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Euler 方程式の大域解接続に関する Beale-Kato-Majda の定理とその発展 (Remarks on the result of Beale-Kato-Majda for the Euler equations)

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Abstract

We prove that the BMO norm of the vorticity controls the blow-up phenomena of smooth solutions to the Euler equations in the whole space $\boldsymbol{R}^n$.

Introduction.

In this paper we prove that the BMO norm of the vorticity controls the blow-up phenomena of smooth solutions to the Euler equations.

the Euler equations in $\boldsymbol{R}^n$ $(n \geq 3)$ are as follows:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \text{div } u = 0 \quad \text{in } x \in \boldsymbol{R}^n,\ t > 0, \\ u\,|_{t=0} = a \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \boldsymbol{R}^n \times (0,\infty)$, respectively, while $a = (a^1(x), a^2(x), \cdots, a^n(x))$ is the given initial velocity vector.

It is proved by Kato-Lai [3] and Kato-Ponce [4] that for every $a \in W^{s,p}_\sigma$ for $s > n/p + 1$, $1 < p < \infty$, there are $T > 0$ and a unique solution u of (E) on the interval $[0,T)$ in the class

$$\text{(CE)}_{s,p} \qquad u \in C([0,T); W^{s,p}_\sigma) \cap C^1([0,T); W^{s-2,p}_\sigma),$$

where subindex σ means the divergence free. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$.

Beale-Kato-Majda [1] proved that under the condition

$$\int_0^T \|\text{rot } u(t)\|_{L^\infty} dt < \infty$$

$u(t)$ can never break down its regularity at $t = T$. (See also [4].) To prove this assertion, in [1] they made use of the logarithmic inequality such as

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|\text{rot } u\|_{L^\infty}(1 + \log^+\|u\|_{W^{s+1,p}}) + \|\text{rot } u\|_{L^2}\right), \quad sp > n \tag{0.1}$$

for all vector functions u with div $u = 0$, where $\log^+ a = \log a$ if $a \ge 1$, $= 0$ if $0 < a < 1$.

The purpose of this paper is to extend these results to BMO which is larger than L^∞. (It is possible to extend these to more general classes, see [7].)

In a forthcoming paper, we will discuss the blow-up of smooth solutions to the Euler equations in a *bounded* domain.

1 Result.

Before stating our result, we introduce some function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with compact support in $\boldsymbol{R}^n$, such that div $\phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over $\boldsymbol{R}^n$, $1 \le r \le \infty$. $W_\sigma^{s,p}$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the $W^{s,p}$-norm.

Our result on (E) reads as follows.

Theorem 1 *Let $1 < p < \infty$, $s > n/p + 1$. Suppose that u is the solution of (E) in the class $(CE)_{s,p}$ on $(0,T)$. If either*

$$\int_0^T \|\text{rot } u(t)\|_{BMO} dt (\equiv M_0) < \infty \tag{1.1}$$

or

$$\int_0^T \|\text{Def } u(t)\|_{BMO} dt (\equiv M_1) < \infty \tag{1.2}$$

holds, then u can be continued to the solution in the class $(CE)_{s,p}$ on $(0,T')$ for some $T' > T$.

Here Def u denotes the deformation tensor of u, i.e., $(\text{Def } u)_{ij} = \partial_i u^j + \partial_j u^i$, $(1 \le j,k \le n)$.

Corollary 1 *Let u be the solution of (E) in the class $(CE)_{s,p}$ on $(0,T)$ for $1 < p < \infty$, $s > n/p + 1$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class $(CE)_{s,p}$ on $(0,T')$ for any $T' > T$. Then both*

$$\int_0^T \|\text{rot } u(t)\|_{BMO} dt = \infty \quad \textit{and} \quad \int_0^T \|\text{Def } u(t)\|_{BMO} dt = \infty$$

2 Preliminaries.

In what follows we shall denote by C various constants. In particular, $C = C(*,\cdots,*)$ denotes constants depending only on the quantities appearing in the parenthesis.

We first recall the Biot-Savart law. By the Biot-Savart law, for solenoidal vectors u, we have the representation

$$(2.1)\quad \frac{\partial u}{\partial x_j} = R_j(R\times\omega),\quad j=1,\cdots,n,\quad \text{where } \omega = \text{rot } u;$$

$$(2.2)\quad \frac{\partial u^l}{\partial x_j} = R_j(\sum_{k=1}^{n} R_k(\text{Def } u)_{kl}),\quad j,l=1,\cdots,n,\quad \text{where } (\text{Def } u)_{kl} = \frac{\partial u^k}{\partial x_l}+\frac{\partial u^l}{\partial x_k}.$$

Here $R = (R_1,\cdots,R_n)$, and $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since R is a bounded operator in BMO, we have by (2.1), (2.2) that

$$(2.3)\quad \|\nabla u\|_{BMO} \le C\|\text{rot } u\|_{BMO};$$

$$(2.4)\quad \|\nabla u\|_{BMO} \le C\|\text{Def } u\|_{BMO}.$$

Now we prove the following lemma which is an extension of (0.1).

Lemma 2.1 *Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n,p,s)$ such that the estimate*

$$(2.5)\quad \|f\|_\infty \le C\left(1+\|f\|_{BMO}(1+\log^+\|f\|_{W^{s,p}})\right)$$

holds for all $f\in W^{s,p}$.

Remark. Compared with (0.1), we do not need to add $\|f\|_{L^2}$ to the right hand side of (2.5). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale-Kato-Majda [1].

Proof of Lemma 2.1.

We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi\in\mathcal{S}$ ($\mathcal{S}$; the Schwartz class) such that supp$\varphi\subset\{2^{-1}\le|\xi|\le 2\}$ and such that $\sum_{k=-\infty}^{\infty}\varphi(2^{-k}\xi)=1$ for $\xi\neq 0$. See Bergh-Löfström [2, Lemma 6.1.7]. Let us define ϕ_0 and ϕ_1 as

$$\phi_0(\xi)=\sum_{k=1}^{\infty}\varphi(2^k\xi)\quad\text{and}\quad\phi_1(\xi)=\sum_{k=-\infty}^{-1}\varphi(2^k\xi),$$

respectively. Then we have that $\phi_0(\xi)=1$ for $|\xi|\le 1/2$, $\phi_0(\xi)=0$ for $|\xi|\ge 1$ and that $\phi_1(\xi)=0$ for $|\xi|\le 1$, $\phi_1(\xi)=1$ for $|\xi|\ge 2$. It is easy to see that for every positive integer N there holds the identity

$$(2.6)\quad \phi_0(2^N\xi)+\sum_{k=-N}^{N}\varphi(2^{-k}\xi)+\phi_1(2^{-N}\xi)=1,\quad \xi\neq 0.$$

Since C_0^∞ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in BMO, implied by $s > n/p$, it suffices to prove (2.5) for $f \in C_0^\infty$. For such f we have the representation

$$f(x) = \int_{y\in \mathbf{R}^n} K(x-y)\cdot \nabla f(y)dy \quad \text{with} \quad K(y) = \frac{1}{n\omega_n}\frac{y}{|y|^n},$$

for all $x \in \mathbf{R}^n$, where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. By (2.6) we decompose f into three parts:

$$\begin{aligned} f(x) &= \int_{y\in\mathbf{R}^n} K(x-y)\times \\ &\quad \times \left(\phi_0(2^N(x-y)) + \sum_{k=-N}^{N} \varphi(2^{-k}(x-y)) + \phi_1(2^{-N}(x-y))\right)\cdot \nabla f(y)dy \\ &\equiv f_0(x) + g(x) + f_1(x) \qquad (2.7) \end{aligned}$$

for all $x \in \mathbf{R}^n$.

We can show that

$$|f_0(x)| \le C2^{-\beta N}\|f\|_{W^{s,p}} \qquad (2.8)$$

for all $x \in \mathbf{R}^n$, where $\beta = \beta(n,p,s)$ is a positive constant. For detail, see [6].

By integration by parts we have

$$g(x) = \sum_{k=-N}^{N} (\mathrm{div}\, \Psi)_{2^k} * f(x), \quad x \in \mathbf{R}^n,$$

where $\Psi(x) = K(x)\varphi(x)$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Since $\Psi \in \mathcal{S}$ with the property that

$$\int_{\mathbf{R}^n} \mathrm{div}\, \Psi(x)dx = 0,$$

it follows from Stein [9, Chap. IV, 4.3.3] that

$$\begin{aligned} \|g\|_\infty &\le \sum_{k=-N}^{N} \|(\mathrm{div}\, \Psi)_{2^k} * f\|_\infty \\ &\le \sum_{k=-N}^{N} \sup_{t>0} \|(\mathrm{div}\, \Psi)_t * f\|_\infty \\ &\le CN\|f\|_{BMO}, \qquad (2.9) \end{aligned}$$

where $C = C(n)$ is independent of N.

Integrating by parts, we have by a direct calculation

$$\begin{aligned} |f_1(x)| &= \left|\int_{y\in\mathbf{R}^n} \mathrm{div}_y \left(K(x-y)\phi_1(2^{-N}(x-y))\right) f(y)dy\right| \\ &\le C2^{-N\cdot\frac{n}{p}}\|f\|_p \qquad (2.10) \end{aligned}$$

for all $x \in \mathbf{R}^n$, where $C = C(n,p)$ is independent of N.

Now it follows from (2.7) and (2.8)-(2.10) that

$$\|f\|_\infty \le C(2^{-\gamma N}\|f\|_{W^{s,p}} + N\|f\|_{BMO}) \tag{2.11}$$

with $\gamma = \mathrm{Min.}\{\beta, n/p\}$, where $C = C(n,s,p)$ is independent of N and f. If $\|f\|_{W^{s,p}} \le 1$, then we may take $N = 1$; otherwise, we take N so large that the first term of the right hand side of (2.11) is dominated by 1, i.e., $N \equiv \left[\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2}\right] + 1$ ($[\cdot]$; Gauss symbol) and (2.11) becomes

$$\|f\|_\infty \le C\left\{1 + \|f\|_{BMO}\left(\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} + 1\right)\right\}.$$

In both cases, (2.5) holds. This proves Lemma 2.1.

3 Proof of Theorem 5.

We follow the argument of Beale-Kato-Majda [1]. It is proved by Kato-Lai [3] and Kato-Ponce [4] that for the given initial data $a \in W^{s,p}_\sigma$ for $s > 1 + n/p$, the time interval T of the existence of the solution u to (E) in the class $(\mathrm{CE})_{s,p}$ depends only on $\|a\|_{W^{s,p}}$. Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for u in $W^{s,p}$ in terms of a, T, M_0 or a, T, M_1 according to (1.1) or (1.2). Indeed, we shall show that the solution $u(t)$ in the class $(\mathrm{CE})_{s,p}$ on $(0,T)$ is subject to the following estimate:

$$\sup_{0<t<T} \|u(t)\|_{W^{s,p}} \le (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(CT\alpha_j) \quad \text{with } \alpha_j = e^{CM_j}, \quad j = 0,1, \tag{3.12}$$

where $C = C(n,p,s)$ is a constant independent of a and T.

We shall first prove (3.12) under (1.1). It follows from the commutator estimate in L^p given by Kato-Ponce [4, Proposition 4.2] that

$$\|u(t)\|_{W^{s,p}} \le \|a\|_{W^{s,p}} \exp\left(C\int_0^t \|\nabla u(\tau)\|_\infty d\tau\right), \quad 0 < t < T, \tag{3.13}$$

where $C = C(n,p,s)$.

By the Biot-Savard law (2.1), we have

$$\|\nabla u\|_{BMO} \le C\|\mathrm{rot}\ u\|_{BMO} \tag{3.14}$$

with $C = C(n)$. Hence it follows from (3.14) and Lemma 2.1 that

$$\|\nabla u(t)\|_\infty \le C\left(1 + \|\mathrm{rot}\ u(t)\|_{BMO}(1 + \log^+ \|u(t)\|_{W^{s,p}})\right) \tag{3.15}$$

for all $0 < t < T$ with $C = C(n,p,s)$. Substituting (3.15) to (3.13), we have

$$\begin{aligned}&\|u(t)\|_{W^{s,p}} + e \\ \le\ &(\|a\|_{W^{s,p}} + e)\exp\left(C\int_0^t \{1 + \|\mathrm{rot}\ u(\tau)\|_{BMO}\log(\|u(\tau)\|_{W^{s,p}} + e)\}d\tau\right)\end{aligned}$$

for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \leq z(0) + CT + C\int_0^t \|\mathrm{rot}\ u(\tau)\|_{BMO} z(\tau)d\tau, \quad 0 < t < T.$$

Now (1.1) and the Gronwall inequality yield

$$\begin{aligned} z(t) &\leq (z(0) + CT)\exp\left(C\int_0^t \|\mathrm{rot}\ u(\tau)\|_{BMO} d\tau\right) \\ &\leq (z(0) + CT)\,\alpha_0 \end{aligned}$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (3.12) for $j = 0$.

Similarly we prove (3.12) for $j = 1$ under (1.2). This proves Theorem 5.

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