitous.

Note. $NS_{\kappa\lambda}$ is the minimal $\kappa$-complete normal ideal over $P_{\kappa}\lambda$.

Theorem 1.4 (Foreman-Magidor) [2].

Unless $\kappa = \lambda = \aleph_1$, $NS_{\kappa\lambda}$ cannot be $\lambda^+$-saturated.

What about $NS_{\kappa\lambda} | X$?
Menas' Conjecture. Every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.

It turned out that Menas' Conjecture is independent of ZFC.

Theorem 1.5. $L \models "\text{Menas' Conjecture holds}"$.

Theorem 1.6 (Gitik) [5]. Suppose that $\kappa$ is supercompact and $\lambda > \kappa$. Then $\exists$ p.o. $\mathbb{P}$ that preserves cardinals $\geq \kappa$ such that $\not\models \text{"}\kappa$ is inaccessible and $\exists$ stationary $X \subseteq \mathcal{P}_\kappa\lambda$ such that $X$ cannot be partitioned into $\kappa^+$ disjoint stationary sets".

2. MAIN RESULTS

Theorem 2.1 (Matsubara-Shelah)[9]. If $\lambda$ is a strong limit singular cardinal then $NS_{\kappa\lambda}$ is nowhere precipitous (i.e. $NS_{\kappa\lambda} \upharpoonright X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$).

Theorem 2.2 [9]. If $\lambda$ is a strong limit singular cardinal then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.

One of the ingredients of the proof is the following lemma.

Lemma 2.3. If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then

(i) every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets and

(ii) $NS_{\kappa\lambda}$ is nowhere precipitous. (Matsubara-Shioya).

Remark.

(1) The hypothesis of Lemma 2.3 is satisfied if $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$.

(2) Under the hypothesis of Lemma 2.3, if $X \subseteq \mathcal{P}_\kappa\lambda$ has size $< 2^\lambda$ then $X$ is bounded and therefore non-stationary.

For the proof of (i) see page 345 of Kanamori [8].

proof of (ii).

Consider the following game $G_\omega$ between two players, Nonempty and Empty.

Nonempty $X_1 X_2 \ldots X_n \ldots$

Empty $Y_1 Y_2 \ldots Y_n \ldots$

Nonempty and Empty alternately choose stationary sets $X_n, Y_n \subseteq \mathcal{P}_\kappa\lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_n$ for $n=1,2,3,\ldots$.

After $\omega$ moves, Empty wins $G_\omega$ if $\bigcap_{n=1}^{\infty} X_n = \emptyset$.

Fact. $NS_{\kappa\lambda}$ is nowhere precipitous iff Empty has a winning strategy in $G_\omega$.

For the proof of this fact, see [4]. Let $\langle f_\alpha | \alpha < 2^\lambda \rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa\lambda$.

For a function $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, let

$$C(f) = \{ s \in \mathcal{P}_\kappa\lambda \mid \bigcup f^n s^{<\omega} \subseteq s \}$$

club set generated by $f$
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**Fact.** $X \subseteq P_{\kappa}\lambda$ is stationary iff $\forall \alpha < 2^\lambda C(f_\alpha) \cap X \neq \emptyset$.

We now describe Empty's strategy. Suppose Nonempty plays $X_1$. Choose a sequence $\langle s^1_\alpha | \alpha < 2^\lambda \rangle$ from $X_1$ by induction on $\alpha$ as follow: Pick an element from $X_1 \cap C(f_0)$ and call it $s^1_0$.

Given $\langle s^1_\alpha | \alpha < \beta \rangle$ for some $\beta < 2^\lambda$, pick $s^1_\beta \in X_1 \cap C(f_\beta) \setminus \{s^1_\alpha | \alpha < \beta\}$.

Let Empty play $Y_1 = \{s^1_\alpha | \alpha < 2^\lambda \}$. Now suppose Nonempty plays $X_n$ immediately following Empty's move $Y_{n-1} = \{s^{n-1}_\alpha | \alpha < 2^\lambda \}$.

Choose $\langle s^n_\alpha | \alpha < 2^\lambda \rangle$ a sequence from $X_n$ as follows:

Pick $s^n_0 \in (X \cap C(f_\beta)) \setminus \{s^{n-1}_\alpha | \alpha \leq \beta\} \cup \{s^n_\alpha | \alpha < \beta\}$.

Let Empty play $Y_n = \{s^n_\alpha | \alpha < 2^\lambda \}$.

**Claim.** This is a winning strategy for Empty

**proof:** We want to show that $\bigcap_{n=1}^{\infty} Y_n = \emptyset$.

Suppose otherwise, say $t \in \bigcap_{n=1}^{\infty} Y_n$. For each $n < \omega$, $\exists \alpha_n < 2^\lambda$ such that $t = s^{n}_{\alpha_n}$.

It is easy to see that $\alpha_n > \alpha_{n+1}$ for each $n$. ($s^n_\beta \notin \{s^n_\alpha | \alpha \leq \beta\}$ etc ...)

We now prove Theorem 2.2 assuming Theorem 2.1 and Lemma 2.3 (i).

**proof of Theorem 2.2:** Let $\lambda$ be a strong limit singular cardinal. If $\text{cf}(\lambda) < \kappa$ then by Lemma 2.3 (i) we are done.

Assume $\text{cf}(\lambda) \geq \kappa$. In this case $\lambda^{<\kappa} = \lambda$. So it is enough to show that $NS_{\kappa\lambda} | X$ is not $\lambda$-saturated for every stationary $X \subseteq P_{\kappa}\lambda$.

But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous. In fact we know that $NS_{\kappa\lambda} | X$ cannot be $\lambda^+$-saturated for every stationary $X \subseteq P_{\kappa}\lambda$.

**proof of Theorem 2.1:** We now tamper with the definition of $P_{\kappa}\lambda$.

From now on we let $P_{\kappa}\lambda = \{s \subseteq \lambda | s \subseteq \kappa, s \cap \kappa \in \kappa\}$. This set is club in $\{s \subseteq \lambda | |s| < \kappa, s \cap \kappa \in \kappa\}$. The following is the advantage of this change:

$X \subseteq P_{\kappa}\lambda$ is stationary iff $\forall f : \lambda^{<\omega} \rightarrow \lambda$ $C[f] \cap X \neq \emptyset$,

where $C[f] = \{s \in P_{\kappa}\lambda | s$ is closed under $f\}$.

Let $\lambda$ be a strong limit singular cardinal. By Lemma 2.3 (ii) we may assume that $\text{cf}(\lambda) \geq \kappa$. Let $\langle \lambda_i | i < \text{cf}(\lambda) \rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to $\lambda$. Let $T = \{i < \text{cf}(\lambda) | \text{cf}(i) < \kappa\}$.

For each $i \in T$, let $E_i = \{s \in P_{\kappa}\lambda | \sup(s) = \lambda_i, \lambda_i \notin s\}$

**Note.**

(i) $|E_i| = 2^{\lambda_i}$

(ii) $\bigcup_{i \in T} E_i$ is club in $P_{\kappa}\lambda$.

For each $i \in T$, let $\langle f_i^\epsilon | \epsilon < 2^{\lambda_i} \rangle$ enumerate all of the functions whose domain $\subseteq \lambda_i^{<\omega}$ and range $\subseteq \lambda_i$. 
Definition 2.4. $C^i[f^i] = \{ s \in E_i \mid s^{<\omega} \subseteq \text{dom}(f^i) \}$ and $s$ is close.

To show $NS_{\kappa\lambda}$ is nowhere precipitous we will present a win in $G_\omega$.

**Empty**

Suppose $W_1$ is **Nonempty**'s first move in $G_\omega$. For each $i \in T$, we a “local game” where each player alternately chooses subsets of $E_i$.

**Local game $G(i)$**

For each $i \in T$, define a game $G(i)$ as follows:

<table>
<thead>
<tr>
<th>$G_\omega$</th>
<th><strong>Nonempty</strong></th>
<th>$W_1$</th>
<th>$W_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Empty</strong></td>
<td>$\bigcup_{i \in T} \tau_i((W \cup E_i))$</td>
<td>$\bigcup_{i \in T} \tau_i((W \cap E_i))$</td>
<td></td>
</tr>
<tr>
<td>$(i \in T)$</td>
<td>$W_1 \cap E_i$</td>
<td>$W_2 \cap E_i$</td>
<td></td>
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</tbody>
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Just as in the proof of Lemma 2.3 (ii) we can show that **Empty**'s strategy, say $\tau_i$, in $G_i$.

Lemma 2.5. Suppose $W \subseteq \mathcal{P}_{\kappa}\lambda$ is stationary. If $U \subseteq \mathcal{P}_{\kappa}\lambda$ satisfies condition ($\ddagger$) then $U$ is stationary.

$G_\omega$:

- $\forall i \in T |U \cap E_i| < 2^{\lambda_i}$
- $C^i[f^i] \cap U \neq \emptyset$

Now we describe **Empty**'s (combined) strategy $\sigma$ in $G_\omega$. Suppose $W_1$ plays $W_1$.

Let **Empty** play $\bigcup_{i \in T} \tau_i((W_1 \cap E_i)) \overset{def}{=} \sigma((W_1))$.

Suppose

- $\sigma((W_1))$
- $\sigma((W_1, W_2))$
- $\ldots$
- $\sigma((W_1, W_2, \ldots, W_n)) \overset{def}{=} \bigcup_{i \in T} \tau_i((W_1 \cap E_i, W_2 \cap E_i, \ldots, W_i)$

is the run of the game $G_\omega$ so far.

Let

- $\sigma((W_1, W_2, \ldots, W_n)) \overset{def}{=} \bigcup_{i \in T} \tau_i((W_1 \cap E_i, W_2 \cap E_i, \ldots, W_i)$

Lemma 2.5 guarantees that $\sigma$ provides **Empty** a legal move i.e. a winning strategy for **Empty**.

The proof of Lemma 2.5 depends upon the following lemma whose

Lemma 2.6. Suppose $U \subseteq \mathcal{P}_{\kappa}\lambda$. If $\forall i \in T |U \cap E_i| < 2^{\lambda_i}$, then $U$ is stationary.
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pcf Fact. $\exists$ club $C \subseteq \operatorname{cf}(\lambda)$ such that $\operatorname{pp}(\lambda_i) = 2^{\lambda_i}$ for every $i \in C$.


REFERENCES
2. M. Foreman, and M. Magidor, Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $\mathcal{P}_\kappa\lambda$.
9. Y. Matsubara, and S. Shelah, Nowhere precipitousness of the non-stationary ideal over $\mathcal{P}_\kappa\lambda$.