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ON THE PAINLEVÉ I HIERARCHY

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Every solution of the first Painlevé equation

\[(I) \quad Z'' = 6Z^2 + 4t \]

\(('= d/dt)\) is meromorphic in \(\mathbb{C}\), that is to say, equation (I) admits the Painlevé property. It is known that the fourth-order equation

\[(I_4) \quad Z^{(4)} = 20ZZ'' + 10(Z')^2 - 40Z^3 + 16t \]

also admits the Painlevé property, which is proved by using Miwa's result concerning the isomonodromic deformation ([1,2,3]). In this note, we show that

1. there exists a hierarchy of systems of nonlinear equations, from which we can derive (I), (I_4) and

\[(I_6) \quad Z^{(6)} = 28ZZ^{(4)} + 56ZZ^{(3)} + 42(Z'')^2 - 280(Z^2Z'' + Z(Z')^2 - Z^4) + 64t, \]

2. all the systems in the hierarchy and the nonlinear equations derived from it such as (I), (I_4), (I_6) admit the Painlevé property.

1. Results

Consider the following formal power series in \(\xi\):

\[Q(\xi) = \sum_{\nu \geq 1} Z_\nu \xi^\nu,\]

\[R(\xi) = \sum_{\nu \geq 1} U_\nu \xi^\nu,\]

\[F(\xi) = 2\xi^{-1}Q(\xi)(1 + Z_1 \xi) + (\xi^{-1}Q(\xi)^2 - R(\xi)^2)(1 - Q(\xi))^{-1} - u_0^2,\]

where \(u_0, Z_\nu, U_\nu (\nu \in \mathbb{N})\) are parameters. Then, \(F(\xi)\) is written in the form

\[F(\xi) = \sum_{\nu \geq 0} F_\nu \xi^\nu\]
\[ F_0 = 2Z_1 - u_0^2, \]
\[ F_\nu = 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1) \quad (\nu \in \mathbb{N}). \]

Here \( G_\nu(Z_j, U_k; ...) \) denotes a polynomial in \( Z_j \) and \( U_k \) \((1 \leq j \leq \nu, 1 \leq k \leq \nu - 1)\).

Let \( m \) be a nonnegative integer and \( t \) a variable. Then the relations
\[
\frac{d}{dt}(u_0 + R(\xi)) \equiv F(\xi) + 2(t - Z_{m+1})\xi^m \quad (\text{mod } \xi^{m+1}),
\]
\[
\frac{d}{dt}Q(\xi) \equiv 2R(\xi) \quad (\text{mod } \xi^{m+1}),
\]
define the following systems:

for \( m = 0 \),

\((S_1)\)
\[ u_0' = 2t - u_0^2; \]

for \( m \geq 1 \),

\[ u_0' = 2Z_1 - u_0^2, \]
\[ Z_\nu' = 2U_\nu, \]

\((S_m)\)
\[ U_\nu' = 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1), \]
\[ Z_m' = 2U_m, \]
\[ U_m' = 2t + G_m(Z_j, U_k; 1 \leq j \leq m, 1 \leq k \leq m - 1) \]
\((1 \leq \nu \leq m - 1)\). Then we have

**Theorem 1.1.** Every solution \((u_0(t), Z_\nu(t), U_\nu(t)) (1 \leq \nu \leq m)\) of \((S_m) (m \geq 0)\) is meromorphic in \( \mathbb{C} \).

As an immediate corollary of this theorem, we have

**Corollary 1.2.** Every solution of \((I_4)\) or \((I_6)\) is meromorphic in \( \mathbb{C} \).

It is known that, for an arbitrary solution \( P(t) \) of \((I)\), every solution of
\[ y'' - 2P(t)y = 0 \]
is meromorphic in \( \mathbb{C} \). Furthermore we have

**Corollary 1.3.** Let \( P_4(t) \) (resp. \( P_6(t) \)) be an arbitrary solution of \((I_4)\) (resp. \((I_6)\)). Then every solution of
\[ y'' - 2P_4(t)y = 0 \quad \text{(resp. } y'' - 2P_6(t)y = 0) \]
is meromorphic in \( \mathbb{C} \).
2. Outline of the proof of Theorem 1.1

Consider the 2 by 2 matrix linear differential equation

\[
\frac{d\Xi}{dx} = A(x)\Xi, \quad A(x) = -\sum_{j=0}^{2(m+1)} A_{-j} x^j + A_1 x^{-1}.
\]

Here \( A_{-\nu} \) are given as below:

\[
A_{-2(m+1)} = J, \quad A_{-(2m+1)} = -u_0 L,
\]
\[
A_{-2m} = v_1 K - w_1 J, \quad A_{-(2m-1)} = -u_1 L,
\]
\[
A_{-2(m+1)+2i} = v_i K - w_i J, \quad A_{-(2m+1)+2i} = -u_i L \quad (1 \leq i \leq m),
\]
\[
A_0 = s(J+K), \quad A_1 = (I-L)/2
\]

with

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Proposition 2.1.** Let \( t, u_0, u_1, \ldots, u_m, v_1, \ldots, v_m \) be arbitrary parameters. System (2.1) admits a formal matrix solution of the form

\[
\Xi(x) = Y(x) \exp T(x),
\]
\[
T(x) = -\frac{J}{2m+3} x^{2m+3} - t J x + \frac{1}{2} \log(1/x), \quad Y(x) = \sum_{j \geq 1} Y_j x^{-j},
\]

if and only if

\[
w_1 = u_0^2/2,
\]
\[
w_\nu = \frac{1}{2} \left( \sum_{j=1}^{\nu-1} w_j w_{\nu-j} - \sum_{j=1}^{\nu-1} v_j v_{\nu-j} + \sum_{j=1}^{\nu} u_{j-1} u_{\nu-j} \right),
\]
\[
s = t - \frac{1}{2} \left( \sum_{j=1}^{m} w_j w_{m+1-j} - \sum_{j=1}^{m} v_j v_{m+1-j} + \sum_{j=1}^{m+1} u_{j-1} u_{m+1-j} \right)
\]

\((1 \leq \nu \leq m)\).

For the deformation parameter \( t \), the deformation equation with respect to (2.1) is written in the form

\[
dA(x) = \frac{\partial}{\partial x} \Omega(x,t) + [\Omega(x,t), A(x)],
\]
\[
\Omega(x,t) = \Phi_{-1}(t)x + \Phi_0(t),
\]

where \( \Phi_{-1}(t) \) and \( \Phi_0(t) \) are 1-forms of \( t \) defined by

\[
\sum_{k=-\infty}^{1} \Phi_{-k}(t) x^k = Y(x)(-xdt)JY(x)^{-1}.
\]
Proposition 2.2. Equation (2.3) is equivalent to

\[ u'_{\nu-1} = 2v_{\nu}, \quad v'_{\nu} = 2u_{\nu} + 2u_{0}w_{\nu}, \quad w'_{\nu} = 2u_{0}v_{\nu}, \]

\[ u'_{m} = 2s, \quad s' = 1 - 2u_{0}s \]

(1 ≤ \(\nu\) ≤ \(m\)), where \(w_{\nu}, s\) are the parameters defined by (2.2).

System (2.1) possesses an apparent singularity at \(x = 0\), and Miwa's theorem [2] is not applicable. To remove it, we employ the Schlesinger transformation

\[ W = \Psi(x)\Xi, \quad \Psi(x) = \begin{pmatrix} 1 & 1/2 \\ u_{0}/2 & u_{0}/2 + x \end{pmatrix}. \]

Then system (2.1) is changed into

\[(2.4)\]

\[ \frac{dW}{dx} = B(x)W, \quad B(x) = - \sum_{j=0}^{2(m+1)} B_{-j}x^{j}, \]

where

\[ B_{-2(m+1)} = J, \]

\[ B_{-(2\nu+1)} = \begin{pmatrix} -u_{m-\nu} - u_{0}(v_{m-\nu} + w_{m-\nu}) & 2(v_{m-\nu} + w_{m-\nu}) \\ -(v_{m+1-\nu} + w_{m+1-\nu})/2 - u_{0}u_{m-\nu} - v_{m-(\nu-1)} & u_{m-\nu} + u_{0}(v_{m-\nu} + w_{m-\nu}) \end{pmatrix}, \]

\[ B_{-2\nu} = \begin{pmatrix} -(v_{m+1-\nu} + w_{m+1-\nu})/2 - u_{0}u_{m-\nu} - s & u_{m} + u_{0}(v_{m} + w_{m}) \\ -u_{0}(v_{m} + w_{m})/2 - u_{0}u_{m} - s & u_{m} + u_{0}(v_{m} + w_{m}) \end{pmatrix}, \]

\[ B_{-1} = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}, \]

\[ B_{0} = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}. \]

(1 ≤ \(\nu\) ≤ \(m\)), \(v_{0} = w_{0} = 0\). Applying Miwa's theorem to (2.4), we can show that \(u_{0}, Z_{\nu} = v_{\nu} + w_{\nu}\) and \(U_{\nu} = u_{\nu} + u_{0}Z_{\nu}\) are meromorphic in \(\mathbb{C}\). Since the isomonodromy property is invariant under the Schlesinger transformation, from (2.2) and Proposition 2.2 we derive the deformation equation with respect to \(Z_{\nu}, U_{\nu}\), which coincides with \((S_{m})\). This completes the proof.

3. Derivation of the corollaries

Eliminating the unknown variables other than \(Z_{1}\), from \((S_{2})\) and \((S_{3})\) we get equations (I4) and (I6), respectively. Thus we have Corollary 1.2.

To show Corollary 1.3, let us consider, for example, system \((S_{3})\). By Corollary 1.2, an arbitrary solution \(Z = P_{6}(t)\) of (I6) is meromorphic in \(\mathbb{C}\), and, around each pole \(t = t_{0}\), it is expanded into one of the following Laurent series:

\[(3.1)\]

\[ (t - t_{0})^{-2} + \cdots, \quad 3(t - t_{0})^{-2} + \cdots, \quad 6(t - t_{0})^{-2}. \]
By Theorem 1.1, every solution of

\[(3.2)\quad u' = 2P_6(t) - u^2,\]

is meromorphic in \(\mathbb{C}\), which is the first equation of \((S_3)\). The transformation \(u = y'/y\) takes (3.2) into

\[(3.3)\quad y'' - 2P_6(t)y = 0.\]

Let \(y(t)\) be an arbitrary solution of (3.3). It is sufficient to show that an arbitrary pole \(t = t_0\) of \(P_6(t)\) is at most a pole of \(y(t)\). To do this, we note that \(u(t) = y'(t)/y(t)\) is written in the form

\[u(t) = c(t - t_0)^{-1} + \cdots,\]

around it, where \(c\) is an integer equal to one of \(-3, -2, -1, 2, 3, 4\). Hence we get an expression of the form

\[y(t) = (t - t_0)^c \sum_{j=0}^{\infty} C_j (t - t_0)^j,\]

from which Corollary 1.3 follows.

REFERENCES

