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<th>Title</th>
<th>ON THE PAINLEVE I HIERARCHY (Analysis of Painleve equations)</th>
</tr>
</thead>
<tbody>
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ON THE PAINLEVÉ I HIERARCHY

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Every solution of the first Painlevé equation

\[(I) \quad Z'' = 6Z^2 + 4t\]

\(' = d/dt) is meromorphic in \(\mathbb{C}\), that is to say, equation (I) admits the Painlevé property. It is known that the fourth-order equation

\[(I_4) \quad Z^{(4)} = 20ZZ'' + 10(Z')^2 - 40Z^3 + 16t\]

also admits the Painlevé property, which is proved by using Miwa’s result concerning the isomonodromic deformation ([1,2,3]). In this note, we show that

1. there exists a hierarchy of systems of nonlinear equations, from which we can derive (I), (I_4) and

\[(I_6) \quad Z^{(6)} = 28ZZ^{(4)} + 56ZZ^{(3)} + 42(Z'')^2 - 280(Z^2Z'' + Z(Z')^2 - Z^4) + 64t,\]

2. all the systems in the hierarchy and the nonlinear equations derived from it such as (I), (I_4), (I_6) admit the Painlevé property.

1. Results

Consider the following formal power series in \(\xi\):

\[Q(\xi) = \sum_{\nu \geq 1} Z_{\nu} \xi^\nu,\]

\[R(\xi) = \sum_{\nu \geq 1} U_{\nu} \xi^\nu,\]

\[F(\xi) = 2\xi^{-1}Q(\xi)(1 + Z_{1}\xi) + (\xi^{-1}Q(\xi)^2 - R(\xi)^2)(1 - Q(\xi))^{-1} - u_0^2,\]

where \(u_0, Z_{\nu}, U_{\nu} (\nu \in \mathbb{N})\) are parameters. Then, \(F(\xi)\) is written in the form

\[F(\xi) = \sum_{\nu \geq 0} F_{\nu} \xi^\nu.\]
\[ F_0 = 2Z_1 - u_0^2, \]
\[ F_\nu = 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1) \quad (\nu \in \mathbb{N}). \]

Here \( G_\nu(Z_j, U_k; ...) \) denotes a polynomial in \( Z_j \) and \( U_k \) \((1 \leq j \leq \nu, 1 \leq k \leq \nu - 1)\).

Let \( m \) be a nonnegative integer and \( t \) a variable. Then the relations

\[ \frac{d}{dt}(u_0 + R(\xi)) \equiv F(\xi) + 2(t - Z_{m+1})\xi^m \quad (\text{mod } \xi^{m+1}), \]
\[ \frac{d}{dt}Q(\xi) \equiv 2R(\xi) \quad (\text{mod } \xi^{m+1}), \]

define the following systems:

for \( m = 0 \),

\[ (S_0) \quad u_0' = 2t - u_0^2; \]

for \( m \geq 1 \),

\[ u_0' = 2Z_1 - u_0^2, \]
\[ Z_{\nu}' = 2U_{\nu}, \]
\[ U_{\nu}' = 2Z_{\nu+1} + G_{\nu}(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1), \]
\[ Z_{m}' = 2U_{m}, \]
\[ U_{m}' = 2t + G_{m}(Z_j, U_k; 1 \leq j \leq m, 1 \leq k \leq m - 1) \]

\((1 \leq \nu \leq m - 1)\). Then we have

**Theorem 1.1.** Every solution \((u_0(t), Z_{\nu}(t), U_{\nu}(t))\) \((1 \leq \nu \leq m)\) of \((S_m)\) \((m \geq 0)\) is meromorphic in \( \mathbb{C} \).

As an immediate corollary of this theorem, we have

**Corollary 1.2.** Every solution of \((I_4)\) or \((I_6)\) is meromorphic in \( \mathbb{C} \).

It is known that, for an arbitrary solution \( P(t) \) of \((I)\), every solution of

\[ y'' - 2P(t)y = 0 \]

is meromorphic in \( \mathbb{C} \). Furthermore we have

**Corollary 1.3.** Let \( P_4(t) \) \((\text{resp. } P_6(t))\) be an arbitrary solution of \((I_4)\) \((\text{resp. } (I_6))\).

Then every solution of

\[ y'' - 2P_4(t)y = 0 \quad (\text{resp. } y'' - 2P_6(t)y = 0) \]

is meromorphic in \( \mathbb{C} \).
2. Outline of the proof of Theorem 1.1

Consider the 2 by 2 matrix linear differential equation

\[
\frac{d\Xi}{dx} = A(x)\Xi, \quad A(x) = -\sum_{j=0}^{2(m+1)} A_{-j}x^j + A_1 x^{-1}.
\]

Here \( A_{-\nu} \) are given as below:

\[
A_{-2(m+1)} = J, \quad A_{-(2m+1)} = -u_0 L,
A_{-2m} = v_1 K - w_1 J, \quad A_{-(2m-1)} = -u_1 L,
A_{-2(m+1)+2i} = v_i K - w_i J, \quad A_{-(2m+1)+2i} = -u_i L \quad (1 \leq i \leq m),
A_0 = s(J + K), \quad A_1 = (I - L)/2
\]

with

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Proposition 2.1.** Let \( t, u_0, u_1, \ldots, u_m, v_1, \ldots, v_m \) be arbitrary parameters. System (2.1) admits a formal matrix solution of the form

\[
\Xi = \Xi(x) = Y(x)\exp T(x),
\]

\[
T(x) = -\frac{J}{2m+3}x^{2m+3} - tJx + \frac{1}{2} \log(1/x), \quad Y(x) = \sum_{j \geq 1} Y_j x^{-j},
\]

if and only if

\[
w_1 = u_0^2/2,
\]

\[
w_\nu = \frac{1}{2} \left( \sum_{j=1}^{\nu-1} w_j w_{\nu-j} - \sum_{j=1}^{\nu-1} v_j v_{\nu-j} + \sum_{j=1}^{\nu} u_{j-1} u_{\nu-j} \right),
\]

\[
s = t - \frac{1}{2} \left( \sum_{j=1}^{m} w_j w_{m+1-j} - \sum_{j=1}^{m} v_j v_{m+1-j} + \sum_{j=1}^{m+1} u_{j-1} u_{m+1-j} \right)
\]

(1 \leq \nu \leq m).

For the deformation parameter \( t \), the deformation equation with respect to (2.1) is written in the form

\[
dA(x) = \frac{\partial}{\partial x} \Omega(x,t) + [\Omega(x,t), A(x)],
\]

\[
\Omega(x,t) = \Phi_{-1}(t)x + \Phi_0(t),
\]

where \( \Phi_{-1}(t) \) and \( \Phi_0(t) \) are 1-forms of \( t \) defined by

\[
\sum_{k=-\infty}^{1} \Phi_{-k}(t)x^k = Y(x)(-xdt)JY(x)^{-1}.
\]
Proposition 2.2. Equations (2.3) is equivalent to

\[ u'_{\nu-1} = 2v_\nu, \quad v'_\nu = 2u_\nu + 2u_0w_\nu, \quad w'_\nu = 2u_0v_\nu, \]

\[ u'_m = 2s, \quad s' = 1 - 2u_0s \]

(1 ≤ \(\nu\) ≤ \(m\)), where \(w_\nu, s\) are the parameters defined by (2.2).

System (2.1) possesses an apparent singularity at \(x=0\), and Miwa's theorem [2] is not applicable. To remove it, we employ the Schlesinger transformation

\[ W = \Psi(x)\Xi, \quad \Psi(x) = \begin{pmatrix} 1 & 1 \\ u_0/2 & u_0/2 + x \end{pmatrix}. \]

Then system (2.1) is changed into

\[ \frac{dW}{dx} = B(x)W, \quad B(x) = -\sum_{j=0}^{2(m+1)} B_{-j}x^j, \]

where

\[ B_{-2(m+1)} = J, \]

\[ B_{-(2\nu+1)} = \begin{pmatrix} -u_0^2(v_{m+1-\nu} + w_{m+1-\nu}) & 2(v_{m-\nu} + w_{m-\nu}) \\ -u_0(v_{m+1-\nu} + w_{m+1-\nu}) - u_{m+1-\nu} & u_{m+1-\nu} + u_0(v_{m-\nu} + w_{m-\nu}) \end{pmatrix}, \]

\[ B_{-2\nu} = \begin{pmatrix} -u_0(v_{m+1-\nu} + w_{m+1-\nu}) - u_{m+1-\nu} & 2(v_{m-\nu} + w_{m-\nu}) \\ -u_0^2(v_{m+1-\nu} + w_{m+1-\nu}) & u_{m+1-\nu} + u_0(v_{m+1-\nu} + w_{m+1-\nu}) \end{pmatrix}, \]

\[ B_{-1} = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}, \]

(1 ≤ \(\nu\) ≤ \(m\)), \(v_0 = w_0 = 0\). Applying Miwa's theorem to (2.4), we can show that \(u_0, Z_\nu = v_\nu + w_\nu\) and \(U_\nu = u_\nu + u_0Z_\nu\) are meromorphic in \(\mathbb{C}\). Since the isomonodromy property is invariant under the Schlesinger transformation, from (2.2) and Proposition 2.2 we derive the deformation equation with respect to \(Z_\nu, U_\nu\), which coincides with \((S_m)\). This completes the proof.

3. Derivation of the corollaries

Eliminating the unknown variables other than \(Z_1\), from \((S_2)\) and \((S_3)\) we get equations \((I_4)\) and \((I_6)\), respectively. Thus we have Corollary 1.2.

To show Corollary 1.3, let us consider, for example, system \((S_3)\). By Corollary 1.2, an arbitrary solution \(Z = P_6(t)\) of \((I_6)\) is meromorphic in \(\mathbb{C}\), and, around each pole \(t = t_0\), it is expanded into one of the following Laurent series:

\[ (t - t_0)^{-2} + \cdots, \quad 3(t - t_0)^{-2} + \cdots, \quad 6(t - t_0)^{-2}. \]
By Theorem 1.1, every solution of
\begin{equation}
(3.2) \quad u' = 2P_6(t) - u^2,
\end{equation}
is meromorphic in $\mathbb{C}$, which is the first equation of $(S_3)$. The transformation $u = y'/y$ takes (3.2) into
\begin{equation}
(3.3) \quad y'' - 2P_6(t)y = 0.
\end{equation}
Let $y(t)$ be an arbitrary solution of (3.3). It is sufficient to show that an arbitrary pole $t = t_0$ of $P_6(t)$ is at most a pole of $y(t)$. To do this, we note that $u(t) = y'(t)/y(t)$ is written in the form
\[ u(t) = c(t - t_0)^{-1} + \cdots , \]
around it, where $c$ is an integer equal to one of $-3, -2, -1, 2, 3, 4$. Hence we get an expression of the form
\[ y(t) = (t - t_0)^c \sum_{j=0}^{\infty} C_j(t - t_0)^j, \]
from which Corollary 1.3 follows.

REFERENCES