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G_2-Geometry of Overdetermined Systems of Second Order

By

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Introduction.

Discovery of E. Cartan in

[C1] *Les systèmes de Pfaff à cinq variables et les équations aux derivèes partielles du second ordre*, Ann. Ec. Normale. 27 (1910), 109–192

Overdetermined (involutive) system :

(A)
$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

Single equation of Goursat type:

(B)
$$9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0,$$

where
$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

14-dimensional Exceptional Simple Lie Algebra G_2

The Plan of This TALK

Main Theme **Contact Geometry of Second Order**

• Grassmannian Construction of Jet Spaces (§1)
$\Longrightarrow$ PD-manifolds (§4).

• Geometry of Linear Differential Systems(Tanaka Theory)(§2)
$\Longrightarrow$ Differential Systems associated with SGLA (Simple Graded Lie Algebras) (§3).

• Link between them
$\Longrightarrow$ Reduction Theorems for PD manifolds (§4, §5)

Together combined to discuss

G_2-Geometry of Overdetermined Systems of Second Order (§6)

§1.Second Order Contact Manifolds.

GrassmannBundle:
M: a manifold of dimension $m+n$

$$J(M,n) = \bigcup_{x\in M} J_x, \quad J_x = \mathrm{Gr}(T_x(M),n).$$

Canonical System C on $J(M,n)$:
$\forall u \in J(M,n)$

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M,n)).$$

Inhomogeneous Grassmann coordinate: $x_o = \pi(u_o) \in U'; (x_1,\cdots,x_n,z^1,\cdots,z^m)$

$$U = \{u \in \pi^{-1}(U') \mid \pi(u) = x \in U' \quad \text{and} \quad dx_1 \wedge\cdots\wedge dx_n \mid_u \neq 0\};$$

Coordinates $(x_1,\cdots,x_n,z^1,\cdots,z^m,p_1^1,\cdots,p_n^m)$ are introduced by

$$dz^\alpha \mid_u = \sum_{i=1}^n p_i^\alpha(u)\ dx_i \mid_u .$$

On a canonical coordinate system $(x_1,\cdots,x_n,z^1,\cdots,z^m,p_1^1,\cdots,p_n^m)$

$$C = \{\varpi^1 = \cdots = \varpi^m = 0\},$$

where $\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha\, dx_i$.

$m = 1 \Longrightarrow$ Contact Manifold

$$\varphi : M \to \hat{M} : \text{diffeomorphism} \Longrightarrow \varphi_* : (J(M,n),C) \to (J(\hat{M},n),\hat{C})$$

Theorem 2.1 (Bäcklund). *M, $\hat{M}$: manifolds of dimension $m+n$.*
Assume $m \geqq 2$. $\Phi : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$; isomorphism
$\Longrightarrow \exists_1 \varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.

$$(J, C) : \text{Contact Manifold} \Longrightarrow (L(J), E)$$

Lagrange-Grassmann Bundle

$$L(J) = \bigcup_{u \in J} L_u \xrightarrow{\pi} J$$

$$L_u = \{\text{Legendrian subspaces of}(C(u), d\varpi)\}.$$

$\forall v \in L(J)$

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J)$$

where $(x_1, \cdots, x_n, z, p_1, \cdots, p_n, p_{11}, \cdots, p_{nn})$ and $p_{ij} = p_{ji}$

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

where

$$\begin{cases} \varpi = dz - \sum_{i=1}^n p_i \, dx_i, \\ \varpi_i = dp_i - \sum_{j=1}^n p_{ij} \, dx_j \end{cases}$$

Theorem 2.2. *$\Phi : (L(J), E) \to (L(\hat{J}), \hat{E})$; isomorphism $\Longrightarrow \exists_1 \varphi : (J, C) \to (\hat{J}, \hat{C})$ such that $\Phi = \varphi_*$.*

§2. Geometry of Linear Differential Systems (Tanaka Theory).

M: a manifold of dimension d
$D \subset T(M)$: subbundle of rank r $(s + r = d)$

$$D = \{\omega_1 = \cdots = \omega_s = 0\}.$$

(M, D): completely integrable
$\Longleftrightarrow D = \{dx_1 = \cdots = dx_s = 0\}$
$\Longleftrightarrow d\omega_i \equiv 0 \pmod{\omega_1, \cdots, \omega_s}$ $(1 \leqq i \leqq s)$
$\Longleftrightarrow [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ where $\mathcal{D} = \Gamma(D)$

Derived System ∂D: $\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$.
Cauchy Characteristic System $Ch(D)$:

$$Ch(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \quad \text{for } i = 1, \ldots, s\},$$

k-th Derived System $\partial^k D$:

$$\partial^k D = \partial(\partial^{k-1} D)$$

k-th Weak Derived System $\partial^{(k)} D$:

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

Symbol Algebras

(M, D) : regular

$(S1)$ $\exists \mu > 0$ such that, for all $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p, q < 0$.

(M, D) : regular such that $T(M) = D^{-\mu}$.

$\forall x \in M$,

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

$\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X} \in \Gamma(D^p), X = \varpi_p(\tilde{X}_x) \in \mathfrak{g}_p(x)$, $\tilde{Y} \in \Gamma(D^q), Y = \varpi_q(\tilde{Y}_x) \in \mathfrak{g}_q(x)$.

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

Conversely given a

Fundamental Graded Lie Algebra :

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

i.e., Nilpotent GLA satisfying the generating condition : $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$

$\Longrightarrow$

$(M(\mathfrak{m}), D_{\mathfrak{m}})$: Standard Differential System of type $\mathfrak{m}$

$\Longrightarrow$

$\mathfrak{g}(\mathfrak{m})$: Prolongation of $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$

Our Problem :

When does $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple ?

Symbol Algebra of $(L(J), E)$:

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

$$\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-3} = W, \mathfrak{c}_{-2} = W \otimes V^*$,

$$\mathfrak{c}_{-1} = V \oplus W \otimes S^2(V^*).$$

Coframe: Dual frame:

$$\{\varpi, \varpi_i, dx_i, dp_{ij}\}, \qquad \{\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial z} + \sum_{j=1}^{n} p_{ij}\frac{\partial}{\partial p_j}$$

$$\left[\frac{\partial}{\partial p_{ij}}, \frac{d}{dx_k}\right] = \delta^i_k\frac{\partial}{\partial p_j} + \delta^j_k\frac{\partial}{\partial p_i},$$

$$\left[\frac{\partial}{\partial p_i}, \frac{d}{dx_k}\right] = \delta^i_k\frac{\partial}{\partial z}.$$

$$\partial E = \{\varpi = 0\} = \pi_*^{-1}C, \quad Ch(\partial E) = \mathrm{Ker}\pi_*.$$

§3. Differential Systems associated with Simple Graded Lie Algebras.

Gradation of $\mathfrak{g}$

$\mathfrak{g}$: Simple Lie Algebra over $\mathbb{C}$

$\mathfrak{h}$: Cartan Subalg. ;$\Phi \subset \mathfrak{h}^*$:Root System

$\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$: Simple Root System

$$\mathfrak{g} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha},$$

$\Delta_1 \subset \Delta$: Fix, $\quad \Phi^+ = \cup_{p\geq 0}\Phi^+_p$,

$$\Phi^+_p = \{\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \mid \sum_{\alpha_i\in\Delta_1} n_i = p\},$$

$$\begin{cases} \mathfrak{g}_p = \bigoplus_{\alpha\in\Phi^+_p} \mathfrak{g}_\alpha, & (p > 0) \\ \mathfrak{g}_0 = \bigoplus_{\alpha\in\Phi^+_0} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+_0} \mathfrak{g}_{-\alpha}, \\ \mathfrak{g}_{-p} = \bigoplus_{\alpha\in\Phi^+_p} \mathfrak{g}_{-\alpha}, \end{cases}$$

Then $\quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \quad$ for $\; p, q \in \mathbb{Z}$.

Generating Condition:$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ $(\star)$ $\quad \mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$

$$\Delta_1 \subset \Delta \quad \Longrightarrow \quad (X_\ell, \Delta_1): \quad \mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$$

where $\mu = \sum_{\alpha_i\in\Delta_1} n_i(\theta)$, $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$

Theorem 4.1. $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$: *Simple Graded Lie Algebra over* $\mathbb{C}$ *satisfying* $(\star)$.
X_ℓ: *Dynkin Diagram of* $\mathfrak{g}$.$\Longrightarrow \exists_1\ \Delta_1 \subset \Delta$ *s.t.* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p \cong (X_\ell, \Delta_1)$

Classification of$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$with$(\star)$

$$\Longleftrightarrow \text{Classification of Parabolic subalgebras} \mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$$

$(X_\ell, \Delta_1) \Longrightarrow M_\mathfrak{g} = G/G'$: R-space $\quad \mu \geqq 2 \quad \mathfrak{g}_{-1} \Longrightarrow D_\mathfrak{g}$ on $M_\mathfrak{g}$

$(M_\mathfrak{g}, D_\mathfrak{g}) \supset (M_\mathfrak{m}, D_\mathfrak{m})$, $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.

Theorem 4.2. $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$: *Simple Graded Lie Algebra over* $\mathbb{C}$ *satisfying* $(\star)$.
Except for (1), (2), (3),

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p \cong \mathfrak{g}(\mathfrak{m}),$$

where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1 $(\mu = 1)$.
(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a contact gradation.*
(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

Corresponding R-spaces
(1) $\Rightarrow$ Compact Hermitian Symmetric Spaces
(2) $\Longrightarrow$ Standard Contact Manifolds
(3) $\quad (A_\ell, \{\alpha_1, \alpha_i\}) \Longrightarrow (J(\mathbb{P}^\ell, i-1), C)$ $(C_\ell, \{\alpha_1, \alpha_\ell\}) \Longrightarrow (L(\mathbb{P}^{2\ell-1}), E)$.

§4. Geometry of PD-manifolds.

$R \subset L(J)$: submanifold satisfying

$$(R.0) \quad p : R \longrightarrow J \text{ ; submersion,}$$

On $L(J)$, $\qquad C^1 = \partial E, \quad C^2 = E$
On R, $\qquad D^1 = C^1 |_R,\ D^2 = C^2 |_R$
$(R; D^1, D^2)$ satisfies :

$(R.1)$ $\quad D^1$: codim. 1, D^2: codim. $n+1$,
$(R.2)$ $\quad \partial D^2 \subset D^1$,

$(R.3)\quad Ch(D^1) \subset D^2$: codim. n,

$(R.4)\quad Ch(D^1) \cap Ch(D^2) = \{0\}$.

Triplet $(R; D^1, D^2)$: **PD-manifold** $\Longleftrightarrow$ $(R.1) \sim (R.4)$

$$\{0\} = Ch(D^1) \cap Ch(D^2) \subset Ch(D^1) \subset D^2 \subset \partial D^2 \subset D^1 \subset T(R)$$

Realization Theorem for PD-manifold

(*i*) $(R.1)$ and $(R.3) \Longrightarrow (J, C)$

$$J = R/Ch(D^1), \qquad D^1 = p_*^{-1}(C),$$

where $p : R \to J = R/Ch(D^1)$.

(*ii*) $(R.1)$ and $(R.2) \Longrightarrow \iota(v) = p_*(D^2(v)(\subset C(u)$: *Legendrian*

(*iii*) $(R.4) \Longrightarrow \iota : R \to L(J)$: *immersion*

Theorem 5.1.
$\Phi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$:*isomorphism* $\Longrightarrow$
$\exists_1 \varphi : (J, C) \to (\hat{J}, \hat{C})$: *contact diffeo s.t.;*

$$\begin{array}{ccc} R & \xrightarrow{\ \iota\ } & L(J) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \hat{R} & \xrightarrow{\ \iota\ } & L(\hat{J}). \end{array}$$

Compatibility Condition (C)

$(C)\quad p^{(1)} : R^{(1)} \to R$ *is onto.*

where $R^{(1)}$: the first prolongation of $(R; D^1, D^2)$.

Theorem 5.2. $(R : D^1, D^2)$:*PD-manifold satisfying the condition* (C).
$\forall v \in R$:

$$\dim D^1(v) - \dim \partial D^2(v) = \dim Ch(D^2)(v).$$

Especially $D^1 = \partial D^2 \iff Ch(D^2) = \{0\}$.

In case rank $Ch(D^2) > 0$,

Geometry of $(R; D^1, D^2) \Longrightarrow$ **Geometry of** (X, D),

where $\quad X = R/Ch(D^2)$, $D^2 = \rho_*^{-1}(D)$, $\rho : R \to X$.

§5. Single Equations of Goursat Type.

$L(J) \supset R = \{F(x_i, z, p_i, p_{ij}) = 0\}$: Hypersurface s.t. $p : R \to J$; submersion,

R is of (weak) parabolic type at each $v \in R$

$\iff$ $(\frac{\partial F}{\partial p_{ij}}(v))$: rank 1 at each $v \in R$

$\iff$ (R, D^2) : regular of type $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1},$$

where $\mathfrak{s}_{-3} = \mathbb{R}$, $\mathfrak{s}_{-2} = V^*$, $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$ $\mathfrak{f} \subset S^2(V^*)$; $(\mathfrak{f})^{\perp} = \langle e^2 \rangle \subset S^2(V)$, $e \in V$.

$\iff$ $\exists$ Coframe $\{\varpi, \varpi_a, \omega_a, \varpi_{1\alpha}, \varpi_{\alpha\beta}\}$ $(1 \leqq a \leqq n, 2 \leqq \alpha \leqq \beta \leqq n)$ on R such that

$$D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \cdots\cdots + \omega_n \wedge \varpi_n & (\text{mod } \varpi), \\ d\varpi_1 \equiv \omega_2 \wedge \varpi_{12} + \cdots + \omega_n \wedge \varpi_{1n} & (\text{mod } \varpi, \varpi_1, \cdots, \varpi_n), \\ d\varpi_\alpha \equiv \omega_1 \wedge \varpi_{\alpha 1} + \cdots\cdots + \omega_n \wedge \varpi_{\alpha n} & (\text{mod } \varpi, \varpi_1, \cdots, \varpi_n). \end{cases}$$

where $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$ $\varpi_{1\alpha} = \varpi_{\alpha 1}$, $2 \leqq \alpha, \beta \leqq n$.

R is a equation of **Goursat** type
$\iff$ R ; (weak) parabolic type s.t. $M(E)$; completely integrable,
where $M(E)$ is the **Monge system** ;

$$M(E) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega_\alpha = \varpi_{1\alpha} = 0 \quad (2 \leqq \alpha \leqq n)\}.$$

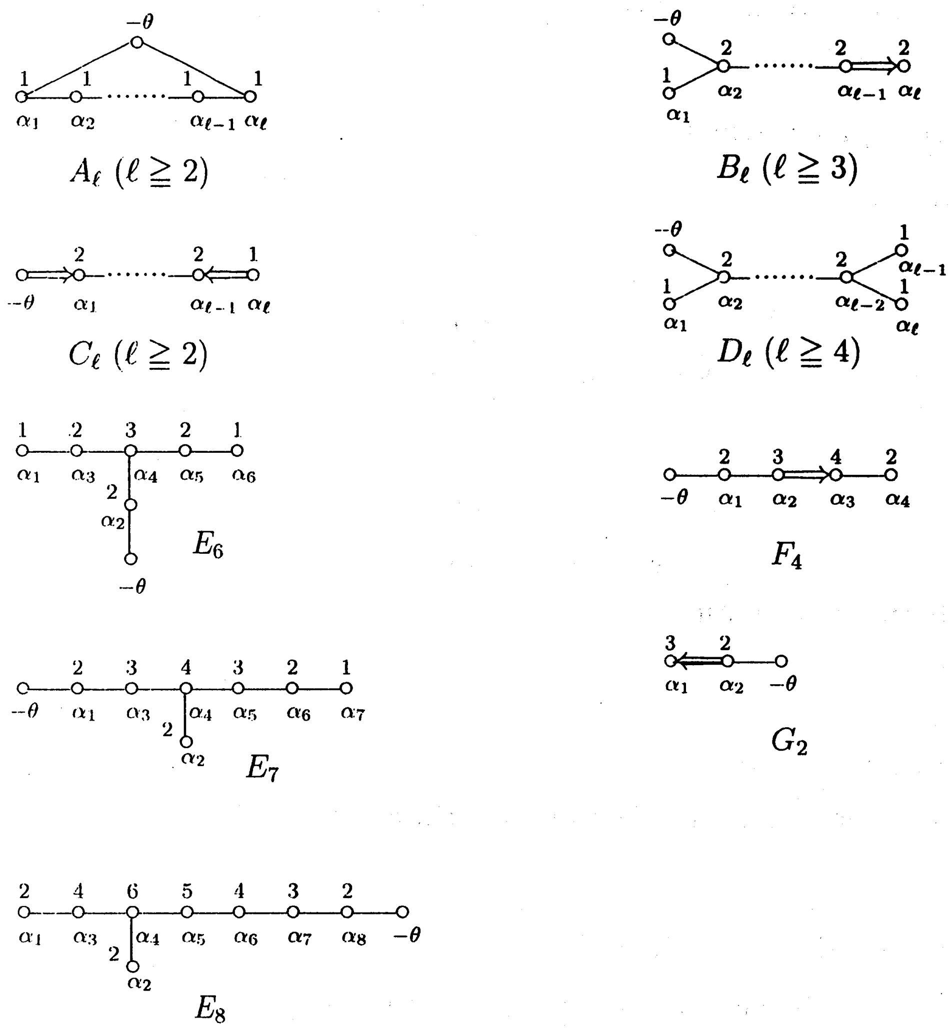
The First Order Covariant System $N(E)$

$$N = N(E) = \{\varpi = \varpi_1 = 0\}.$$

By Two Step Reductions

Geometry of (R, D^2); **Goursat Type** $\Longrightarrow$ **Geometry of** (Y, D_N); **Type** $\mathfrak{c}^1(n-1, 2)$,

where $Y = R/Ch(N)$, $N = \rho_*^{-1}(D_N)$, $\rho : R \to Y$.



Extended Dynkin Diagrams with the coefficient of the highest root

§6. G_2-geometry.

6.1. Standard Contact Manifolds

$\mathfrak{g}$: Simple Lie Algebra over $\mathbb{C}$

θ: Highest Root

(X_ℓ, Δ_θ) : Contact Gradation $\Longrightarrow$ $\quad \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$

$(J_{\mathfrak{g}}, C_{\mathfrak{g}})$:**Standard Contact Manifolds** $\Leftarrow$ Boothby

[Projectiviation of the (co-)ajoint orbit through the highest root vector]

$\Delta_\theta \Longleftrightarrow$ **Extended Dynkin Diagram**

6.2. Gradation of G_2.

$$\underset{\alpha_1}{\odot} \Lleftarrow \underset{\alpha_2}{\odot}, \qquad \theta = 3\alpha_1 + 2\alpha_2.$$

$$\Delta_1 \subset \Delta = \{\alpha_1, \alpha_2\}$$

(G1) $\Delta_1 = \{\alpha_1\}$. $\mu = 3$,

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

where $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$, $\dim \mathfrak{g}_{-2} = 1$.

(G2) $\Delta_1 = \{\alpha_2\}$. $\mu = 2$

$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$: **Contact Gradation**

(G3) $\Delta_1 = \{\alpha_1, \alpha_2\}$. $\mu = 5$,

$$\mathfrak{m} = \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

where $\dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_p = 1$ for others.

Root System G_2

$\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h} = 2$

$\Delta = \{\alpha_1, \alpha_2\}$: **Simple Root System**

Φ^+ consists of the following roots

α_1, α_2,

$\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$, $\alpha_2 + 3\alpha_1$,

$2\alpha_2 + 3\alpha_1$

Type G_2

$(J_{\mathfrak{g}}, C_{\mathfrak{g}})$: **Standard Contact Manifold**

$$\dim J_{\mathfrak{g}} = 5$$

$L(J_{\mathfrak{g}})$: **Lagrange-Grassmann Bundle**

$$\dim L(J_{\mathfrak{g}}) = 8$$

Orbits Decomposition

$$L(J_{\mathfrak{g}}) = O \cup R_1 \cup R_2,$$

(1) O: Open orbit,

(2) R_1: Codim 1, the Global Model of (B),

(3) R_2: Codim 2, the Global Model of (A).

$$R_2\text{: compact} \quad \rightleftharpoons \quad (G_2, \{\alpha_1, \alpha_2\})$$

$$X_\ell \not\cong A_\ell \quad \Longrightarrow \quad \Delta_\theta = \{\alpha_\theta\}$$

For **Exceptional Simple Lie Algebras**, $\exists_1 \quad \alpha_G$: **3 next to** α_θ

$$(X_\ell, \{\alpha_G\})\text{: } \mu = 3 \quad \Longleftrightarrow \quad (M_{\mathfrak{g}}, D_{\mathfrak{g}})$$

$$\mathfrak{g}_{-3} = W, \quad \mathfrak{g}_{-2} = V \quad \mathfrak{g}_{-1} = W \otimes V^*.$$

$$\dim \mathfrak{g}_{-3} = 2$$

i.e.,$(M_{\mathfrak{g}}, \partial D_{\mathfrak{g}})$: regular of type $\mathfrak{c}^1(r, 2)$.

$$(J_{\mathfrak{g}}, C_{\mathfrak{g}}) \quad \Longleftrightarrow \quad (X_\ell, \{\alpha_\theta\})$$

$$\uparrow$$

$$L(J_{\mathfrak{g}}) \supset R_2 \quad \rightleftharpoons \quad (X_\ell, \{\alpha_\theta, \alpha_G\})$$

$$\downarrow$$

$$(X, D) \quad \Longleftrightarrow \quad (X_\ell, \{\alpha_G\})$$

$$L(J_{\mathfrak{g}}) \supset R_1 \quad \Longleftarrow \quad (M_{\mathfrak{g}}, \partial D_{\mathfrak{g}})$$

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[C1] E.Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Ec. Normale **27** (1910), 109–192.

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