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NONREFLECTING STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

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ABSTRACT. A nonreflecting stationary subset of $\mathcal{P}_\kappa\kappa^+$ is constructed e.g. when κ is the successor of a regular uncountable cardinal.

1. INTRODUCTION

Let $\kappa > \omega$ be a regular cardinal. The reflection principle for stationary subsets of $\mathcal{P}_\kappa\lambda$, where $\lambda > \kappa$ is a cardinal, was introduced and shown consistent relative to a supercompact cardinal in case $\kappa = \omega_1$ by Foreman, Magidor and Shelah [3]. The corresponding principle for $\kappa > \omega_1$ was refuted in ZFC by Feng and Magidor [1] when κ is a successor cardinal, and in general by Foreman and Magidor [2]. Specifically, "combinatorialization" of the latter argument (see Section 4 below) yields

Theorem 1. *$\mathcal{P}_\kappa\lambda$ has a nonreflecting stationary subset when $\kappa > \omega_1$ and $\lambda \geq 2^{\kappa^+}$.*

What about $\mathcal{P}_\kappa\kappa^+$? In this note, we give a parallel result for $\kappa = \nu^+$ with $\nu > \omega$ regular. More generally, we show

Theorem 2. *Assume $\operatorname{cf}[\lambda]^\kappa = \lambda$, $\omega < \nu < \kappa$ is regular and $\operatorname{cf}[\gamma]^{<\nu} < \kappa$ for all $\gamma < \kappa$. Then $\mathcal{P}_\kappa\lambda$ has a nonreflecting stationary subset.*

2. PRELIMINARIES

Our terminology generally follows Kanamori [5] with the following exceptions. For the rest of this paper, κ denotes a regular cardinal $> \omega_1$, λ a cardinal $> \kappa$, μ a cardinal from $\lambda - \kappa$ and ν a regular cardinal from $\kappa - \omega_1$. We let $[\lambda]^\mu = \{x \subset \lambda : |x| = \mu\}$, $\operatorname{cf}[\lambda]^\mu$ the minimal size of its unbounded subsets and $S^\nu_\kappa = \{\gamma < \kappa : \operatorname{cf}\gamma = \nu\}$. Also $\lim A$ denotes the set of limit points of a set A of ordinals, and for a map f defined on a subset of $\lambda^{<\omega}$, $\operatorname{cl}_f x$ the closure of $x \in \mathcal{P}_\kappa\lambda$ under

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f and $C(f)$ the set $\{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in \kappa \wedge \mathrm{cl}_f x = x\}$. The reflection principle we consider states that for all $\lambda > \kappa$ and $S \subset \mathcal{P}_\kappa\lambda$ stationary there is $\kappa \subset A \in [\lambda]^\kappa$ such that $S \cap \mathcal{P}_\kappa A$ is stationary. A stationary set witnessing its failure is called nonreflecting. More generally, $S \subset \mathcal{P}_\kappa\lambda$ is called μ-nonreflecting if $S \cap \mathcal{P}_\kappa A$ is nonstationary for all $\mu \subset A \in [\lambda]^\mu$. A μ^+-complete filter on $[\lambda]^\mu$ extending $\{\{x \in [\lambda]^\mu : \alpha \in x\} : \alpha < \lambda\}$ is called fine. The specific example relevant to us was introduced in [7]:

Lemma 1. *The sets* $\{\bigcup_{i<\omega} A_i : \{A_i : i < \omega\} \subset [\lambda]^\mu \wedge \forall n < \omega(\tau(\langle A_i : i < n\rangle) \subset A_n)\}$, *where* $\tau : ([\lambda]^\mu)^{<\omega} \to [\lambda]^\mu$, *generate a fine filter on* $[\lambda]^\mu$.

We need an analogue of Ulam's result (see [4] or [6] for a proof):

Lemma 2. *Let F be a fine filter on $[\lambda]^\mu$. Then λ many mutually disjoint F-positive sets exist.*

3. Main Theorem

In this section we prove Theorem 2 in an even more general form:

Theorem 3. *Let* $\omega < \nu < \kappa \le \mu < \lambda$ *be as in Section 2. Assume* $\{\bigcup_{\alpha\in a} E_\alpha : a \in [\lambda]^{<\nu}\}$ *is unbounded in* $[\lambda]^\mu$ *for some* $\{E_\alpha : \alpha < \lambda\} \subset [\lambda]^\mu$ *and* $\{z \in \mathcal{P}_\kappa\mu : \{c_\xi : \xi \in z\}$ *is unbounded in* $[z]^{<\nu}\}$ *has a stationary subset T of size μ for some* $\{c_\xi : \xi < \mu\} \subset [\mu]^{<\nu}$. *Then $\mathcal{P}_\kappa\lambda$ has a μ-nonreflecting stationary subset.*

Proof. Define $e : \lambda \times \mu \to \lambda$ so that $e``\{\alpha\} \times \mu = E_\alpha$. Let F be the filter on $[\lambda]^\mu$ as in Lemma 1. Fix a mutually disjoint $\{X_z : z \in T\} \subset F^+$ by Lemma 2. We show that $S = \{x \in C(e) : x \cap \mu \in T \wedge \exists b \in [x]^{<\nu}(x \subset \mathrm{cl}_e(b \cup \mu) \in X_{x\cap\mu})\}$ is as desired.

To show S stationary, fix $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$. Consider the following game $G_{f,z}$ for $z \in T$: I and II take in turn $\mu \subset A_n \in [\lambda]^\mu$ and a triple of $b_n \in [\lambda]^{<\nu}$, a bijection $\pi_n : \mu \to \mathrm{cl}_e(b_n \cup \mu)$ and $x_n \in C(f)$ respectively so that $A_n \subset \mathrm{cl}_e(b_n \cup \mu) \subset A_{n+1}$, $b_n \subset x_n \subset \mathrm{cl}_e(b_n \cup \mu)$, $\pi_n``(x_n \cap \mu) = x_n$ and $\langle b_i : i < \omega\rangle$ and $\langle x_i : i < \omega\rangle$ are increasing. II wins iff $x_n \cap \mu = z$ for all $n < \omega$. We first claim $T \cap D \subset \{z \in T :$ II has a winning strategy in $G_{f,z}\}$ for some club $D \subset \mathcal{P}_\kappa\mu$.

Suppose to the contrary $T' = \{z \in T :$ II has no winning strategy in $G_{f,z}\}$ is stationary. For $z \in T'$, we have a winning strategy σ_z for I in $G_{f,z}$, since the game is closed for II and hence determined. By induction on $n < \omega$, build (b_n, π_n, x_n^z) for $z \in T'$ as follows: First take $b_{n-1} \subset b_n \in [\lambda]^{<\nu}$ with $\bigcup_{z\in T'} \sigma_z(\langle (b_i, \pi_i, x_i^z) : i < n\rangle) \subset \mathrm{cl}_e(b_n \cup \mu) \in C(f)$. Next take a bijection $\pi_n : \mu \to \mathrm{cl}_e(b_n \cup \mu)$. For $z \in T'$, take $b_n \subset x_n^z \in C(f) \cap \mathcal{P}_\kappa \mathrm{cl}_e(b_n \cup \mu)$ with $\pi_n``(x_n^z \cap \mu) = x_n^z$, and if possible,

$x_n^z \cap \mu = z$. Now set $b = \bigcup_{n<\omega} b_n \in [\lambda]^{<\nu}$ and $A = \mathrm{cl}_e(b \cup \mu) \in [\lambda]^\mu$. Take $b \subset x \in C(f) \cap \mathcal{P}_\kappa A$ with $\pi_n``(x \cap \mu) = x \cap \mathrm{cl}_e(b_n \cup \mu)$ for all $n < \omega$ and $z = x \cap \mu \in T'$. Then $x_n^z = x \cap \mathrm{cl}_e(b_n \cup \mu)$ for all $n < \omega$, since $x \cap \mathrm{cl}_e(b_n \cup \mu)$ is the unique set satisfying all the requirements for x_n^z including the extra one. Thus II wins the game $G_{f,z}$ with the moves $\langle (b_n, \pi_n, x_n^z) : n < \omega \rangle$, yet I plays according to the winning strategy σ_z. Contradiction.

Now fix $z \in T$ with a winning strategy σ for II in $G_{f,z}$. Define $\tau : ([\lambda]^\mu)^{<\omega} \to [\lambda]^\mu$ by $\tau(t) = \mathrm{cl}_e(b \cup \mu)$, where $\sigma(t) = (b, \pi, x)$. Since $X_z \in F^+$, we have $\{A_i : i < \omega\} \subset [\lambda]^\mu$ such that $\bigcup_{i<\omega} A_i \in X_z$ and $\tau(\langle A_i : i < n \rangle) \subset A_n$ for all $n < \omega$. Set $(b_n, \pi_n, x_n) = \sigma(\langle A_i : i \leq n \rangle)$ for $n < \omega$. Then $x = \bigcup_{n<\omega} x_n \in S \cap C(f)$ as desired, since $x \cap \mu = z \in T$ and $x \subset \mathrm{cl}_e(b \cup \mu) = \bigcup_{n<\omega} \mathrm{cl}_e(b_n \cup \mu) = \bigcup_{i<\omega} A_i \in X_z$, where $b = \bigcup_{n<\omega} b_n \in [x]^{<\nu}$.

To show S μ-nonreflecting, suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary for some $\mu \subset A \in [\lambda]^\mu$. Then $e``(A \times \mu) \subset A$, since $C(e) \cap \mathcal{P}_\kappa A$ is unbounded in $\mathcal{P}_\kappa A$. We next give $a \in [A]^{<\nu}$ with $\mathrm{cl}_e(a \cup \mu) = A$.

Fix a bijection $\pi : \mu \to A$. Set $B = \pi^{-1}\mu \in [\mu]^\mu$. Define $h : \mu \times B \to \mu$ by $\pi(h(\xi, \zeta)) = e(\pi(\xi), \pi(\zeta))$. Let S' be the stationary $\{x \cap \mu : \pi``(x \cap \mu) = x \in S \cap \mathcal{P}_\kappa A\} \subset T$. For $z \in S'$, take $\xi_z \in z$ with $z \subset \mathrm{cl}_h(c_{\xi_z} \cup B)$ by $\pi``z \in S$. Take $\xi < \mu$ so that $S^* = \{z \in S' : \xi_z = \xi\}$ is stationary. Then $\mu = \mathrm{cl}_h(c_\xi \cup B)$, since $z \subset \mathrm{cl}_h(c_\xi \cup B)$ for all $z \in S^*$. Hence $A = \mathrm{cl}_e(\pi``c_\xi \cup \mu)$, as desired.

Now we have the desired contradiction to the mutual disjointness of $\{X_z : z \in T\}$: $A \in X_{x \cap \mu}$ for all $x \in S \cap \mathcal{P}_\kappa A$ with $a \subset x$, since for some $b \in [x]^{<\nu}$, $A = \mathrm{cl}_e(a \cup \mu) = \mathrm{cl}_e(b \cup \mu)$ by $a \subset \mathrm{cl}_e(b \cup \mu)$. □

4. Remarks

Let us first deduce Theorem 2 from Theorem 3: Assume $\mathrm{cf}\,[\gamma]^{<\nu} < \kappa$ for all $\gamma < \kappa$. Then we have $\{c_\xi : \xi < \kappa\} \subset [\kappa]^{<\nu}$ and $f : \kappa \to \kappa$ such that $\{c_\xi : \xi < f(\gamma)\}$ is unbounded in $[\gamma]^{<\nu}$. Then $T = \{\gamma \in S_\kappa^\nu : f``\gamma \subset \gamma\}$ is the desired stationary subset of $\{\gamma < \kappa : \{c_\xi : \xi < \gamma\}$ is unbounded in $[\gamma]^{<\nu}\}$.

The rest of the section is devoted to the

Proof of Theorem 1. Fix a bijection $\pi_\gamma : \kappa \to \gamma$ for $\gamma \in \kappa^+ - \kappa$ and a surjection $g : \lambda \to {}^{\kappa^+}\kappa$. Define $h : [\kappa^+]^2 \to \mathcal{P}_\kappa \kappa^+$ by $h(\alpha, \beta) = \lim \pi_\beta``\pi_\beta{}^{-1}(\alpha)$. Let $D$ be the club $\{x \in C(h) : \forall \gamma \in x \cap (\kappa^+ - \kappa)(\pi_\gamma``(x \cap \kappa) = x \cap \gamma) \wedge \forall \xi \in x(x \in C(g(\xi)))\}$. We show that $S = \{x \in \mathcal{P}_\kappa \lambda : \sup\{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\} < \kappa^+\}$ is as

To show S stationary, suppose otherwise. By induction on $n < \omega$, build $f_n : \lambda^{<\omega} \to \lambda$ closed under composition so that $C(f_0) \subset D - S$ and for all $m < \omega$ there is $n < \omega$ such that $f_m(t * \langle\gamma\rangle) = g_{f_n(t)}(\gamma)$ for all $t \in \lambda^{<\omega}$ and $\gamma < \kappa^+$ with $f_m(t * \langle\gamma\rangle) < \kappa$. Define $f : \lambda^{<\omega} \to \mathcal{P}_{\omega_1}\lambda$ by $f(t) = \{f_n(t) : n < \omega\}$. Fix $x \in C(f)$. We claim that $\sup\{\sup(z \cap \kappa^+) : x \subset z \in C(f) \wedge z \cap \kappa = x \cap \kappa\} = \kappa^+$.

Fix $\alpha < \kappa^+$. By $x \notin S$, we have $x \subset y \in D$ with $y \cap \kappa = x \cap \kappa$ and $\alpha < \gamma \in y \cap \kappa^+$. Then $z = \mathrm{cl}_f(x \cup \{\gamma\})$ is as desired: To see $z \cap \kappa \subset y \cap \kappa$, fix $\beta \in z \cap \kappa$. Then $\beta = f_m(t * \langle\gamma\rangle)$ for some $m < \omega$ and $t \in x^{<\omega}$. Hence $\beta = g_{f_n(t)}(\gamma) \in y$ for some $n < \omega$, since $\{f_n(t), \gamma\} \subset y \in D$.

For $i < 2$, build an increasing and continuous $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ so that $x^i_\xi \cap \kappa = x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$, $\sup(x^0_\xi \cap \kappa^+) < \sup(x^1_\xi \cap \kappa^+) < \sup(x^0_{\xi+1} \cap \kappa^+)$ and $x^0_1 \cap \sup(x^1_0 \cap \kappa^+) \neq x^1_0 \cap \sup(x^1_0 \cap \kappa^+)$ as follows: First fix $x^0_0 \in C(f)$ with $\mathrm{cf}\,(x^0_0 \cap \kappa) = \omega_1$. Take $x^0_0 \subset x^0_1 \in C(f)$ with $x^0_1 \cap \kappa = x^0_0 \cap \kappa$ so that $\sup(x^0_1 \cap \kappa^+)$ is the κ-th element of $\{\sup(z \cap \kappa^+) : x^0_0 \subset z \in C(f) \wedge z \cap \kappa = x^0_0 \cap \kappa\}$. Take $x^0_0 \subset x^1_0 \in C(f)$ with $x^1_0 \cap \kappa = x^0_0 \cap \kappa$ so that $\sup(x^0_0 \cap \kappa^+) < \sup(x^1_0 \cap \kappa^+) \in \sup(x^0_1 \cap \kappa^+) - \lim(x^0_1 \cap \kappa^+)$. The rest of the construction is routine.

Now set $x^i = \bigcup_{\xi<\omega_1} x^i_\xi$. Then $x^0 \cap \kappa^+ \neq x^1 \cap \kappa^+$, since $x^i_\xi \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$. Next to show $x^i \cap \kappa^+$ countably closed in $\sup(x^0 \cap \kappa^+) = \sup(x^1 \cap \kappa^+)$, fix $a \subset x^i \cap \kappa^+$ of order type ω. We have $a \subset \beta \in x^i \cap \kappa^+$ by $\mathrm{cf}\,\sup(x^i \cap \kappa^+) = \omega_1$, and $\alpha \in x^i \cap \beta = \pi_\beta``(x^i \cap \kappa)$ with $\pi_\beta{}^{-1}a \subset \pi_\beta{}^{-1}(\alpha)$ by $x^i \cap \kappa \in S^{\omega_1}_\kappa$. Then $\sup a \in h(\alpha, \beta) \subset x^i$, as desired. Now we have the desired contradiction $x^i \cap \kappa^+ = \bigcup_{\gamma \in C} \pi_\gamma``(x^i \cap \kappa) = \bigcup_{\gamma\in C} \pi_\gamma``(x^0_0 \cap \kappa)$, where $C \subset x^0 \cap x^1 \cap \kappa^+$ is unbounded in $\sup(x^0 \cap \kappa^+) = \sup(x^1 \cap \kappa^+)$.

To show S nonreflecting, suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary for some $\kappa \subset A \in [\lambda]^\kappa$. Fix a bijection $\pi : \kappa \to A$. Then $S' = \{\gamma < \kappa : \pi``\gamma \in S \wedge \pi``\gamma \cap \kappa = \gamma\}$ and $\{y \cap \kappa^+ : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in S'\}$ are stationary in $\kappa$ and $\mathcal{P}_\kappa \kappa^+$ respectively. Hence $\sup\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in S'\} = \kappa^+$. Thus we have $\gamma \in S'$ such that $\sup\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa = \gamma\} = \kappa^+$, contradicting $\pi``\gamma \in S$. □

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