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Iterated Forcing with $^\omega \omega$-bounding and Semiproper Preorders (Axiomatic Set Theory)

Author(s)
Miyamoto, Tadatoshi

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Iterated Forcing with $\omega$-bounding and Semiproper Preorders

Tadatoshi MIYAMOTO

Abstract

Assume the Continuum Hypothesis (CH) in the ground model. If we iteratively force with preorders which are $\omega$-bounding and semiproper taking suitable limits, then so is the final preorder constructed. Therefore we may show that the Cofinal Branch Principle (CBP) of [F] is strictly weaker than the Semiproper Forcing Axiom (SPFA).

§0 Introduction

We formulate a theory of iterated forcing and propose a construction of a limit stage in [M2]. We call it the *simple* limit. This limit is designed to preserve the semiproperness of preorders. Namely, if we construct an iterated forcing of semiproper preorders which takes this limit at every limit stage, then so is the iterated forcing thus constructed. We move on to consider the $\omega$-boundingness together with the semiproperness in this note. Assuming CH in the ground model, we show the preservation of these two properties combined. It is known that the $\omega$-boundingness together with the properness is preserved under countable support. Our proof is a straightforward modification of [B] in the current context.

§1 Preliminary

Our approach to forcing is based on the notion of preorders. We say $(P, \leq, 1)$ is a preorder, if $\leq$ is a reflexive and transitive binary relation on $P$ with a greatest element 1. So a preorder may not be anti-symmetric. A preorder $P$ (we use this type of abbreviations) is separative, if for any $p, q \in P$, we have $q \leq p$ iff $q \Vdash_{\mathcal{P}} p \in G$, where $\mathcal{G}$ denotes the canonical $P$-name of the generic filters. For $p, q \in P$, we write $p \equiv q$ for short, if $p \leq q \leq p$. Since $P$ is a preorder, this relation $\equiv$ is an equivalence relation. But we never take the quotient.

We review the following technical but important structures and notions from [M1] and [M2]. For more details, we may consult [M1] and [M2].
• Iterations $I = \langle P_\alpha \mid \alpha < \nu \rangle$ and dynamical stages $\langle \delta_k \mid k < \omega \rangle$.

• The simple limit and simple iterations.

• Nested antichains $T$.

• $(T, I)$-nice sequences.

• A fusion structure $\mathcal{F}$ and its fusions.

Let us first recall our axiomatic approach to the theory of iterated forcing. The actual construction of a relevant iterated forcing $\langle P_\alpha \mid \alpha < \nu \rangle$ together with the $P_\alpha$-names $\dot{Q}_\alpha$ is done by recursion.

1.1 Definition. Let $I = \langle (P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha < \nu \rangle$ be a sequence of separative preorders s.t. for any $\alpha < \nu$, $P_\alpha$ is a set of sequences of length $\alpha$. We say $I$ is an iteration, if for all $\alpha \leq \beta < \nu$, we have

- $1_\beta \lceil \alpha = 1_\alpha$ and if $p \in P_\beta$, then $p \lceil \alpha \in P_\alpha$.
- If $p \in P_\beta$, $a \in P_\alpha$ and $a \leq_\alpha p \lceil \alpha$, then $a \equiv p \lceil (\alpha, \beta) \in P_\beta$ and $a \equiv p \lceil (\alpha, \beta) \leq_\beta p$.
- If $p \leq_\beta q$, then $p \lceil \alpha \leq_\alpha q \lceil \alpha$ and $p \leq_\beta p \lceil \alpha \equiv q \lceil (\alpha, \beta)$.
- If $\beta$ is a limit ordinal, then for any $x, y \in P_\beta$, we have $y \leq_\beta x$ iff $\forall \alpha < \beta \ y \lceil \alpha \leq_\alpha x \lceil \alpha$.

Let $I = \langle P_\alpha \mid \alpha < \nu \rangle$ (we allow this type of abbreviations) be an iteration and $\alpha \leq \beta < \nu$. Let $G_\alpha$ be a $P_\alpha$-generic filter over the ground model $V$. In $V[G_\alpha]$, the quotient preorder $P_{\alpha\beta}$ is defined as follows:

- $P_{\alpha\beta} = \{ x \lceil (\alpha, \beta) \mid x \in P_\beta \text{ with } x \lceil \alpha \in G_\alpha \}$.
- For $t, s \in P_{\alpha\beta}$, we set $t \leq_{\alpha\beta} s$, if there is $a \in G_\alpha$ s.t. $a \lhd t \leq a \lhd s$ in $P_\beta$.

We consider conditions in the limit stages which have a sort of their own Boolean-valued $\omega$-stages. The stages are dependent to given generic objects in the manner given below.

1.2 Definition. Let $\nu$ be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be an iteration. We say $p \in P_\nu$ has $P_\nu$-dynamical stages $\langle \delta_k \mid k < \omega \rangle$, if for all
\begin{itemize}
\item \( \dot{\delta}_k \) is a \( P_\nu \)-name s.t.
\begin{itemize}
\item \( \models_{P_\nu} " \dot{\delta}_k \leq \dot{\delta}_{k+1} \leq \nu ". \)
\item For any \( x \in P_\nu \), if \( x \models_{P_\nu} " \dot{\delta}_k = \check{\xi} " \), then \( x[\xi \sim 1_{\nu}][\xi, \nu] \models_{P_\nu} " \dot{\delta}_k = \check{\xi} ". \)
\item \( p \models_{P_\nu} " \dot{\delta}_k < \nu " \).
\end{itemize}
\item \( \models_{P_\nu} " \text{if } \dot{\delta} = \sup\{\dot{\delta}_k | k < \omega \} \text{ and } p[\dot{\delta} \in \dot{G}_\nu][\dot{\delta}] \text{, then } p \in \dot{G}_\nu ". \)
\end{itemize}

The conditions with dynamical stages constitute a limit stage of an iterated forcing.

1.3 Definition. Let \( \nu \) be a limit ordinal and \( I = \langle P_\alpha | \alpha < \nu \rangle \) be an iteration. We write \( I^* \) for the inverse limit of \( I \). By this we mean \( I^* = \{ x | x \) is a sequence of length \( \nu \) s.t. for all \( \alpha < \nu \), we demand \( x[\alpha \in P_\alpha] \). The simple limit \( X \) of \( I \) is a subpreorder of \( I^* \) and its universe is defined by \( X = \{ x \in I^* | \) there exist \( I^* \)-dynamical stages \( \langle \dot{\alpha}_k | k < \omega \rangle \) for \( x \}. \)

In the following the former requirement is what we call the fullness of successor stages. This requirement is satisfied by the usual recursive constructions. We demand to take the simple limit at every relevant limit stages.

1.4 Definition. We say an iteration \( I = \langle P_\alpha | \alpha < \nu \rangle \) is simple, if
\begin{itemize}
\item For any \( \alpha \) with \( \alpha + 1 < \nu \) and any \( P_\alpha \)-name \( \tau \), if \( a \models_{P_\alpha} " \tau \in P_{\alpha+1} \) with \( \tau[\alpha \in \dot{G}_\alpha] \), then there is \( b \in P_{\alpha+1} \) s.t. \( b[\alpha = a \text{ and } a \models_{P_\alpha} " b[\alpha, \alpha + 1] \equiv \tau[\alpha, \alpha + 1] \in P_{\alpha+1} "]. \)
\item For any limit \( \alpha < \nu \), \( P_\alpha \) is the simple limit of \( I[\alpha]. \)
\end{itemize}

We sum up what we got here. The following is 3.5 Lemma and 3.12 Proposition in [M2]. The last two items say we have some freedom when we choose dynamical stages in the simple limit. We caution that to define the simple limit we use \( I^* \)-dynamical stages. But once we got the limit \( X \), then every \( x \in X \) has \( X \)-dynamical stages. This takes some proof and these two should not be confused.

1.5 Lemma. Let \( I = \langle P_\alpha | \alpha < \nu \rangle \) be a simple iteration. For any limit ordinal \( j < \nu \), let us write \( (I[j])_* \) for the direct limit of \( I[j] \) and \( (I[j])^* \) for the inverse limit of \( I[j] \). Then we have
• $(I[j])_* ⊆ P_j ⊆ (I[j])^*$.

• For any $x \in (I[j])^*$, we have $x \in P_j$ iff $x$ has $(I[j])^*$-dynamical stages.

• And so if $cf(j) = \omega$, then $P_j = (I[j])^*$. If $cf(j) \geq \omega_1$, then $P_j$ might be bigger than $(I[j])_*$.

• For any $x, y \in P_j$, we have $x \leq_j y$ iff $\forall \alpha < j \ x[\alpha] \leq \alpha y[\alpha]$.

• For any $i < j$ and $x \in P_i$, we may construct $P_j$-dynamical stages $\langle \delta_k | k < \omega \rangle$ for $x$ s.t. $\mathcal{P}_j"\delta_0 = \check{i}"$.

• For any $x, y \in P_j$ with $y \leq_j x$, if $\langle \delta_k(x) | k < \omega \rangle$ are $P_j$-dynamical stages for $x$ and $\langle \delta_k(y) | k < \omega \rangle$ are $P_j$-dynamical stages for $y$, then we may assume for all $k < \omega$, $\mathcal{P}_j"\delta_{k+1}(x) \leq \delta_k(y)"$.

The simple iterations enjoy the fullness not only at the successor stages but at the limit stages. We state this precisely in 1.11 Lemma. And for that we introduce a kind of generalized names.

1.6 Definition. Let $\nu$ be a limit ordinal and $I = \langle P_\alpha | \alpha < \nu \rangle$ be an iteration. A nested antichain $T$ in $I$ is $\langle \langle T_n \mid n < \omega \rangle, \langle \text{suc}^n_T \mid n < \omega \rangle \rangle$ s.t. for all $n < \omega$ and all $a \in T_n$, we have

• $T_0 = \{a_0\}$ for some $\alpha_0 < \nu$ and $a_0 \in P_{\alpha_0}$.

• $T_n \subseteq \bigcup\{P_\alpha \mid \alpha < \nu\}$ and $\text{suc}^n_T : T_n \to P(T_{n+1})$.

• For any $b \in \text{suc}^n_T(a)$, we have $l(a) \leq l(b)$ and $b[l(a)] \leq a$.

• $\{b[l(a)] \mid b \in \text{suc}^n_T(a)\}$ is a maximal antichain below $a$ in $P_{l(a)}$.

• For any $b, b' \in \text{suc}^n_T(a)$, if $b[l(a)] = b'[l(a)]$, then $b = b'$.

• $T_{n+1} = \bigcup\{\text{suc}^n_T(a) \mid a \in T_n\}$.

We consider a relation between conditions and nested antichains. We intend to identify conditions and nested antichains in the simple iterations via this relation.

1.7 Definition. Let $\nu$ be a limit ordinal and $I = \langle P_\alpha \mid \alpha < \nu \rangle$ be an iteration. Let $T$ be a nested antichain in $I$. We say $x \in I^*$ (the inverse limit of $I$) is $(T, I)$-nice, if the following hold.
\[ x[l(a_0)] \equiv a_0, \text{ where } \{a_0\} = T_0. \]

- For any \( a \in T_n \), we have \( a \leq x[l(a)] \).
- For any \( a \in T_n \) and any \( b \in \text{suc}_T^n(a) \), we have \( b \equiv b[l(a) \cap x[l(a), l(b))] \).
- For any \( \alpha < \nu \) and \( w \in P_\alpha \) with \( w \leq x[\alpha] \), if \( w \models_{P_\alpha} \) "there is a sequence \( \langle a_n \mid n < \omega \rangle \) s.t. for all \( n < \omega \), we have \( a_n \in T_n \), \( a_{n+1} \in \text{suc}_T^n(a_n) \), \( l(a_n) \leq \alpha \) and \( a_n \in \dot{G}_\alpha[l(a_n)]" \), then we have \( w \models x[[\alpha, \nu)] \equiv w \upharpoonright 1_\nu[[\alpha, \nu)] \).

In particular, in this case, for any limit \( Y \) of \( I \) with \( x \in Y \), we have

- \( \models_Y \) "\( x \in \dot{G}_Y \) iff there is a sequence \( \langle a_n \mid n < \omega \rangle \) s.t. for all \( n < \omega \), we have \( a_n \in T_n \), \( a_{n+1} \in \text{suc}_T^n(a_n) \) and \( a_n \in \dot{G}_Y[l(a_n)]" \).

We are going to perform a diagonal argument by constructing a nested antichain with associated objects. The following is from [M1].

1.8 Definition. Let \( \nu \) be a limit ordinal and \( J = \langle P_\alpha \mid \alpha \leq \nu \rangle \) be an iteration. Let \( T \) be a nested antichain in \( I = J[\nu] \). We say a structure \( \mathcal{F} = \langle (a, n) \models (x(a, n), T^{(a, n)}) \mid a \in T_n, n < \omega \rangle \) is a fusion structure in \( J \), if for all \( n < \omega \) and all \( a \in T_n \)

- \( T^{(a, n)} \) is a nested antichain in \( I \).
- \( x^{(a, n)} \in P_\nu \) is \( (T^{(a, n)}, I) \)-nice.
- \( a \leq x^{(a, n)}[l(a)] \) and if \( r \in T_0^{(a, n)} \), then \( l(r) = l(a) \).
- For any \( b \in \text{suc}_T^n(a) \), we have \( x^{(b, n+1)} \leq x^{(a, n)} \) and \( T^{(b, n+1)} \angle T^{(a, n)} \). By this we mean that for all \( i < \omega \) and all \( e \in T_i^{(b, n+1)} \), there is \( f \in T_i^{(a, n)} \) s.t. \( l(f) \leq l(e) \) and \( e[l(f)] \leq f \).

We are interested in the identification between a condition and the nested antichain \( T \) which served as indices of the fusion structure \( \mathcal{F} \). The condition, if any, is to work kind of a master condition by generically descending through \( T \).

1.9 Definition. Let \( \mathcal{F} \) be a fusion structure in \( J \), \( y \in P_\nu \) and write \( I = J[\nu] \). We say \( y \) is a fusion of \( \mathcal{F} \), if \( y \) is \( (T, I) \)-nice.
We have the following as in 4.9 Lemma of [M2].

1.10 Lemma. Let $\nu$ be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration. Let us write $I = J[\nu]$. If $F = \langle (a, n) \mapsto (x^{(\alpha, n)}, T^{(\alpha, n)}) \mid a \in T_n, n < \omega \rangle$ is a fusion structure in $J$ with fusion $y \in P_\nu$, then $y \forces \lceil \nu \{ \text{there is } \langle \dot{a}_n \mid n < \omega \rangle \text{ s.t. for all } n < \omega, \text{ we have } \dot{a}_n \in T_n, \dot{a}_{n+1} \in suc^T_n(\dot{a}_n), \dot{a}_n \in G_\nu \lceil l(\dot{a}_n) \text{ and } x^{(\dot{a}_n, n)} \in \dot{G}_\nu \} \}.$

The following are 4.10 Theorem and Claim 1 of 6.4 Theorem in [M2]. The former says every nested antichain is equivalent to a condition in the limit stage. The latter says the conditions which have equivalent nested antichains are dense in the limit stage. Hence we may assume every condition in any limit stage of any simple iteration has not only dynamical stages but also an associated nested antichain. Whenever we want a condition in the simple limit, we construct a nested antichain through the iteration with associated objects, i.e., a fusion structure.

1.11 Lemma. Let $\nu$ be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration. Let us write $I = J[\nu]$. Then we have

(1) For every nested antichain $T$ in $I$, there is $x \in P_\nu$ s.t. $x$ is $(T, I)$-nice.

(2) For every $p \in P_\nu$, there is a nested antichain $T$ in $I$ s.t. if $q \in P_\nu$ is $(T, I)$-nice, then $q \leq p$.

We quote the preservation theorem for the semiproperness with the simple iterations from [M2].

1.12 Theorem. Let $I = \langle P_i \mid i < \nu \rangle$ be a simple iteration of semiproper preorders, i.e., $I$ is a simple iteration and for all $i$ with $i + 1 < \nu$, we assume $\lceil \nu \{ \text{there is } \langle \dot{a}_n \mid n < \omega \rangle \text{ s.t. for all } n < \omega, \text{ we have } \dot{a}_n \in T_n, \dot{a}_{n+1} \in suc^T_n(\dot{a}_n), \dot{a}_n \in G_\nu \lceil l(\dot{a}_n) \text{ and } x^{(\dot{a}_n, n)} \in \dot{G}_\nu \} \}.$

Suppose we have $\alpha \leq \beta < \nu$, $P_\alpha$-names $\dot{M}$, $\dot{y}$ and $d \in P_\alpha$ s.t.

(1) $d \forces \lceil \nu \{ \text{there is } \langle \dot{G}_\alpha \} \subseteq \dot{M} \prec H_\theta^{V[\dot{G}_\alpha]}, \text{ } \dot{M} \text{ is countable and } \beta \in \dot{M} \}$

(2) $d \forces \lceil \nu \{ \text{there is } \dot{y} \in \dot{M} \cap P_\beta \text{ and } \dot{y}[\alpha \in \dot{G}_\alpha \}$

Then there is $d^* \in P_\beta$ s.t.

(3) $d^* \lceil \alpha = d.$
(4) \( d^* \models \vdash_{P_\beta} \dot{y} \in \dot{G}_\beta \).

(5) \( d^* \models \vdash_{P_\beta} \dot{M}[\dot{G}_\beta[[\alpha, \beta]] \cap \omega_1^\nu = \dot{M} \cap \omega_1^\nu \).

Proof. We consider a maximal antichain below \( d \) to decide the values of \( \dot{y} \). Then take a mixture of conditions gotten by 6.2 Theorem in [M2] to form \( d^* \).

\[ \square \]

§2 Main Theorem

Let us recall that a preorder \( P \) is \( \omega^\omega \)-bounding, if for any \( P \)-generic filter \( G \) over \( V \) and any \( f \in V[G] \) with \( f : \omega \to \omega \), there is \( g \in V \) with \( g : \omega \to \omega \) s.t. \( f <^* g \) holds. Namely, there is \( k < \omega \) s.t. for all \( n \) with \( k \leq n < \omega \), we have \( f(n) < g(n) \). We write this \( f <_k g \) for short. So \( f <_0 g \) means for all \( n < \omega \), we have \( f(n) < g(n) \). We may abusively use this notation to finite sequences of natural numbers.

We also recall that a preorder \( P \) is semiproper, if for all sufficiently large regular cardinals \( \theta \), all countable elementary substructures \( N \) of \( H_\theta \) with \( P \in N \) and all \( p \in P \cap N \), there is \( q \preceq p \) s.t. we have \( q \models \vdash_{P} \dot{N} \cap \omega_1^\nu = N[\dot{G}] \cap \omega_1^\nu \). For a cardinal \( \lambda \geq \omega_1 \) and \( S \subseteq [\lambda]^\omega \), we say \( S \) is semistationary, if \( \{ Y \in [\lambda]^\omega \mid \exists X \in S \ X \subseteq Y \text{ with } X \cap \omega_1 = Y \cap \omega_1 \} \) is stationary in \( [\lambda]^\omega \). It is known that \( P \) is semiproper iff \( P \) preserves not only \( \omega_1 \) but also every semistationary subset \( S \subseteq [\lambda]^\omega \) for every cardinal \( \lambda \geq \omega_1 \).

We consider a simple iteration \( I = \langle P_i \mid i < \nu \rangle \) of semiproper preorders. By this we mean that for every \( i < \nu \), we assume \( \models \vdash_{P_i} \dot{P}_{i+1} \) is semiproper". Namely, we force with a semiproper preorder each time. We know that the \( \omega^\omega \)-boundingness is trivially preserved by the 2-step iterations. And we have

2.1 Lemma. (CH) Let \( \nu \) be a limit ordinal and \( J = \langle P_i \mid i \leq \nu \rangle \) be a simple iteration of semiproper preorders. If for all \( i < \nu \), \( P_i \) is \( \omega^\omega \)-bounding, then so is \( P_\nu \).

So we immediately have

2.2 Theorem. (CH) If \( I = \langle P_i \mid i < \nu \rangle \) is a simple iteration of \( \omega^\omega \)-bounding and semiproper preorders, i.e., for all \( i \) with \( i + 1 < \nu \), we assume \( \models \vdash_{P_i} \dot{P}_{i+1} \) is \( \omega^\omega \)-bounding and semiproper, then for all \( i < \nu \), so are \( P_i \).
Proof of 2.1 Lemma. Let \( \dot{f} \) be a \( P_\nu \)-name s.t. \( \vdash_{P_\nu} "\dot{f} : \omega \rightarrow \omega." \) We want to find \( q \leq p \) in \( P_\nu \) and \( g : \omega \rightarrow \omega \) s.t. \( \vdash_{P_\nu} "\dot{f} \preceq g." \) To this end we define \( D = D(J, p, \dot{f}) \) and fix an operation \( \Pi = \Pi(J, p, \dot{f}) \) in the ground model. We write \( I = J[\nu] \) for short in the following.

\[ X_0 = (\alpha, x, T^x, \dot{f}_0, (\dot{s}_k | k < \omega)) \in D, \text{ if} \]

1. \( \alpha < \nu. \)
2. \( x \leq p \) in \( P_\nu \) and \( T^x \) is a nested antichain in \( I \) s.t. \( x \) is \( (T^x, I) \)-nice and for \( a_0^x \in T_0^x \), we assume \( l(a_0^x) = \alpha. \)
3. \( \dot{f}_0 \) is a \( P_\alpha \)-name s.t.
   
   \[ x[\alpha] \vdash_{P_\alpha} "\dot{f}_0 : \omega \rightarrow \omega." \]
4. For all \( k < \omega, \dot{s}_k \) are \( P_\alpha \)-names s.t. given any \( P_\alpha \)-generic filter \( G_\alpha \) over \( V \) with \( x[\alpha] \in G_\alpha \), we have the following in \( V[G_\alpha] \). We first calculate
   
   \[ f_0 = f_0[G_\alpha] \text{ and } s_k = \dot{s}_k[G_\alpha]. \]
   
   We then have two items
   
   \[ \bullet s_0 = x, s_{k+1} \leq s_k \text{ in } P_\nu \text{ and } s_k[\alpha] \in G_\alpha. \]
   
   \[ \bullet s_k[\nu] \vdash_{P_\nu} "\dot{f}[k] = (f_0[k])." \]

So we are looking at \( \dot{f} \) standing in \( V[G_\alpha] \) and it looked like \( f_0[k] \) with the condition \( s_k \).

For \( X_0 \in D \) and \( a_1^x \in T_1^x \), we consider

\[ \Pi(\alpha, a_1^x, x, T^x, \dot{f}_0, (\dot{s}_k | k < \omega)) \]

\[ = (l(a_1^x), x, S^x, \dot{f}_1, (i_k | k < \omega), (\dot{w}_k, \dot{g}_k | k < \omega), \dot{g}_\omega) \]

Let us write \( \beta = l(a_1^x) \) for short. Then concerning the image of \( \Pi \), we demand

\[ (\beta, x, S^x, \dot{f}_1, (i_k | k < \omega)) \in D. \]

5. Given any \( P_\beta \)-generic filter \( G_\beta \) over \( V \) with \( x[\beta] \in G_\beta \), let us calculate
   
   \[ f_1 = \dot{f}_1[G_\beta] \text{ and } t_k = i_k[G_\beta]. \]
   
   Let us write \( G_\alpha = G_\beta[\alpha] \) and calculate
   
   \[ f_0 = f_0[G_\alpha], s_k = \dot{s}_k[G_\alpha]. \]
   
   Then
   
   \[ \bullet \text{ If } s_k[\beta] \in G_\beta, \text{ then } t_k = s_k \text{ and so } f_0[k] = f_1[k]. \]
(6) For all $k < \omega$, $\dot{w}_k$, $\dot{g}_k$ and $\dot{g}_\omega$ are $P_\alpha$-names. Given any $P_\alpha$-generic filter $G_\alpha$ over $V$ with $\alpha \in G_\alpha$, let us calculate $s_k = \dot{s}_k[G_\alpha]$, $w_k = \dot{w}_k[G_\alpha]$, $g_k = \dot{g}_k[G_\alpha]$ and $g_\omega = \dot{g}_\omega[G_\alpha]$. Then we have the following in $V[G_\alpha]$.

- $w_k \leq s_k[\alpha], a_1^k$ and $w_k[\alpha] \in G_\alpha$.
- $g_k \in (\omega)^V$ and $w_k \models \neg_{P_\beta} \dot{f}_1 < \check{g}_k$.
- $w_k[\alpha]$ decides the value of $\dot{s}_k$.
- $g_\omega : \omega \to \omega$ s.t. $g_\omega(m) > \max\{g_k(m) \mid k \leq m\}$ for all $m < \omega$ and so $g_k < m g_\omega$ for all $m$ with $k < m < \omega$.

So we have the following three Facts in the generic extension $V[G_\alpha]$ as in (6) for all $k < \omega$ and all $g \in (\omega)^V$.

**Fact 1.** $w_k[\alpha] \models \neg_{P_\alpha} \dot{s}_k = \check{\dot{s}}_k$ and so $w_k \models \neg_{P_\alpha} \dot{i}_k = \check{\dot{s}}_k$.

**Proof.** Since $w_k[\alpha]$ decides the value of $\dot{s}_k$, $w_k[\alpha] \in G_\alpha$ and $\dot{s}_k[G_\alpha] = s_k$, we must have $w_k[\alpha] \models \neg_{P_\alpha} \dot{s}_k = \check{\dot{s}}_k$.

To observe the latter half, let $G_\beta$ be any $P_\beta$-generic filter over $V$ with $w_k \in G_\beta$. Let us write $G_\alpha$ for $G_\beta[\alpha]$. Since $w_k \in G_\beta[\alpha]$, we have $w_k[\alpha] \in G_\alpha$. Since $w_k[\alpha] \models \neg_{P_\alpha} \dot{s}_k = \check{\dot{s}}_k$, we have $w_k \leq s_k[\alpha] = \dot{s}_k[G_\alpha][\alpha]$ and so $\dot{s}_k[G_\alpha][\beta] \in G_\beta$. Hence by the definition of $i_k$, we have $i_k[G_\beta] = \dot{s}_k[G_\alpha] = s_k$. $\square$

**Fact 2.** If $f_0[k] < g[k]$ and $g_\omega < k g$, then $w_k \models \neg_{P_\beta} \dot{f}_1 < \check{g}$.

**Proof.** Suppose $f_0[k] < g[k]$ and $g_\omega < k g \in (\omega)^V$. Let $G_\beta$ be any $P_\beta$-generic filter over $V$ with $w_k \in G_\beta$. Since $w_k \models \neg_{P_\beta} \dot{f}_1 < \check{g}_k$, we have $f_1[G_\beta] < g_k < k g < g$. Since $w_k \models \neg_{P_\beta} \dot{i}_k = \check{\dot{s}}_k$, we have $i_k[G_\beta] = s_k$.

Since $w_k \leq x[\beta] \in G_\beta$, we have $i_k[G_\beta] \models \neg_{P_\beta} \dot{f}^k = (f_0[k])$. Since $s_k \models \neg_{P_\beta} \dot{f} = (f_0[k])$, we have $f_1[G_\beta]k = f_0[k] < g[k]$. So we have $\dot{f}_1[G_\beta] < g$. $\square$

**Fact 3.** If $f_0[k] < g[k]$, then $w_k \models \neg_{P_\nu} s_k[\beta, \nu] \models \neg_{P_\nu} \dot{f}^k < (g[k])$.

**Proof.** Suppose $f_0[k] < g[k]$. Let $G_\nu$ be any $P_\nu$-generic filter over $V$ with $w_k \models s_k[\beta, \nu] \in G_\nu$. Since $s_k \models \neg_{P_\nu} \dot{f}^k = (f_0[k])$, we have $f_1[G_\nu]k = f_0[k] < g[k]$. $\square
On (5): Given $G_\beta$ with $x|\beta \in G_\beta$, we descend along the $s_k[\beta]$ as long as $s_k[\beta] \in G_\beta$. And we set $t_k = s_k$. But once we hit $k < \omega$ s.t. $s_k[\beta] \notin G_\beta$, then we forget about $s_k$ and start to construct the rest of the $t_k$.

On (6): Suppose we have fixed $f_1$ and $\langle i_k | k < \omega \rangle$. Since we assume $P_\beta$ is $\omega$-bounding, it is routine to get $w_k$ and $g_k$.

The following is from 2.11 Lemma in [M1].

**Lemma.** Let $x \in P_\nu$ be $(T^x, I)$-nice, $a_1^x \in T^x_1$, $\beta = l(a_1^x)$, $y \in P_\nu$ and $y \leq a_1^x \text{^} x[[\beta, \nu])$. Then there is a nested antichain $T^y$ in $I$ s.t. $y$ is $(T^y, I)$-nice, $T^y_0 = \{a_0^y\}$, $l(a_0^y) = \beta$, $a_0^y \leq a_1^x$ and $T^y \triangleleft T^x$. Namely, we mean for all $n < \omega$ and all $e \in T^y_n$, there is $f \in T^x_{n+1}$ s.t. $l(f) \leq l(e)$ and $e[l(f)] \leq f$. $\square$

The following is concerned with a closure property of $D$.

**Subclaim.** Let $x \in P_\nu$ be $(T^x, I)$-nice and $a_1^x \in T^x_1$. Let us write $\beta$ for $l(a_1^x)$. Let $X_1 = (\beta, x, S^x, f_1, \langle i_k | k < \omega \rangle) \in D$, $k^* < \omega$, $w \in P_\beta$ and $t \in P_\nu$ s.t.

- $w \leq a_1^x$.
- $w \models_{P_\beta} "i_{k^*} = t"$.

Then there are $y$, $T^y$, $\langle \dot{u}_k | k < \omega \rangle$ s.t. $Y_0 = (\beta, y, T^y, f_1, \langle \dot{u}_k | k < \omega \rangle) \in D$ and for all $k < \omega$

- $y = w \text{^} t[[\beta, \nu]) \leq a_1^x \text{^} x[[\beta, \nu), t$.
- $y[[\beta] \models_{P_\beta} "\dot{u}_k [[\beta, \nu) = i_{k^*+k}[[\beta, \nu)]"$.
- $T^y \triangleleft T^x$.

**Proof.** Since $w \leq t[[\beta$, we may consider $y = w \text{^} t[[\beta, \nu) \in P_\nu$. We have $y \leq a_1^x \text{^} x[[\beta, \nu), t$. So by Lemma, we may take a nested antichain $T^y$ in $I$ s.t. $y$ is $(T^y, I)$-nice, $\{a_0^y\} = T^y_0$, $l(a_0^y) = \beta$, $a_0^y \leq a_1^x$ and $T^y \triangleleft T^x$. To define $\langle \dot{u}_k | k < \omega \rangle$, let $G_\beta$ be any $P_\beta$-generic filter over $V$ with $y[[\beta) \in G_\beta$. We define a sequence $\langle u_k | k < \omega \rangle$ by recursion in $V[G_\beta]$. Let $t_k = i_k[G_\beta]$ and $f_1 = f_1[G_\beta]$. So we have $t_{k^*} = t$. We first set $u_0 = y$. So we clearly have $u_0[[\beta, \nu) = t[[\beta, \nu) = t_{k^*+0}[[\beta, \nu)$. Suppose we have constructed $u_k$ so that $u_k \leq y$, $u_k[[\beta, \nu) \in G_\beta$ and $u_k[[\beta, \nu) = t_{k^*+k}[[\beta, \nu)$. Since $t_{k^*+k+1}[[\beta$, $u_k[[\beta \in G_\beta$,
we have a common extension \( e \in G_{\beta} \). Let \( u_{k+1} = e \frown t_{k} \frown k_{1}[[\beta, \nu]] \in P_{\nu} \).

Then we have \( u_{k+1} \leq u_{k}, t_{k} \frown k_{1} \). And so \( u_{k+1} \Vdash P_{\nu} \) \( f[k + 1] = (f[k]) \) "\( f[k + 1] = (f[k]) \)". This completes the construction. Now back in \( V \), let \( \dot{u}_{k} \) be a \( P_{\beta} \)-name of \( u_{k} \). This \( \langle \dot{u}_{k} | k < \omega \rangle \) together with \( y \) work.

Let \( \theta \) be a sufficiently large regular cardinal and \( N \) be a countable elementary substructure of \( H_{\theta} \) with \( p, \dot{f}, J, D, \Pi \in N \). Let us fix \( g : \omega \to \omega \) so that \( N \cap \omega <^* g \) which is meant that for any \( h \in N \cap \omega \), we have \( h <^* g \).

The following construction is crucial.

**Main Claim.** Suppose we have given \( X_{0} = (\alpha, x, T^{x}, \dot{f}_{0}, \langle \dot{s}_{k} | k < \omega \rangle) \in D, M, a \in P_{\alpha} \) and \( K < \omega \) s.t.

\[
(7) \quad a \leq x\lceil \alpha.
\]

\[
(8) \quad \dot{M} \text{ is a } P_{\alpha}-\text{name s.t.}
\]

- \( a \Vdash_{P_{\alpha}} "N \cup \{\dot{G}_{\alpha}, \check{X}_{0}\} \subseteq \dot{M} < H_{\theta}^{V[\dot{G}_{\alpha}]} \text{ and } \dot{M} \text{ is countable}"."
- \( a \Vdash_{P_{\alpha}} "N \cap \omega_{1}^{V} = \dot{M} \cap \omega_{1}^{V}"."

Since we assume CH, an enumeration of \( \omega \) in the order type \( \omega_{1} \) exists in \( N \). Since we assume \( P_{\alpha} \) is \( \omega \)-bounding, we consequently have

- \( a \Vdash_{P_{\alpha}} "N \cap (\omega) = \hat{M} < \omega_{1}^{V} \text{ and } \hat{M} \cap (\omega) <^* \check{y}"."

\[
(9) \quad x\lceil \alpha \Vdash_{P_{\alpha}} "\check{y} \in \hat{M}".
\]

Then the following \( d \in P_{\alpha} \) are dense below \( a \).

There are \( Y_{0} = (\beta, y, T^{y}, \dot{f}_{1}, \langle \dot{s}_{k} | k < \omega \rangle) \in D, a_{1}^{y} \in T_{1}^{y}, \dot{M}_{1}, d^{*} \in P_{\beta} \) and \( k^{*} \) s.t.

\[
(0) \quad y \leq a_{1}^{y} \frown x[[\beta, \nu]] \text{ in } P_{\nu} \text{ and that}
\]

- If \( \{a_{0}^{y}\} = T_{0}^{y} \), then \( \alpha \leq l(a_{0}^{y}) = l(a_{1}^{x}) = \beta \) and \( T^{y} \subseteq T^{x} \).

\[
(0^{+}) \quad d \Vdash_{P_{\alpha}} "\check{y}, \check{\beta} \in \hat{M}".
\]

\[
(0^{++}) \quad d^{*}[\alpha = d \leq y][\alpha \leq a_{1}^{y}[\alpha].
\]
\(d^* \leq y[\beta].\)

(8) \(M_1\) is a \(P_\beta\)-name s.t.
\[
\begin{align*}
\bullet d^* \models_{P_\beta} & \{N \cup \{\dot{G}_\beta, \check{Y}_0\} \subseteq \dot{M}_1 = \dot{M}[\dot{G}_\beta \lfloor [\alpha, \beta)] \prec H^{V(G_\beta)} \text{ and } \dot{M}_1 \text{ is countable}\}.
\end{align*}
\]

(9) \(y[\beta] \models_{P_\beta} \check{g}^{k*} >_{0} \check{g}\).

(10) \(K \leq k^* < \omega \) and \(y \models_{P_\nu} \check{g}^{k*} >_{0} \check{g}\).

Proof. Take any \(d\) (we abusively denote it) below \(a\). Let us write \(T^x_0 = \{a^x_0\}\). Fix any \(P_\alpha\)-generic filter \(G_\alpha\) over \(V\) with \(d \in G_\alpha\). We argue in \(V[G_\alpha]\).

(Step 1) Since \(d \leq a \leq x[\alpha \equiv a^x_0]\), we have \(a^x_0 \in G_\alpha\). Since \(\{b[\alpha] \mid b \in \text{suc}_P(a^x_0)\}\) is a maximal antichain below \(a^x_0\) in \(P_\alpha\), there is \(a^x_1 \in \text{suc}_P(a^x_0)\) s.t. \(a^x_1[\alpha] \in G_\alpha\). Let us write \(M = M[G_\alpha]\). Since \(T^x_0, a^x_0, G_\alpha \in M\), we have \(a^x_1 \in M\) and so \(l(a^x_1) \in M\).

(Step 2) Let \(X = (\alpha, a^x_1, x, T^x_0, \check{f}_0, \langle \check{s}_k \mid k < \omega\rangle)\) and
\[
Y = \Pi(X) = (\beta, x, S^x, \check{f}_1, \langle \check{t}_k \mid k < \omega\rangle, \langle \check{w}_k, \check{g}_k \mid k < \omega\rangle, \check{g}_\omega).
\]
Since \(\Pi \in N \cup \{X_0\} \subseteq M\), we have \(X, Y \in M\).

(Step 3) Let us write \(f_0 = \check{f}_0[G_\alpha], s_k = \check{s}_k[G_\alpha], w_k = \check{w}_k[G_\alpha], g_k = \check{g}_k[G_\alpha], g_\omega = \check{g}_\omega[G_\alpha]\). Since \(X, Y, G_\alpha \in M\), we have \(f_0, \langle s_k \mid k < \omega\rangle, \langle w_k, g_k \mid k < \omega\rangle, g_\omega \in M\). Since \(g_\omega \in M \cap (\omega^\omega)^{V[G_\alpha]}\), there is \(k^*\) s.t. \(K \leq k^* < \omega\) and \(g_\omega <_{k^*} g\). Since \(f_0 <_0 g\), we have

- \(f_0[k^*] <_0 g[k^*]\) and \(g_\omega <_{k^*} g\). Hence by Facts 1 through 3, we have

- \(w_{k^*} \models_{P_\nu} \check{f}_1 <_{0} \check{g}\).

- \(w_{k^*} \models_{P_\nu} \check{f}_1 <_{0} (g[k^*])\). We set \(y = w_{k^*} \models_{P_\nu} \check{f}_1 <_{0} (g[k^*])\). Then we have

- \(y \in M\). By Subclaim, we have \(T^y\) and \(\langle \check{u}_k \mid k < \omega\rangle\) s.t.
\( \bullet \ Y_0 = (\beta, y, T^y, \dot{f}_1, \langle \dot{u}_k \mid k < \omega \rangle) \in D \cap M. \)

\( \bullet \ y \leq a^\ast_{\check{x}} \sim x[[\beta, \nu]] \) and \( y[\alpha] \in G_\alpha. \)

\( \bullet \ T^y \triangleleft T^x. \)

(9) \( y[\beta] \not\models_{P_\beta}^{V} \dot{f}_1 <_0 \check{g}. \)

(10) \( K \leq k^* < \omega \) and \( y \models_{P_\nu}^{V} \dot{f}[k^*] <_0 (g[\check{k}^*]). \)

(Step 4) Now back in \( V \), by extending \( d \), we may decide the values of \( a^\ast_{\check{x}}, k^* \) and \( Y_0 \in D \cap M \). So we may assume

\( \bullet \ d \models_{P_\alpha}^{\dot{\alpha}} \check{\beta}, \check{y} \in \dot{M} \) and \( \check{y}[\check{\alpha}] \in \dot{G}_\alpha. \)

Then by the iteration theorem for the semiproperness, we have \( d^* \in P_\beta \) s.t.

\( \bullet \ d^*[\alpha] = d, \ d^* \leq y[\beta]. \)

\( \bullet \ d^* \models_{P_\beta}^{\dot{M}} \dot{M} \cap \omega_1^V = \dot{M}[\dot{G}_\beta[[\check{\alpha}, \check{\beta}]]] \cap \omega_1^V. \)

(Step 5) Let \( \dot{M}_1 \) be a \( P_\beta \)-name s.t.

\( \bullet \ d^* \models_{P_\beta}^{\dot{M}_1} \dot{M}_1 = \dot{M}[\dot{G}_\beta[[\check{\alpha}, \check{\beta}]]]. \)

Then we certainly have (8). In particular,

\( \bullet \ d^* \models_{P_\beta}^{\dot{M}_1 \cup \{\dot{G}_\beta, \check{Y}_0\}} \subseteq \dot{M}_1 < H_{\theta}^{\dot{G}_\beta} \) and \( \dot{M}_1 \) is countable.

\( \bullet \ d^* \models_{P_\beta}^{N \cap \omega_1^V} = \dot{M}_1 \cap \omega_1^V. \)

This establishes Main Claim. \( \square \)

We now carry a recursive construction of a fusion structure

\( F = \langle \langle a, n \rangle \mapsto (x^{(a,n)}, T^{(a,n)}) \mid a \in T_n, n < \omega \rangle \)

together with

\( \langle \langle a, n \rangle \mapsto (x^{(a,n)}, \dot{f}^{(a,n)}, \langle \dot{s}^{(a,n)}_k \mid k < \omega \rangle, \dot{M}^{(a,n)}) \mid a \in T_n, n < \omega \rangle \)

such that if we denote \( X^{(a,n)} = (l(a), x^{(a,n)}, T^{(a,n)}, \dot{f}^{(a,n)}, \langle \dot{s}^{(a,n)}_k \mid k < \omega \rangle) \)

for short, then
\(*\) $X^{(a,n)} \in D$.

- For any $b \in \text{suc}^n_T(a)$, we have $x^{(b,n+1)} \leq x^{(a,n)}$ and $T^{(b,n+1)} \angle T^{(a,n)}$.

(7) $a \leq x^{(a,n)} \vert l(a)$.

(8) $\dot{M}^{(a,n)}$ is a $P_{l(a)^{-}}$-name s.t.

- $a \vdash_{P_{l(a)}} "N \cup \{\check{G}_{l(a)}, X^{(\check{a},n)}\} \subseteq \dot{M}^{(a,n)} \prec H_{\theta}^{V[\check{G}_{l(a)}]}$ and $\dot{M}^{(a,n)}$ is countable”.

- $a \vdash_{P_{l(a)}} "N \cap \omega^V_1 = \dot{M}^{(a,n)} \cap \omega^V_1"$.

(9) $x^{(a,n)} \vert l(a) \vdash_{P_{\nu}} "j^{(a,n)} <_0 \check{g}"$.

(10) For $b \in \text{suc}^n_T(a)$, we have $x^{(b,n+1)} \vdash_{P_{\nu}} "j <_0 \check{g} <_0 \check{g}"$.

If $q$ is a fusion of $\mathcal{F}$, then $q \leq p$ in $P_\nu$ and $q \vdash_{P_{\nu}} "\check{f} <_0 \check{g}"$. So we would be done.

Here is the construction. We first take $f : \omega \rightarrow \omega$ and $\langle s_k \mid k < \omega \rangle$ s.t. $s_0 = p$ and for all $k < \omega$, we have $s_{k+1} \leq s_k$ and $s_k \vdash_{P_{\nu}} "f[k] = (f \upharpoonright k)"$. We may assume $f$, $\langle s_k \mid k < \omega \rangle \in N$. We may also fix a nested antichain $T^p$ s.t. $p$ is $(T^p, I)$-nice and $T^p_0 = \{\emptyset\}$. We may assume $T^p \in N$.

Now we let $T_0 = \{\emptyset\}$. And set $x^{(0,0)} = p$, $T^{(0,0)} = T^p$, $\check{j}^{(0,0)} = \check{f}$, $\langle s_k^{(0,0)} \mid k < \omega \rangle = \langle s_k \mid k < \omega \rangle$ and $M^{(0,0)} = \check{N}$.

Since we may assume $g : \omega \rightarrow \omega$ s.t. $f <_0 g$ and $N \cap \omega <^* g$, we are done at $T_0$.

For successor stages, suppose we have gotten $T_n$ and the associated objects. Fix any $a \in T_n$ and apply Main Claim to $X^{(a,n)}$, $\dot{M}^{(a,n)}$, $a$ and $n$. It is routine to form $\text{suc}^n_T(a)$ and attach associated objects to each $b \in \text{suc}^n_T(a)$.

This completes the construction.

\[\square\]

\section{Applications}

We consider an application of our main theorem. We pick up a forcing axiom from [F].
3.1 Definition. ([F]) The Cofinal Branch Principle (CBP) says: Every tree of height $\omega_1$ which preserves every stationary subset of $\omega_1$ has a cofinal branch.

We have the following.

3.2 Proposition. ([F]) The Semiproper Forcing Axiom (SPFA) implies CBP.

We give a forcing construction which establishes the consistency of CBP and more.

3.3 Theorem. (CH) Let $\kappa$ be a supercompact cardinal. We may construct a notion of forcing $P_\kappa$ which is semiproper, has the $\kappa$-c.c. and we have the following in $V^{P_\kappa}$.

- For any $\dot{f} : \omega \rightarrow \omega$, there exists $g \in (^{\omega}\omega)^V$ s.t. $\dot{f} <^* g$.
- CBP holds.

3.4 Corollary. CBP does not imply SPFA.

To give a proof to the theorem, we make preparations.

3.5 Lemma. The following are equivalent.

(1) CBP.

(2) Every tree of height $\omega_1$ which is semiproper has a cofinal branch.

Proof. Since semiproper preorders preserve every stationary subset of $\omega_1$, we know that (1) implies (2). To show the converse the following suffice.

Claim 1. (2) implies the Strong Reflectin Principle (SRP).

Claim 2. SRP implies that if a preorder $P$ preserves every stationary subset of $\omega_1$, then $P$ is semiproper.

Proof of Claim 1. By p. 58 of [Be], the notion of forcing which forces a strong reflection sequence is a tree of height $\omega_1$ and is semiproper. Hence (2) implies SRP.

Proof of Claim 2. Let $\lambda \geq \omega_1$ be a cardinal and $S \subseteq [\lambda]^{\omega}$ be a semistationary
subset. We show $\neg p$ "$S$ remains semistationary in $[\lambda]^\omega$". To this end suppose $p \text{ forces } \langle \dot{\pi} \colon \omega \lambda \to \lambda \rangle$. We want to find $q \leq p$, $X \in S$ and $\dot{Y}$ s.t. $q \text{ forces } X \subseteq \dot{Y} \in [\lambda]^\omega$, $X \cap \omega_1 \dot{Y} = \dot{Y} \cap \omega_1 \dot{Y}$ and $\dot{Y}$ is $\dot{\pi}$-closed".

Let $\theta$ be a sufficiently large regular cardinal. Then by SRP, we have $\langle N_i \mid i < \omega_1 \rangle$ s.t.

- $N_i$ is a countable elementary substructure of $H_\theta$.
- $P, \dot{\pi} \in N_0$.
- If $j < \omega_1$, then $\langle N_i \mid i \leq j \rangle \in N_{j+1}$.
- If $j < \omega_1$ is a limit ordinal, then $N_j = \bigcup \{N_i \mid i < j\}$.
- Either there is $X \in S$ s.t. $X \subseteq N_i \cap \lambda$ and $X \cap \omega_1 = N_i \cap \omega_1$.
- Or, for any countable elementary substructure $N$ of $H_\theta$ s.t. $N_i \subseteq N$ and $N_i \cap \omega_1 = N \cap \omega_1$, we have no $X \in S$ s.t. $X \subseteq N \cap \lambda$ and $X \cap \omega_1 = N \cap \omega_1$.

**Subclaim.** $S_0 = \{i < \omega_1 \mid \exists X \in S X \subseteq N_i \cap \lambda \text{ and } X \cap \omega_1 = N_i \cap \omega_1\}$ is stationary in $\omega_1$.

**Proof.** By contradiction. Suppose $\omega_1 \setminus S_0$ contained a club subset $C_0$ of $\omega_1$. Since $S$ is semistationary, we may take a sufficiently large regular cardinal $\chi$ and a countable elementary substructure $M$ of $H_\chi$ so that $C_0, H_\theta, \langle N_i \mid i < \omega_1 \rangle \in M$ and there is $X \in S$ s.t. $X \subseteq M \cap \lambda$ and $X \cap \omega_1 = M \cap \omega_1$.

Let $i_1 = M \cap \omega_1$. Since $C_0$ is a club and $C_0 \in M$, we have $i_1 \in C_0$. Since $\langle N_i \mid i < \omega_1 \rangle \in M$, each $N_i$ is countable and continuously increasing, we have $N_{i_1} = \bigcup \{N_i \mid i < i_1\} \subseteq M \cap H_\theta \prec H_\theta$ and $N_{i_1} \cap \omega_1 = i_1 = (M \cap H_\theta) \cap \omega_1$.

Since we have $X \in S$ and $X \subseteq (M \cap H_\theta) \cap \lambda$ and $X \cap \omega_1 = (M \cap H_\theta) \cap \omega_1$, and since $\langle N_i \mid i < \omega_1 \rangle$ is a strong reflection sequence, we must have some $\tilde{X} \in S$ s.t. $\tilde{X} \subseteq N_{i_1} \cap \lambda$ and $\tilde{X} \cap \omega_1 = N_{i_1} \cap \omega_1$. But $i_1 \in C_0 \subseteq \omega_1 \setminus S_0$, so $i_1 \notin S_0$. This contradicts the definition of $S_0$. This establishes Subclaim.

Now since $P$ preserves every stationary subset of $\omega_1$, we have $p \text{ forces } S_0 \text{ remains stationary}$. However, $p \text{ forces } \dot{C} = \{i < \omega_1 \mid N_i \cap \omega_1 = i = N_i[G] \cap \omega_1\}$ is a club". Therefore, $p$ forces the following. There is $i_0 \in S_0 \cap \dot{C}$. Since
We have $X \in S$ s.t. $X \subseteq N_{i_0} \cap \lambda$ and $X \cap \omega_1 = N_{i_0} \cap \omega_1$. Since $i_0 \in C$, we have $N_{i_0} \cap \omega_1 = i_0 = N_{i_0}[G] \cap \omega_1$. Let $Y = N_{i_0}[G] \cap \lambda$. Then $X \subseteq Y$, $X \cap \omega_1 = Y \cap \omega_1$ and $Y$ is $\pi$-closed. So we may take $q \leq p$ which decides the value of $X \in S$. This establishes Claim 2 and 3.5 Lemma.

**Proof** of 3.3 Theorem. Let $\kappa$ be a supercompact cardinal. We may assume CH. We construct a simple iteration $J = \langle P_\alpha \mid \alpha \leq \kappa \rangle$. The construction is exactly the same as when we force SPFA. So $P_\kappa$ is semiproper and has the $\kappa$-c.c. However, since we force with $\sigma$-Baire preorders each time in this construction, we are iterating with $\omega$-bounding and semiproper preorders. Therefore by our main theorem, $P_\kappa$ is $\omega$-bounding.

**References**


[M1]: T. Miyamoto, A Note on Nice Iterations, 1993.