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線形偏微分方程式の CAUCHY 問題の解の 解析接続についての或注意 (A REMARK ON THE ANALYTIC CONTINUATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS)

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1. Introduction and Results.

J. Leray [L] and L. Gårding, T. Kotake and J. Leray [GKL] have studied the singularities and an analytic continuation of the solution of the Cauchy problem in the complex domain.
[P], [PW] and [HLT] have studied analytic continuations in the case of differential operators with coefficients of entire fuctions or polynomial coefficients.

Let $x=(x_0,x')\,[x'=(x_1,\cdots,x_n)]$ be a point of $\mathbf{C}^{n+1}$. We consider $a(x,D)$ a differential operator of order m, with coefficients of entire functions on $\mathbf{C}^{n+1}$. We denote its principal part by $g(x,D)$ and suppose that $g(x;1,0,\cdots,0)=1$.

Let S be the hyperplane $x_0=0$, therefore non-chraracteristic for g.

We study the Cauchy problem

$$a(x,D)u(x)=v(x),\ D_0^h u(0,x')=w_h(x'),\ 0\leq h\leq m-1, \tag{1.1}$$

where $v(x),w_h(x'),0\leq h\leq m-1$, are entire functions on $\mathbf{C}^{n+1}$ and $\mathbf{C}^n$ respectively.
By the Cauchy-Kowalewski theorem, there exists a unique holomorphic solution in a neighborhood of S in $\mathbf{C}^{n+1}$. How far can this local solution be continued analytically ? In general, the various complicated phenomena happen.
In [H1], by applying a result of L. Bieberbach and P. Fatou to this problem, we have constructed an example such that the domain of holomorphy of the solution

has the nonempty exterior in $\mathbf{C}^{n+1}$, that is, it does not contain a ball in $\mathbf{C}^{n+1}$, for the differential operator with coefficients of entire functions. In [H2], we have given an example such that, roughly speaking, the domain of holomorphy of the ramified solution has an exterior point, for the differential operator with polynomial coefficients.

In this talk, we give a complement to [H2].

First, in order to explicate this situation, we recall a result of [HLT].

Theorem [HLT]. *Suppose that*

$g(x,D) = D_0^m + \sum_{k=1}^{m} L_k(x,D_{x'})D_0^{m-k}$, *where* $L_k(x,D_{x'}), 1 \leq k \leq m$, *is of order* k *in* $D_{x'}$ *and polynomial in* x' *of degree* μk, μ *being an integer* ≥ 0. *Then there exists a constant* $C(0 < C \leq 1)$ *depending only on* $M(R)$ *such that the solution is holomorphic on* $\left\{x \in C^{n+1}; |\, x_0 \,| \leq C \min\left[(1+ \| \, x' \, \|)^{-\max(\mu-1,0)}, R\right]\right\}$,
where M(R) is the maximum modulus on $\{x_0; |\, x_0 \,| \leq R\}$ *of coefficients of polynomials in* x' *of* $g(x,D)$ *and* $\| \, x' \, \| = \max_{1\leq i \leq n} |\, x_i \,|$. (See also [H2]).

Therefore in the case of $\mu = 0, 1$, the solution is an entire function on $\mathbf{C}^{n+1}$. This has been already shown in [P], [PW] and [HLT].

In this talk, we give some examples such that the domain of holomorphy of the solution is schlicht and it has an exterior point, for the differential operatos with polynomial coefficients.

In fact, J. Chazy [C] has studied ordinary differential equations of third order and Darboux-Halphen's system of ordinary differential equations. (Also see [AF]). We employ these results.

Consider the Cauchy problems

$$\{D_0 + \sum_{i=1}^{3} H_i(x')D_i\}U_{1,j}(x) = 0,\ U_{1,j}(0,x') = x_j,\ 1 \leq j \leq 3, \tag{1.2}$$
$$[x = (x_0, x'), x' = (x_1, x_2, x_3)]$$

where

$$\begin{aligned} H_1(x') &= \frac{1}{2}[(x_2 + x_3)x_1 - x_2x_3], \\ H_2(x') &= \frac{1}{2}[(x_1 + x_3)x_2 - x_1x_3], \\ H_3(x') &= \frac{1}{2}[(x_1 + x_2)x_3 - x_1x_2]. \end{aligned} \tag{1.3}$$

This concerns Darboux- Halphen's system of ordinary differential equations ([C], [AF]).

Consider the Cauchy problems

$$(1.4)\qquad \{D_0 + x_2D_1 + x_3D_2 + (2x_1x_3 - 3x_2^2)D_3\}U_{2,j}(x) = 0,$$
$$U_{2,j}(0,x') = x_j,\ 1 \le j \le 3.$$

This concerns Chazy's ordinary differential equation. ([C], [AF]).

J. Leray [L] and L. Gårding, T. Kotake and J. Leray [GKL] have studied the Cauchy problem, when the initial surface has characteristic points. In [H2], we have studied an exceptional case in [L] and [GKL]. We give here a complement to the results of [H2].

Consider the Cauchy problems

$$(1.5)\qquad \{\sum_{i=0}^{3} A_i(x')D_i\}U_{3,j}(x) = 0,\ U_{3,j}(0,x') = x_j,\ 1 \le j \le 3,$$
$$[x = (x_0, x'), x' = (x_1, x_2, x_3)]$$

where

$$(1.6)\qquad \begin{aligned} A_0(x') &= 2x_1^2(1-x_1)^2x_2,\\ A_1(x') &= A_0(x')x_2,\\ A_2(x') &= A_0(x')x_3,\\ A_3(x') &= 3x_1^2(1-x_1)^2x_3^2 - (1 - x_1 + x_1^2)x_2^4. \end{aligned}$$

This concerns the ordinary differential equation of modular function. ([C], [Hi], [AF]).

Then we have

Proposition 1.1. *The domains of holomorphy* $\mathcal{D}_i, 1 \le i \le 3$, *of the solutions* $U_{i,j}(x), 1 \le i,j \le 3$, *of the problems* (1.2), (1.4) *and* (1.5) *are schlicht domains in* C^4. *They have the nonempty exteriors in* C^4.

By a birational mapping and an algebraic mapping, the problems (1.2) and (1.4) are transformed to the problems (1.5) respectively.

2. Sketch of the proof of the Proposition 1.1.

The modular function $w = \lambda(z)$ is holomorphic on $\{z; \Im z > 0\}$ and its inverse $z = \nu(w)$ is holomorphic on the universal covering space $\mathcal{R}[\mathbf{C} \setminus \{0,1\}]$ of the domain $\mathbf{C} \setminus \{0,1\}$. The domain of existence of $\lambda(z)$ has $\{z; \Im z = 0\}$ as a natural boundary.

$W = \lambda\left(\frac{at+b}{ct+d}\right)$, a, b, c, d being constants, $ad - bc = 1$, satisfies the equation $\{W;t\} = -R(W)\left(\frac{dW}{dt}\right)^2$, where $\{W;t\}$ is the Schwarzian derivative:

$$\{W;t\} = \frac{d}{dt}\left(\frac{d^2W}{dt^2}\Big/\frac{dW}{dt}\right) - \frac{1}{2}\left(\frac{d^2W}{dt^2}\Big/\frac{dW}{dt}\right)^2,$$

$R(W) = (1 - W + W^2)/2W^2(1-W)^2$. ([Hi])
Therefore, $x_1 = W, x_2 = dW/dt, x_3 = d^2W/dt^2$ satisfy

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = x_3, \frac{dx_3}{dt} = \frac{3x_3^2}{2x_2} - \frac{(1 - x_1 + x_1^2)x_2^3}{2x_1^2(1-x_1)^2}.$$

Then, with the initial conditions $x'(0) = y'$, we have

$$x_1 = \lambda\left(\frac{a(y')t + b(y')}{c(y')t + d(y')}\right),$$

where the functions

$$b(y') = \nu(y_1)d(y'), \quad d(y') = \frac{\lambda'(\nu(y_1))^{1/2}}{y_2^{1/2}},$$

$$c(y') = \frac{\lambda''(\nu(y_1))y_2^{1/2}}{2\lambda'(\nu(y_1))^{3/2}} - \frac{\lambda'(\nu(y_1))^{1/2}y_3}{2y_2^{3/2}},$$

$$a(y') = \frac{1 + b(y')c(y')}{d(y')},$$

are holomorphic in a neighborhood of a point
$y'^{(0)} = (y_1^{(0)}, y_2^{0)}, y_3^{(0)}) \in (\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\}) \times \mathbf{C}$ and they are continued analytically to $\mathcal{R}[(\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\})] \times \mathbf{C}$.
The solutions $U_{3,j}(x), 1 \le j \le 3$, of the problems (1.5) are holomorphic in a neighborhood of a point $(0, x'^{(0)})$ of $\{x; x_0 = 0, x' \in (\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\}) \times \mathbf{C}\}$ and we obtain

$$U_{3,1}(x) = \lambda\left(\frac{a(x')x_0 - b(x')}{c(x')x_o - d(x')}\right), U_{3,2}(x) = -D_0U_{3,1}(x), U_{3,3}(x) = -D_0U_{3,2}(x).$$

By observing the ramification of $\nu(w)$, we see that
$Q(x') = \Im[a(x')\overline{c(x')}], M(x') = \overline{a(x')}d(x') - b(x')\overline{c(x')}$ and $N(x') = \Im[a(x')\overline{d(x')}]$ are analytic functions of real variables $\Re x', \Im x'$ at the point $x'^{(0)}$ and they are uniform and analytic functions of $\Re x', \Im x'$ on $(\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\}) \times \mathbf{C}$.
Then $P(x') = iM(x')/2Q(x'), R(x') = 1/2 \mid Q(x') \mid$ are uniform and analytic functions of $\Re x', \Im x'$ on $\{x' \in (\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\}) \times \mathbf{C};\ Q(x') \neq 0\}$.

Define the following domain

$\mathcal{D}_3 = \{x = (x_0, x') \in \mathbf{C} \times (\mathbf{C} \setminus \{0,1\}) \times (\mathbf{C} \setminus \{0\}) \times \mathbf{C},$

$| x_0 - P(x') |> R(x')$ for $Q(x') > 0$,
$| x_0 - P(x') |< R(x')$ for $Q(x') < 0$,
$\Im[M(x')x_0] - N(x') < 0$ for $Q(x') = 0\}$.

The domain $\mathcal{D}_3$ is schlicht and it has the nonempty exterior in $\mathbf{C}^4$.

By the Cauchy-Kowalewski theorem, the representations of the solutions and using a technique of analytic continuations in [HLT] (Proposition 7.1 in [HLT]), we see that the domains of holomorphy $U_{3,j}(x), 1 \leq j \leq 3$, are $\mathcal{D}_3$. This proves the Proposition 1.1 for $U_{3,j}(x), 1 \leq j \leq 3$.

Next, we study $U_{1,j}(x), 1 \leq j \leq 3$.

Consider a birational mapping from
$\{x' = (x_1, x_2, x_3) \in \mathbf{C}^3, x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3\}$ onto
$\{X' = (X_1, X_2, X_3) \in \mathbf{C}^3, X_1 \neq 0, 1, X_2 \neq 0\}$:

$$X_1 = X_1(x') = (x_1 - x_3)/(x_1 - x_2),$$
$$X_2 = X_2(x') = (x_2 - x_3)(x_1 - x_3)/(x_1 - x_2) = (x_2 - x_3)X_1(x'),$$
$$X_3 = X_3(x') = (x_1 + x_2 - x_3)(x_2 - x_3)(x_1 - x_3)/(x_1 - x_2) = (x_1 + x_2 - x_3)X_2(x'),$$

and therefore we get

$$x_1 = x_1(X') = \frac{X_3}{X_2} - \frac{X_2}{X_1},$$
$$x_2 = x_2(X') = \frac{X_3}{X_2} + \frac{X_2}{1 - X_1},$$
$$x_3 = x_3(X') = \frac{X_3}{X_2} + \frac{X_2}{1 - X_1} - \frac{X_2}{X_1}.$$

By this mapping, the Cauchy problems (1.2) is transformed to the following Cauchy problems.

$$\{\sum_{i=0}^{3} A_i(X')D_{X_i}\}\hat{U}_{3,j}(X) = 0, [X = (X_0, X'), X' = (X_1, X_2, X_3)]$$

with the initial data

$$\hat{U}_{3,1}(0, X') = \frac{X_3}{X_2} - \frac{X_2}{X_1},$$
$$\hat{U}_{3,2}(0, X') = \frac{X_3}{X_2} + \frac{X_2}{1 - X_1},$$
$$\hat{U}_{3,3}(0, X') = \frac{X_3}{X_2} + \frac{X_2}{1 - X_1} - \frac{X_2}{X_1},$$

$$U_{1,j}(x) = \hat{U}_{3,j}(x_0, X'(x')),\ 1 \leq j \leq 3.$$

From this, it follows that

$$\hat{U}_{3,1}(X) = \frac{U_{3,3}(X)}{U_{3,2}(X)} - \frac{U_{3,2}(X)}{U_{3,1}(X)},$$

$$\hat{U}_{3,2}(X) = \frac{U_{3,3}(X)}{U_{3,2}(X)} + \frac{U_{3,2}(X)}{1 - U_{3,1}(X)},$$

$$\hat{U}_{3,3}(X) = \frac{U_{3,3}(X)}{U_{3,2}(X)} + \frac{U_{3,2}(X)}{1 - U_{3,1}(X)} - \frac{U_{3,2}(X)}{U_{3,1}(X)}.$$

Set $\mathcal{E}_1 = \{x = (x_0, x') \in \mathbf{C}^4, x' = (x_1, x_2, x_3), x_1 \neq x_2, x_2 \neq x_3, x_3 \neq x_1, X_0 = x_0, (X_0, X'(x')) \in \mathcal{D}_3\}$, then $U_{1,j}(x), 1 \leq j \leq 3$, are holomorphic on $\mathcal{E}_1$.

Denote by $\mathcal{D}_1 = (\overline{\mathcal{E}_1})^{(o)}$ the interior of the closure $\overline{\mathcal{E}_1}$ of $\mathcal{E}_1$. We obtain then $\mathcal{D}_1 \setminus \{x_k = x_l, 1 \leq k < l \leq 3\} = \mathcal{E}_1$. On the other hand, by the Cauchy-Kowalewski theorem, $U_{1,j}(x), 1 \leq j \leq 3$, are holomorphic in a neighborhood of $S \cap \{x_k = x_l, 1 \leq k < l \leq 3\}$. Therefore by Hartogs's theorem, $U_{1,j}(x), 1 \leq j \leq 3$, are holomorphic on $\mathcal{D}_1$. We can easily see that the domain of holomorphy of $U_{1,j}(x), 1 \leq j \leq 3$, is $\mathcal{D}_1$. $\mathcal{D}_1$ is a schlicht domain and it has an exterior point in $\mathbf{C}^4$. This proves the Proposition 1.1 for $U_{1,j}(x), 1 \leq j \leq 3$.

Finally we study the problems (1.4).

Consider the mapping of $\mathbf{C}^3$ onto $\mathbf{C}^3$:

$$x_1 = x_1(X') = X_1 + X_2 + X_3,$$

$$x_2 = x_2(X') = \frac{1}{2}(X_1X_2 + X_2X_3 + X_3X_1),$$

$$x_3 = x_3(X') = \frac{3}{2}X_1X_2X_3.$$

Let $X_j(x'), 1 \leq j \leq 3$, be the branches of the algebraic function defined by

$$\tau^3 - x_1\tau^2 + 2x_2\tau - \frac{2}{3}x_3 = 0,$$

at a point $x'^{(0)}$ of $\{x' = (x_1, x_2, x_3) \in \mathbf{C}^3, \Delta(x') \neq 0\}, \Delta(x')$ being the discriminant of this algebraic equation.

$X_j(x'), 1 \leq j \leq 3$, are continued analytically to their Riemann surfaces $\mathcal{R}_\tau$, that is, the covering spase of the domain $\{x' = (x_1, x_2, x_3) \in \mathbf{C}^3, \Delta(x') \neq 0\}$.

$X_j = X_j(\tilde{x}')$, $\tilde{x}' \in \mathcal{R}_\tau$, $1 \leq j \leq 3$, maps $\mathcal{R}_\tau$ onto $\{(X_1, X_2, X_3) \in \mathbf{C}^3, X_1 \neq X_2, X_2 \neq X_3, X_3 \neq X_1\}$. By the mapping $x_0 = X_0, x_j = x_j(X'), 1 \leq j \leq 3$, the problems (1.4) are transformed to the following problems

$$\{D_{X_0} + \sum_{i=1}^{3} H_i(X')D_{X_i}\}\mathcal{U}_{1,j}(X) = 0,\ 1 \leq j \leq 3,$$

with the initial conditions

$$\begin{aligned} \mathcal{U}_{1,1}(0, X') &= X_1 + X_2 + X_3, \\ \mathcal{U}_{1,2}(0, X') &= \frac{1}{2}(X_1X_2 + X_2X_3 + X_3X_1), \\ \mathcal{U}_{1,3}(0, X') &= \frac{3}{2}X_1X_2X_3. \end{aligned}$$

Then we have, in a neighborhood of a point $(0, x'^{(0)})$ of $\{x_0 = 0, x' \in \mathbf{C}^3, \Delta(x') \neq 0\}$,

$$U_{2,j}(x) = \mathcal{U}_{1,j}(x_0, X'(x')), 1 \leq j \leq 3.$$

Therefore we obtain, in a neighborhood of the point $(0, x'^{(0)})$,

$$\begin{aligned} U_{2,1}(x) &= \sum_{j=1}^{3} U_{1,j}(x_0, X'(x')), \\ U_{2,2}(x) &= \frac{1}{2}\{\sum_{1\leq j<k\leq 3} U_{1,j}(x_0, X'(x'))U_{1,k}(x_0, X'(x'))\}, \\ U_{2,3}(x) &= \frac{3}{2}U_{1,1}(x_0, X'(x'))U_{1,2}(x_0, X'(x'))U_{1,3}(x_0, X'(x')). \end{aligned}$$

Take an arbitray point x' of $\{x'; \Delta(x') \neq 0\}$ and a path γ in $\{x'; \Delta(x') \neq 0\}$ from the fixed point $x'^{(0)}$ to x'. Continue analytically all $(X_i(x'), X_j(x'), X_k(x'))$, $(1 \leq i, j, k \leq 3, i \neq j, j \neq k, k \neq i)$, along γ and define the following domain $\mathcal{E}_2 = \{x; \Delta(x') \neq 0, (x_0, X_i(x'), X_j(x'), X_k(x')) \in \mathcal{D}_1, 1 \leq i, j, k \leq 3,$ $i \neq j, j \neq k, k \neq i\}$

Denote by $\mathcal{D}_2 = (\overline{\mathcal{E}_2})^{(o)}$ the interior of the closure $\overline{\mathcal{E}_2}$ of $\mathcal{E}_2$. We get $\mathcal{D}_2 \setminus \{x'; \Delta(x') = 0\} = \mathcal{E}_2$. For each point x of $\mathcal{D}_2 \cap \{x'; \Delta(x') = 0\}$, there exists then a neighborhood $W(x)$ in $\mathcal{D}_2$ such that the functions $U_{2,j}, 1 \leq j \leq 3$, are holomorphic, uniform and bounded in $W(x) \setminus \{x'; \Delta(x') = 0\}$, and therefore by

holomorphic, uniform and bounded in $W(x) \setminus \{x'; \Delta(x') = 0\}$, and therefore by virtue of the Riemann removable singularities theorem, they are holomorphic on $\mathcal{D}_2$. Of course, as in $U_{1,j}, 1 \leq j \leq 3$, we can also show it, by using the Cauchy-Kowalewski theorem and Hartogs's theorem. We can easily see that the domain of holomorphy of $U_{2,j}, 1 \leq j \leq 3$, are $\mathcal{D}_2$. $\mathcal{D}_2$ is a schlicht domain and it has an exterior point in $\mathbf{C}^4$.
This proves the Poposition 1.1 for $U_{2,j}, 1 \leq j \leq 3$.

The detailed proof of our results will be published elswhere.

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