A polynomial time approximation scheme
for the minimum maximal matching problem in planar graphs

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Abstract: Given an undirected graph \( G \), the minimum maximal matching problem asks for a
minimum matching in \( G \) where no two adjacent edges have both endpoints in the
same connected component of \( G \). The problem is known to be \( \mathbf{NP} \)-hard even if the graph is planar. We consider the problem for planar graphs, and show that
a polynomial time approximation scheme (PTAS) can be obtained by a divide-and-conquer
method based on the planar separator theorem. For a given \( \epsilon > 0 \), our scheme delivers in
\( O(n \log n + \alpha(\alpha^2+\epsilon)n) \) time a solution with size at most \((1+\epsilon)\) times
the optimal value, where \( n \) is the number of vertices in \( G \) and \( \alpha \) is a constant number.

Keywords: graph algorithm, approximation algorithm, matching, planar graph, separator.

1 Introduction

Given an undirected graph \( G = (V,E) \), a matching is a subset \( M \) of \( E \) containing no two adjacent edges. A matching \( M \) is said to be maximal if there is
no matching \( M' \) which strictly contains \( M \). The minimum maximal matching problem asks for a maximal matching containing the minimum number of edges. The problem is one of the \( \mathbf{NP} \)-hard problems included in the list of \( \mathbf{NP} \)-complete problems [3, p.192], and the problem remains \( \mathbf{NP} \)-hard for planar graphs and for bipartite graphs, in both cases even if no vertex degree exceeds 3 [10]. As to approximability, the problem is shown to be \( \mathbf{APX} \)-hard for general graphs [1, p.374]. In this paper, we consider the complexity status of the minimum maximal matching problem for planar graphs.

An algorithm is called an \( \alpha \)-approximation algorithm to a minimization problem if it outputs a solution whose weight is at most \( \alpha \) times the weight of an optimal solution. A polynomial time approx-
general method for providing PTASs for a variety of the optimization problems on planar graphs, which includes the minimum vertex cover problem and the maximum independent set problem. The paper also pointed out some NP-hard problems to which her method cannot be applied in a straightforward way. For example, it says that the minimum maximal matching problem is one of such problems because the restriction of an optimal solution $M$ in $G$ on a vertex subset $X$ (i.e., the set of edges in $M$ whose endvertices belong to $X$) may not be a maximal matching in the graph induced by $X$. (We remark that the similar difficulty necessarily arises for the minimum edge dominating set problem, which is claimed to admit a PTAS in [2].)

In this paper we use the divide-and-conquer approach to solve the minimum maximal matching problem for planar graphs. However, a naive application of this approach does not yield a PTAS to the problem. One of the reasons is that the size of a minimum maximal matching can be arbitrarily small, compared with the size $|V|$ of a graph $G = (V, E)$. For this, we reduce an arbitrary planar graph $G = (V, E)$ to a particular planar graph, called an irreducible planar graph, so that the size of a minimum maximal matching is $O(|V|)$ (this property is important to obtain a PTAS by the divide-and-conquer approach). Another difficulty of the problem is that the restriction of an optimal solution $M$ in $G$ on a vertex subset $X$ may not be feasible. We overcome this by a careful analysis of the performance of our divide-and-conquer method. As a result, for a given $\epsilon > 0$, our scheme delivers a $(1+\epsilon)$-approximate solution in $O(n \log n + n^{3/2} \epsilon^{-1} n)$ time, where $n$ is the number of vertices in a given planar graph and $\alpha$ is a constant number.

## 2 Preliminaries

Let $G = (V, E)$ stand for a simple undirected graph with a vertex set $V$ and an edge set $E$. The vertex set (resp., edge set) of a graph $G$ may be denoted by $V(G)$ (resp., $E(G)$). For a subset $X \subseteq V(G)$, $G - X$ denotes the graph obtained from $G$ by removing the vertices in $X$ together with edges incident to them. Let $V[e]$ be the set of endpoints of an edge $e$. Let $d_G(v)$ denote the degree of a vertex $v \in V$. Let $\delta(G) = \min_{v \in V} d_G(v)$. A vertex $v$ with $d_G(v) = 1$ is called a leaf vertex. An edge incident to a leaf vertex is called a leaf edge. An non-leaf edge one of whose endpoints is incident to only leaf edges is called a fringe edge. A maximum matching is a matching of the maximum size. Let $\mu(G)$ denote the size of a maximum matching of $G$ and $\rho(G)$ denote the size of a minimum maximal matching.

We introduce lower bounds on the size of a minimum maximal matching in a graph $G$. It is easy to see that the size of any maximal matching is at least half of the size of a maximum matching. That is,

$$\rho(G) \geq \frac{1}{2} \mu(G).$$

(1)

Thus we can use any lower bound on the size of a maximum matching as that on the size of a minimum maximal matching within a constant factor. We always assume that a given planar graph is equipped with a fixed plane embedding. Namely $G$ is a plane graph. The following fact about planarity is known.

**Theorem 2.1** [4] Every planar graph with $n \geq 3$ vertices contains no more than $3n - 6$ edges.

A following lower bound on $\mu(G)$ for planar graphs is known.

**Theorem 2.2** [7] If $G = (V, E)$ is a connected planar graph with $\delta(G) \geq 3$ and $n = |V|$, then $\mu(G) \geq \min \{ \lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{3} \rceil \}$.

Our algorithm uses the following partition of a vertex set of a planar graph.

**Theorem 2.3** [5] Let $G$ be a planar graph with $n$ vertices. Then the vertices of $G$ can be partitioned into three sets $A$, $B$, $C$ such that no edge joins a vertex in $A$ with a vertex in $B$, neither $A$ nor $B$ contains more than $\frac{3}{4}n$ vertices, and $C$ contains no more than $2(2n)^{1/2}$ vertices. Such a partition can be found in linear time.

A vertex set $C$ in the theorem is called a planar separator. In the following sections, we prove the next theorem.

**Theorem 2.4** Given a connected planar graph $G = (V, E)$ and $\epsilon > 0$, the minimum maximal matching problem is $(1 + \epsilon)$-approximable in $O(n \log n + \alpha^{1/2} \epsilon^{-1}n)$ time, where $n = |V|$ and $\alpha$ is a constant number.

A subset $D$ of edges in $G = (V, E)$ is called an edge dominating set if every edge in $E - D$ is adjacent to an edge in $D$. The edge dominating set problem asks to find an edge dominating set of the minimum size. As pointed out in [10], the size of a minimum maximal matching of a graph $G$ is equal
to that of a minimum edge dominating set in $G$ and, from any maximal matching $M$, an edge dominating set $D$ with $|D| = |M|$ can be constructed in linear time. Then the above theorem implies the next result.

**Corollary 1** The edge dominating set problem in a planar graph $G$ admits a PTAS with the same performance in Theorem 2.4. □

# 3 Algorithm

## 3.1 Preprocess

For an arbitrary planar graph $G$, $\rho(G)$ cannot be bounded from below by $c|V(G)|$ for any constant $c$. In this subsection, we present how to process a given graph to obtain a graph $G'$ with $\rho(G') = \Omega(|V(G')|)$ without losing the optimality of the problem.

**Definition 1** A graph $G$ (not necessarily planar) is called irreducible if

(i) $G$ is simple and connected,

(ii) $G$ has no fringe edges,

(iii) each vertex $v \in V(G)$ has at most one leaf vertex adjacent to it,

(iv) any two vertices $u, v \in V(G)$ have at most two common neighbors of degree 2. □

The following procedure converts a given graph $G$ into an irreducible one without changing the optimality of the minimum maximal matching problem.

**Algorithm REDUCE**

**Input:** A connected graph $G$.

**Output:** An irreducible graph $G'$ and a matching $M'$ of $G$ such that $\rho(G) = \rho(G') + |M'|$.

Let $M' := \emptyset$.

while there is a fringe edge $e$ do

Choose a fringe edge $e$ and let $M' := M' \cup \{e\}$, discarding all edges adjacent to $e$.

end while

while there is a vertex $u$ to which at least two leaf vertices are adjacent do

Choose such a vertex $u$.

Choose one leaf vertex adjacent to $u$ and discard the rest of all leaf vertices adjacent to $u$.

end while

while there is a pair of vertices $u$ and $v$ which have at least three common neighbors of degree 2 do

Choose such a pair of vertices $u$ and $v$.

Choose two vertices of degree 2 adjacent to $u$ and $v$ and discard the rest of all vertices of degree 2 adjacent to $u$ and $v$.

end while

Let $G'$ be the resulting graph. □

Then we have the following result (the proof is omitted). We denote by $E^{opt}(G)$ a minimum maximal matching in $G$.

**Lemma 1** For a given graph $G = (V, E)$, let $G'$ and $M'$ be the graph and the matching obtained from $G$ by Algorithm REDUCE. Then for any $E^{opt}(G')$, $E^{opt}(G') \cup M'$ is a minimum maximal matching in $G$, and $G'$ is irreducible. REDUCE can be implemented to run in $O(n+m)$ time, where $n = |V|$ and $m = |E|$. □

For a planar graph $G$ with $n$ vertices, Algorithm REDUCE runs in $O(n)$ time by Theorem 2.1. In what follows, we consider how to find an approximation solution to an irreducible planar graph $G$. If $|V(G)| \leq 36$, then we find a minimum maximal matching $E^{opt}(G)$ by checking every subset of $E$. Otherwise (i.e., $|V(G)| \geq 37$), we use the property that $\mu(G) = \Omega(n)$ in an irreducible planar graph $G$.

**Lemma 2** Let $G = (V, E)$ be an irreducible planar graph with $n = |V| \geq 37$. Then, $\mu(G) \geq \frac{1}{42}n + \frac{13}{21}$.

**Proof:** See Appendix. □

## 3.2 Approximation algorithm

For a graph $G$ with a sufficiently small number of vertices, we find a minimum maximal matching by using the next lemma.

**Lemma 3** For a graph $G$ with $n$ vertices and $m$ edges, a minimum maximal matching can be found in $O(2^n \sqrt{nm})$ time.

**Proof:** Omitted. □

Now we are ready to describe our approximation algorithm.

**Algorithm DIVIDE**

**Input:** An irreducible planar graph $G$ and a real number $\epsilon > 0$. □
Output: A maximal matching \( E^{\text{opt}}(G) \) of \( G \) such that \( |E^{\text{opt}}(G)| \leq (1 + \epsilon)\rho(G) \).

1. Let \( L := \frac{(1943/\epsilon)^2}{2} \), and \( C^* := \emptyset \).

2. while \( G - C^* \) has a connected component with more than \( L \) vertices do
   Choose such a connected component \( G' \),
   Find a planar separator \( C \subseteq V(G') \) by applying Theorem 2.3 to \( G' \),
   \( C^* := C^* \cup C \).
end while

3. For each of connected components \( G_i = (V_i, E_i) \)
   \( i = 1,2, \ldots, p \) of \( G - C^* \) (where \( |V_i| \leq L \) for all \( i \)),
   find a maximum cardinal matching \( E^{\text{opt}}(G_i) \) by using Lemma 3.

4. Extend a matching \( \bigcup_{1 \leq i \leq p} E^{\text{opt}}(G_i) \) to a maximal one in \( G \) by adding a set \( M^* \) of independent edges.
   Let \( E^{\text{apx}}(G) = \bigcup_{1 \leq i \leq p} E^{\text{opt}}(G_i) \cup M^* \).

Notice that each edge in \( M^* \) must be incident to a vertex in the final \( C^* \) by the maximality of each \( E^{\text{opt}}(G_i) \). Thus \( |M^*| \leq |C^*| \).

### 3.3 Analysis

We first analyze the approximation ratio of algorithm \textsc{Divide}.

**Lemma 4** Let \( E^{\text{apx}}(G) \) be a maximal matching obtained from an irreducible planar graph \( G \) with \( |V(G)| \geq 37 \) by Algorithm \textsc{Divide}. Then \( |E^{\text{apx}}(G)| \leq (1 + \epsilon)\rho(G) \) holds.

**Proof:** Let \( n = |V(G)| \). By the inequality (1) and Lemma 2, it holds

\[
|E^{\text{opt}}(G)| = \rho(G) \geq \frac{1}{2} \mu(G) \geq \frac{1}{84} n. \tag{2}
\]

By \( |M^*| \leq |C^*| \),

\[
|E^{\text{apx}}(G)| = \sum_i |E^{\text{opt}}(G_i)| + |M^*| \leq \sum_i |E^{\text{opt}}(G_i)| + |C^*|. \tag{3}
\]

We now compare \( \sum_i |E^{\text{opt}}(G_i)| \) with \( |E^{\text{opt}}(G)| \). It should be noted that \( E^{\text{opt}}(G) \cap E_i \) is not necessarily a maximal matching in \( G_i \) in Step 3, and we may need to add some edges from \( E(G_i) \) to \( E^{\text{opt}}(G) \cap E_i \) in order to make it maximal in \( G_i \). Then, each of these edges joins two vertices \( u \) and \( v \) that are adjacent to vertices in \( C^* \) via some edges \( e_u, e_v \in E^{\text{opt}}(G) \). Thus the number of such edges \( e_u, e_v \in E^{\text{opt}}(G) \) is at most \( |C^*| \). Hence the number of edges to be added to \( E^{\text{opt}}(G) \cap E_i \) over all \( G_i \) is at most \( \frac{1}{2} |C^*| \). Therefore we have

\[
\sum_i |E^{\text{opt}}(G_i)| \leq |E^{\text{opt}}(G)| + \frac{1}{2} |C^*|. \tag{4}
\]

By (3) and (4), we get

\[
|E^{\text{apx}}(G)| \leq |E^{\text{opt}}(G)| + \frac{3}{2} |C^*|. \tag{5}
\]

Now, we claim that \( |C^*| \leq \delta n \) holds for a constant number \( \delta \). Consider all the connected components which appeared during an execution of the above procedure. Assign a level to each component as follows: the final components (with at most \( L \) vertices) have level 0; and each of the components has a level one greater than the maximum level of the components produced from it. Obviously any two components of the same level are disjoint.

Since a component of level \( i \) has at least \( (\frac{3}{2})^i \) vertices, the maximum level \( \ell \) must satisfy \( (\frac{3}{2})^{\ell} \leq n \) or \( \ell \leq \log_3 n \). Since every component of level 1 has at least \( L \) vertices, every components of level \( i \) has at least \((\frac{3}{2})^{i-1} L \) vertices. Therefore the number \( c_i \) of components of level \( i \) is at most \((\frac{3}{2})^{i-1} \) because of \( c_i (\frac{3}{2})^{i-1} L \leq n \).

Now we can bound the size of \( C^* \) as follows. Let \( n_j, 1 \leq j \leq c_i \), be the number of vertices in the \( j \)th component of level \( i \). Then we have

\[
|C^*| \leq \sum_{1 \leq i \leq \ell_1 \leq i \leq \ell} 2(2n_j)^{1/2} \leq 2(2)^{1/2} \sum_{1 \leq i \leq \ell} (c_i \sum_{1 \leq j \leq \ell} n_j)^{1/2} \leq 2(2)^{1/2} \sum_{1 \leq i \leq \ell} c_i^{1/2} n^{1/2} \leq 2\sqrt{6} \frac{1 - (\sqrt{\frac{3}{2}})^{\ell - 1}}{\sqrt{3 - \sqrt{2}}} \cdot \frac{n}{\sqrt{L}} \leq 6\sqrt{2} + 4\sqrt{3} \cdot \frac{\epsilon n}{1943}. \tag{6}
\]

The approximate ratio is evaluated by (2), (5) and (6) as follows.

\[
\frac{|E^{\text{apx}}(G)|}{|E^{\text{opt}}(G)|} \leq 1 + \frac{3}{2} \frac{|C^*|}{|E^{\text{opt}}(G)|} \leq 1 + \frac{n}{84} \cdot \frac{6\sqrt{2} + 4\sqrt{3} \cdot \epsilon n}{1943} \leq 1 + \epsilon. \tag{7}
\]
We finally evaluate the running time of Algorithm DIVIDE for a planar graph $G = (V, E)$, where $n = |V|$ and $m = |E|$. First, we consider the running time by Step 2. A planar separator $C$ is found in linear time by Theorem 2.3 and the number of recurrences during Step 2 is $O(\log n)$ since the separator partitions a graph into two graphs so that the size of these graphs decrease by a constant factor. Therefore it takes $O(n \log n)$ time to decompose a given graph into subgraphs with at most $L$ vertices. 

Next, we consider the running time to find optimal solutions for all subgraphs. By applying Lemma 3 to $G_i$, where $|E(G_i)| = O(|V(G_i)|)$, an $E^{opt}(G_i)$ can be found in $O(2^L L^{3/2})$ time. Thus, the time to compute all $E^{opt}(G_i)$ is $O(2^L L^{3/2} \frac{n}{L}) = O(2^L \sqrt{L}n)$.

Thus Algorithm DIVIDE can be implemented to run in $O(n \log n + 2^L \sqrt{L}n) = O(n \log n + \alpha^2 \epsilon^{-1} n)$ time for $\alpha = 2^{(1943)^2}$.

This completes the proof of Theorem 2.4.

4 Conclusion

In this paper, we proved that the minimum maximal matching problem in planar graphs admits a PTAS. However, the current trade-off of the PTAS between the running time and the approximation ratio is not effective. Thus it is a future work to design a PTAS with a better trade-off.

References


Appendix

Proof of Lemma 2: Let $G$ be a given irreducible planar graph. To prove the lemma via Lemma 2.2, we convert $G$ into graphs $G_1, G_2, G_3, G_4$ in order to obtain a planar graph with the minimum degree at least 3. We first construct a graph $G_1$ from $G$ by applying the following procedures 1 and 2.

1. If there is a pair of leaf edges $(u, u')$ and $(v, v')$ which are adjacent to the same edge, say $(u', v')$, then add three new edges $(u, v), (u, v'), (v, u')$ to the graph (where the resulting graph remains simple and planar, and each of $u$ and $v$ has degree 3). We repeat this until there is no such pair of leaf edges.

2. If there is a leaf edge $(u, v)$ with a leaf vertex $u$, then add new edges $(u, w), (u, w')$ with two neighbors $w, w'(\neq u)$ of $v$ by choosing $w, w'$ so that the augmented graph remains planar (such a pair $w, w'$ exists since the current graph has no fringe edge and no two leaf edges adjacent to the same edge). The resulting graph remains simple and the degree of $u$ becomes 3. We repeatedly apply this until there is no leaf edge.

Claim 1 $G_1$ remains irreducible and planar, and satisfies $V(G_1) = V(G), \delta(G_1) \geq 2, \mu(G_1) \leq 2\mu(G)$.

Proof: Omitted.

We next augment $G_1$ to a graph $G_2$ by adding a maximal set of new edges such that the resulting graph remains simple and planar and has the same size of a maximum matching of $G_1$.

Claim 2 $G_2$ remains irreducible and planar, and satisfies $V(G_2) = V(G_1), \delta(G_2) \geq 2, \mu(G_2) = \mu(G_1)$. In $G_2$, $

\square$
(i) the two neighbors of a vertex of degree 2 are joined by an edge,
(ii) no two vertices of degree 2 are adjacent,
(iii) each vertex of degree 2 is adjacent to two vertices of degree at least 4, and
(iv) if two vertices of degree 2 are adjacent to the same two vertices \( u \) and \( v \), then both \( u \) and \( v \) have degree at least 5.

**Proof:** Omitted. \( \Box \)

For a graph \( H \), let \( V_2(H) \) denote the set of vertices of degree 2 in \( H \). Let us call an edge \( e \) covered if there is a vertex of degree 2 adjacent to both endpoints of \( e \), and uncovered otherwise.

The graph \( G_2 - V_2(G_2) \) may have a vertex of degree at most 2. Let \( u \) be such a vertex. By the irreducibility and (i) of Claim 2, the degree of \( u \) in \( G_2 - V_2(G_2) \) is exactly 2. Let \( v, w \) be the neighbors of \( u \) in \( G_2 - V_2(G_2) \). Let \( t_i, 1 \leq i \leq p \) denote the all vertices in \( V_2(G_2) \) that are adjacent to \( u \) in \( G_2 \); \( t_i \neq v, w \) by \( t_i \in V_2(G_2) \). By (i) and (ii) of Claim 2, each \( t_i \) is adjacent to \( v \) or \( w \). By the irreducibility and (iii) and (iv) of Claim 2, we see that \( 2 \leq p \leq 4 \), and there exist \( t_1 \) and \( t_2 \) which are adjacent to \( v \) and \( w \), respectively, and we can assume that \( t_3 \) and \( t_4 \) (if any) are adjacent to \( v \) and \( w \), respectively (note that if \( p = 2 \) and both \( t_1 \) and \( t_2 \) are adjacent to \( u \) and \( v \) then \( u \) would be of degree 4, contradicting (iv) of Claim 2). Notice that in \( G_2 \) such a set of vertices \( u, v, w \) and \( t_i, 1 \leq i \leq p \) induces a connected subgraph \( S_u \) in which only vertices \( v \) and \( w \) are adjacent to the rest of the vertices. We then apply the following procedure to each of such induced subgraphs. For the above vertices \( u, v, w \) and \( t_i, 1 \leq i \leq p \), we remove \( t_3 \) and \( t_4 \) (if any), and add two new edges \( (v, t_2), (w, t_2) \). Observe that the resulting graph remains irreducible and planar, and the degrees of \( v \) and \( w \) never decrease (hence Claim 2 remains valid). Also it is easy to check that the size of a maximum matching never increases. In particular, each vertex of \( t_1, t_2 \) has degree 3, and each of the eight edges that are incident to \( t_1, t_2 \) or \( u \) is uncovered. We repeat applying this procedure to a subgraph \( S_u \) as long as \( G' - V_2(G') \) has no vertex \( u \) of degree 2 in the current graph \( G' \). Let \( G_3 \) be the resulting graph. We then obtain the next property.

**Claim 3** \( G_3 \) remains irreducible and planar, and satisfies \( \mu(G_3) \leq \mu(G_2) \). For \( n^* = |V(G_2)| - |V(G_3)| \), \( G_3 \) has at least \( 4n^* \) uncovered edges. \( \Box \)

Thus, the graph \( G_4 = G_3 - V_2(G_3) \) satisfies \( \delta(G_4) \geq 3 \) and \( |V_2(G_3)| \leq 2(|E(G_2)| - 4n^*) \).

We are now ready to prove Lemma 2. Let \( n_i = |V_i| \) for \( i = 1, 2, 3, 4 \). By Theorem 2.1, \( |E(G_4)| \leq 3n_4 - 6 \). From this and \( |V_2(G_3)| \leq 2(|E(G_4)| - 4n^*) \), we have \( |V_2(G_3)| \leq 6n_4 - 12 - 8n^* \). By \( n_4 = n_3 - |V_2(G_3)| \), we have

\[
\begin{align*}
n_4 & \geq n_3 - 6n_4 + 12 + 8n^* \\
& \geq (n_2 - n^*) - 6n_4 + 12 + 8n^*.
\end{align*}
\]

Therefore, we get \( n_4 \geq \frac{n_2 + 12}{7} \). By Theorem 2.2, \( \mu(G_4) \geq \min\{\frac{n_2 + 12}{7}, \frac{n_2}{3} + 2\} = \frac{n_2}{3} + 2 \) (since \( n_4 \geq 7 \) by \( n_2 = n_1 = n \geq 37 \), \( n_4 \geq 7 \)). Then, we obtain

\[
\begin{align*}
\mu(G_1) & \geq \mu(G_2) \geq \mu(G_3) \geq \mu(G_4) \\
& \geq \frac{n_2}{21} + \frac{26}{21} = \frac{n}{21} + \frac{26}{21}.
\end{align*}
\]

Finally, by \( \mu(G) \geq \frac{1}{2} \mu(G_1) \), we obtain

\[
\mu(G) \geq \frac{n}{42} + \frac{13}{21}.
\]