WHAT IF $\lambda$ IS A STRONG LIMIT SINGULAR CARDINAL? (Axiomatic Set Theory)

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1. BACKGROUND

Let $\kappa$ denote a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa$. Let $\mathcal{P}_\kappa \lambda$ denote the set $\{x \subseteq \lambda ||x| < \kappa\}$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $\mathcal{P}_\kappa \lambda$.

Suppose $I$ is an ideal over $\mathcal{P}_\kappa \lambda$. Let $I^+ = \{X \subseteq \mathcal{P}_\kappa \lambda | X \notin I\}$. Let $\mathbb{P}_I$ denote the p.o. of members of $I^+$ ordered by $X \leq \mathbb{P}_I Y \iff X \subseteq Y$.

Definition 1.1. We say that an ideal $I$ is precipitous if $\Vdash_{\mathbb{P}_I} \text{"Ult}(V; G)$ is wellfounded".

Let $NS_{\kappa \lambda} = \{X \subseteq \mathcal{P}_\kappa \lambda | X$ is the non-stationary ideal over $\mathcal{P}_\kappa \lambda$}. For a stationary $X \subseteq \mathcal{P}_\kappa \lambda$, let $NS_{\kappa \lambda} | X$ denote the ideal over $\mathcal{P}_\kappa \lambda$ defined by $Y \in NS_{\kappa \lambda} | X \iff Y \cap X \in NS_{\kappa \lambda}$.

Can $NS_{\kappa \lambda}$ or $NS_{\kappa \lambda} | X$ be precipitous?

Answer. : Yes (sometimes assuming ...).

Note The existence of a precipitous ideal has the strength of some large cardinal because it provides us with a "generic" elementary embedding of $V$.

Theorem 1.2 (Foreman, Magidor, Shelah, Goldring) [3][6].
If $\lambda$ is regular and $\delta$ is a Woodin cardinal $> \lambda$, then $\Vdash_{Coll(\lambda, < \delta)} \text{"NS}_{\kappa \lambda}$ is precipitous". ($Coll(\lambda, < \delta)$ is the Levy collapse of $\delta$ to $\lambda^+$.)

Question. What if $\lambda$ is singular?

Burke and Matsubara [1] conjectured that $NS_{\kappa \lambda}$ cannot be precipitous if $\lambda$ is singular.

Definition 1.3. Let $\delta$ be a cardinal. We say that an ideal $I$ is $\delta$-saturated if $\mathbb{P}_I$ satisfies the $\delta$ chain condition.

Fact. If $I$ is a $\lambda^+$-saturated $\kappa$-complete normal ideal over $\mathcal{P}_\kappa \lambda$, then $I$ is precipitous.

Note. $NS_{\kappa \lambda}$ is the minimal $\kappa$-complete normal ideal over $\mathcal{P}_\kappa \lambda$.

Theorem 1.4 (Foreman-Magidor) [2].
Unless $\kappa=\lambda=\aleph_1$, $NS_{\kappa \lambda}$ cannot be $\lambda^+$-saturated.

What about $NS_{\kappa \lambda} | X$?
Menas' Conjecture. Every stationary subset of $\mathcal{P}_\kappa \lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.

It turned out that Menas' Conjecture is independent of ZFC.

**Theorem 1.5.** $L \vDash "\text{Menas' Conjecture holds}"$.

**Theorem 1.6 (Gitik) [5].** Suppose that $\kappa$ is supercompact and $\lambda > \kappa$. Then $\exists p.o. \mathcal{P}$ that preserves cardinals $\geq \kappa$ such that $\parallel \mathcal{P} \parallel$ is inaccessible and $\exists$ stationary $X \subseteq \mathcal{P}_\kappa \lambda$ such that $X$ cannot be partitioned into $\kappa^+$ disjoint stationary sets.

2. MAIN RESULTS

**Theorem 2.1 (Matsubara-Shelah)[9].** If $\lambda$ is a strong limit singular cardinal then $NS_{\kappa \lambda}$ is nowhere precipitous (i.e. $NS_{\kappa \lambda} \mid X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa \lambda$).

**Theorem 2.2** [9]. If $\lambda$ is a strong limit singular cardinal then every stationary subset of $\mathcal{P}_\kappa \lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.

One of the ingredients of the proof is the following lemma.

**Lemma 2.3.** If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then

(i) every stationary subset of $\mathcal{P}_\kappa \lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets and

(ii) $NS_{\kappa \lambda}$ is nowhere precipitous (Matsubara-Shioya).

**Remark.**

(1) The hypothesis of Lemma 2.3 is satisfied if $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$.

(2) Under the hypothesis of Lemma 2.3, if $X \subseteq \mathcal{P}_\kappa \lambda$ has size $< 2^\lambda$ then $X$ is bounded and therefore non-stationary.

For the proof of (i) see page 345 of Kanamori [8].

**proof of (ii).**

Consider the following game $G_\omega$ between two players, Nonempty and Empty.

Nonempty $X_1 \quad X_2 \quad \ldots \quad X_n \quad \ldots$

Empty $Y_1 \quad Y_2 \quad \ldots \quad Y_n \quad \ldots$

Nonempty and Empty alternately choose stationary sets $X_n, Y_n \subseteq \mathcal{P}_\kappa \lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_n$ for $n=1,2,3,\ldots$.

After $\omega$ moves, Empty wins $G_\omega$ if $\bigcap_{n=1}^\infty X_n = \emptyset$

**Fact.** $NS_{\kappa \lambda}$ is nowhere precipitous iff Empty has a winning strategy in $G_\omega$.

For the proof of this fact, see [4]. Let $\langle f_\alpha \mid \alpha < 2^\lambda \rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa \lambda$.

For a function $f: \lambda^{<\omega} \to \mathcal{P}_\kappa \lambda$, let

$$C(f) = \{ s \in \mathcal{P}_\kappa \lambda \mid \bigcup f^"s^{<\omega} \subseteq s \}$$

club set generated by $f$
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Fact. $X \subseteq \mathcal{P}_\kappa \lambda$ is stationary iff $\forall \alpha < 2^\lambda C(f_\alpha) \cap X \neq \emptyset$.

We now describe Empty’s strategy. Suppose Nonempty plays $X_1$. Choose a sequence $\langle s_\alpha^1 \mid \alpha < 2^\lambda \rangle$ from $X_1$ by induction on $\alpha$ as follow: Pick an element from $X_1 \cap C(f_0)$ and call it $s_0^1$.
Given $\langle s_\alpha^1 \mid \alpha < \beta \rangle$ for some $\beta < 2^\lambda$, pick $s_\beta^1 \in X_1 \cap C(f_\beta) \setminus \{s_\alpha^1 \mid \alpha < \beta\}$.

Let Empty play $Y_1 = \{s_\alpha^1 \mid \alpha < 2^\lambda\}$. Now suppose Nonempty plays $X_n$ immediately following Empty’s move $Y_{n-1} = \{s_\alpha^{n-1} \mid \alpha < 2^\lambda\}$.
Choose $\langle s_\alpha^n \mid \alpha < 2^\lambda \rangle$ a sequence from $X_n$ as follows:
Pick $s_\alpha^n \in (X \cap C(f_\beta)) \setminus (\{s_\alpha^{n-1} \mid \alpha \leq \beta\} \cup \{s_\alpha^1 \mid \alpha < \beta\})$.

Let Empty play $Y_n = \{s_\alpha^n \mid \alpha < 2^\lambda\}$.

Claim. This is a winning strategy for Empty

proof: We want to show that $\bigcap_{n=1}^\infty Y_n = \emptyset$.
Suppose otherwise, say $t \in \bigcap_{n=1}^\infty Y_n$. For each $n < \omega$, $\exists! \alpha_n < 2^\lambda$ such that $t = s_\alpha^n$.
It is easy to see that $\alpha_n > \alpha_{n+1}$ for each $n$. ($s_\beta^1 \not\in \{s_\alpha^n \mid \alpha \leq \beta\}$ etc …)
We now prove Theorem 2.2 assuming Theorem 2.1 and Lemma 2.3 (i).

proof of Theorem 2.2.: Let $\lambda$ be a strong limit singular cardinal. If $\text{cf}(\lambda) < \kappa$ then by Lemma 2.3 (i) we are done.
Assume $\text{cf}(\lambda) \geq \kappa$. In this case $\lambda^{<\kappa} = \lambda$. So it is enough to show that $NS_{\kappa\lambda} | X$ is $\lambda$-saturated for every stationary $X \subseteq \mathcal{P}_\kappa \lambda$.
But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous. In fact we know that $NS_{\kappa\lambda} | X$ cannot be $\lambda^+$-saturated for every stationary $X \subseteq \mathcal{P}_\kappa \lambda$.

proof of Theorem 2.1.: We now tamper with the definition of $\mathcal{P}_\kappa \lambda$.
From now on we let $\mathcal{P}_\kappa \lambda = \{s \subseteq \lambda \mid |s| < \kappa, s \cap \kappa \in \kappa\}$. This set is club in $\{s \subseteq \lambda \mid |s| < \kappa\}$. The following is the advantage of this change:
$X \subseteq \mathcal{P}_\kappa \lambda$ is stationary iff $\forall f : \lambda^{<\omega} \rightarrow \lambda$ $C[f] \cap X \neq \emptyset$ where $C[f] = \{s \in \mathcal{P}_\kappa \lambda \mid s \text{ is closed under } f\}$.
Let $\lambda$ be a strong limit singular cardinal. By Lemma 2.3 (ii) we may assume that $\text{cf}(\lambda) \geq \kappa$. Let $\langle \lambda_i \mid i < \text{cf}(\lambda) \rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to $\lambda$. Let $T = \{i < \text{cf}(\lambda) \mid \text{cf}(i) < \kappa\}$.
For each $i \in T$, let $E_i = \{s \in \mathcal{P}_\kappa \lambda \mid \text{sup}(s) = \lambda_i, \lambda_i \not\in s\}$

Note.

(i) $|E_i| = 2^{\lambda_i}$
(ii) $\bigcup_{i\in T} E_i$ is club in $\mathcal{P}_\kappa \lambda$.

For each $i \in T$, let $\langle f_\epsilon^i \mid \epsilon < 2^{\lambda_i} \rangle$ enumerate all of the functions whose domain $\subseteq \lambda_i^{<\omega}$ and range $\subseteq \lambda_i$. 
**Definition 2.4.** \( C[i[f^i] = \{ s \in E_i \mid s^{<\omega} \subseteq \text{dom}(f^i) \} \) and \( s \) is close.

To show \( NS_{\kappa\lambda} \) is nowhere precipitous we will present a win.

**Empty** in \( G_\omega \).

Suppose \( W_1 \) is **Nonempty**'s first move in \( G_\omega \). For each \( i \in T \), we a "**local game**" where each player alternately chooses subsets of \( E_i \), **Nonempty**'s first move is \( W_1 \cap E_i \).

**Local game** \( G(i) \)

For each \( i \in T \), define a game \( G(i) \) as follows:

**Nonempty** and **Empty** alternately choose \( X_n, Y_n \subseteq E_i \) resp.

1, 2, \ldots, so that \( X_n \supseteq Y_n \supseteq X_{n+1} \) and \( \forall \epsilon < 2^{\lambda_i} \) ( \( |C[i[f^i] \cap C[i[f^i] \cap Y_n \neq \emptyset \)).

**Empty** wins \( G(i) \) iff \( \bigcap_{n=1}^{\infty} X_n = \emptyset \).

Just as in the proof of Lemma 2.3 (ii) we can show that **Empty**'s strategy, say \( \tau_i \) in \( G_i \).

The following lemma tells us that we can combine \( \tau_i \)’s for \( i \in T \) for \( G_\omega \).

**Lemma 2.5.** Suppose \( W \subseteq P_{\kappa\lambda} \) is stationary. If \( U \subseteq P_{\kappa\lambda} \) satisfy condition (\( \# \)) then \( U \) is stationary.

(\( \# \)) For each \( i \in T \), \( \forall \epsilon < 2^{\lambda_i} \) ( \( |C[i[f^i] \cap W| = 2^{\lambda_i} \rightarrow C[i[f^i] \cap U \neq \emptyset \).

Now we describe **Empty**'s (combined) strategy \( \sigma \) in \( G_\omega \). Suppose \( W \).

Let **Empty** play \( \bigcup_{i \in T} \tau_i((W_1 \cap E_i)) \equiv \sigma((W_1)) \).

Suppose

\[
\begin{align*}
W_1 & \quad W_2 & \quad \ldots & \quad W_n \\
\sigma((W_1)) & \quad \sigma((W_1, W_2)) & \quad \ldots
\end{align*}
\]

is the run of the game \( G_\omega \) so far.

Let

\[
\sigma((W_1, W_2, \ldots, W_n)) \equiv \bigcup_{i \in T} \tau_i((W_1 \cap E_i, W_2 \cap E_i, \ldots, W_i,
\]

Lemma 2.5 guarantees that \( \sigma \) provides **Empty** a legal move i.e. of **Nonempty**'s last move. This \( \sigma \) is a winning strategy for **Emp**.

The proof of Lemma 2.5 depends upon the following lemma whole theory.

**Lemma 2.6.** Suppose \( U \subseteq P_{\kappa\lambda} \). If \( \forall i \in T \mid U \cap E_i \mid < 2^{\lambda_i} \), then \( U \) is

To prove the last lemma, we need the following fact from pcf theo.
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$\text{pcf Fact. } \exists \text{ club } C \subseteq cf(\lambda) \text{ such that } pp(\lambda_i) = 2^{\lambda_i} \text{ for every } i \in C.$


REFERENCES

2. M. Foreman, and M. Magidor, Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\kappa \lambda$.