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最大値過程について

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1 Introduction

In this paper we are concerned with a maximum process generated through independent and identically distributed random variables via its summation process. Both the i.i.d. process and the summation process are Markov chains. The probabilistic behavior of both is well known [12]. However, the maximum process is not Markov. We are interested in how to make it Markov on suitable state spaces. Further we focus our attention on a recursive computation of expected value, variance and probability distribution of the maximum random variable.

We present a new method "Markovization", which is a *stochastic* realization of invariant imbedding approach [2–7, 9, 11] —"dynamic programming without optimization" [1, p.115], [2, p.72], [4, p.203], [8, p.23], [9,10], [13, Chap.14]. We show that the computation is based upon the Markovization.

2 Profit Process

2.1 Random Walk

Let $X_1,\ X_2, \ldots,\ X_i, \ldots$ be independent and identically distributed with

$$P(X_i = 1) = p, \quad P(X_i = -1) = q \qquad (0 \le p \le 1,\ p + q = 1).$$

In a game of flipping a coin — probability of head is p — infinitely, $X_i = 1\ (-1)$ denotes a *win* (*loss*) at i-th flipping. When a player wins (loses), he get (loses) one-unit.

Then we define the *sum-to-date* as follows :

$$Y_0 = 0, \quad Y_i = \sum_{j=1}^{i} X_j. \tag{1}$$

The *additive process* $\{Y_i,\ i \ge 0\}$ is a random walk [12, p.143]. It is also called a *profit process.* In the game of coin flipping, Y_i denotes a net profit, which is equal to "*the number of wins* $-$ *the number of losses*" up to i flippings. It is a difference between wins and losses.

Further we define the *maximum-to-date* :

$$Z_i = \max_{1 \le j \le i} Y_j.$$

The *maximum process* $\{Z_i, \ i \geq 1\}$ is called a *maximum profit process.* In the game, Z_i denotes the maximum value of differences between wins and losses up to i flippings.

Let N be any positive integer. We are concerned with the maximum profit random variable over the N-stage Z_N. Our problem is to find its expected value, variance and probability distribution :

$$E[Z_N], \quad V[Z_N], \quad P(Z_N = k).$$

2.2 Profit Process

Let us take an integer $n(\geq 2)$. We consider a random variable U, which follows the Binomial distribution $Bi(n;p)$:

$$P(U = k) = {}_nC_k \, p^k q^{n-k} \qquad k = 0, 1, \dots, n.$$

It is easily shown that

$$E[U] = np, \quad V[U] = npq.$$

We return to the random variable Y_n defined in (1). Let us consider the events $Y_n = 1 \cdot k + (-1) \cdot (n-k), \ k = 0, 1, \dots, n$. Then we have

$$P(Y_n = 2k - n) = {}_nC_k \, p^k q^{n-k}.$$

Thus we see that $Y_n = 2U - n$ holds almost surely.

As a summary we obtain

Lemma 2.1.

(i) $Y_n = 2U - n \quad a.s.$

(ii) $P(Y_n = 2k - n) = {}_nC_k \, p^k q^{n-k} \qquad k = 0, 1, \dots, n$

(iii) $E[Y_n] = n(p - q), \quad V[Y_n] = 4npq.$

Thus Y_n takes values $-n, \ -n+2, \ -n+4, \ \dots, \ n-2, \ n$. Let us define state spaces $\{S_n\}$ as follows :

$$S_n := \{-n, \ -n+2, \ -n+4, \ \dots, \ n-2, \ n\} \qquad n = 0, 1, \dots \tag{2}$$

where

$$S_0 = \{0\}.$$

Therefore, the profit process $\{Y_n, \ n = 0, 1, \dots\}$ is a Markov chain on the state spaces $\{S_n\}$ with the transition probability $p = \{p_n(j \mid i)\}$ (Figure 1) :

$$p_n(j \mid i) = \begin{cases} p & \text{if } j = i + 1 \\ q & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases} \qquad n = 0, 1, \dots . \tag{3}$$

Then the recursion $Y_{n+1} = X_{n+1} + Y_n, \ n = 1, 2, \dots$ together with i.i.d. property in $\{X_n\}$

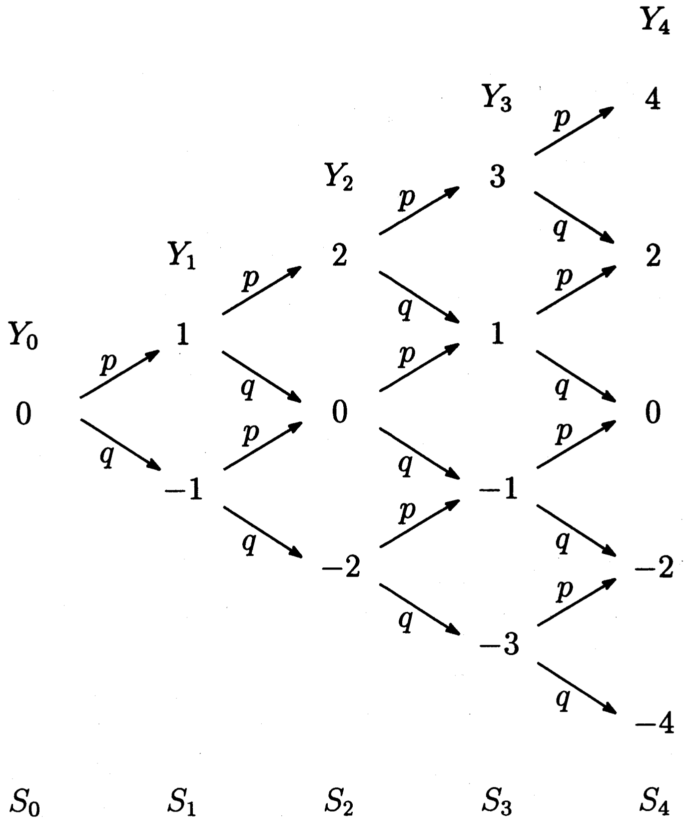

Figure 1: Profit process $Y = \{Y_n\}$

Lemma 2.2.

$$\begin{aligned} &\text{(i)} \quad E[Y_{n+1}] = p - q + E[Y_n] \qquad n = 1, 2, \dots \\ &\qquad E[Y_1] = p - q. \\ &\text{(ii)} \quad E[Y_{n+1}^2] = 1 + 2(p-q)E[Y_n] + E[Y_n^2] \qquad n = 1, 2, \dots \\ &\qquad E[Y_1^2] = 1. \end{aligned}$$

Thus we have

$$E[Y_n] = n(p-q), \quad E[Y_n^2] = n + n(n-1)(p-q)^2.$$

This also implies $V[Y_n] = 4npq$.

3 Maximum Process

It is shown that expected value and variance of profit Y_n has been calculated through the Binomial distribution. In this section we consider expected value and variance of maximum profit Z_n.

Lemma 3.1. (i) *For* $n = 1, 2, \dots,$

$$Y_{2n+1} > Z_{2n}$$

is equivalent to

$$\begin{aligned} X_2 + X_3 + X_4 + X_5 + \cdots + X_{2n-4} + X_{2n-3} + X_{2n-2} + X_{2n-1} &\geq 0 \\ X_4 + X_5 + \cdots + X_{2n-4} + X_{2n-3} + X_{2n-2} + X_{2n-1} &\geq 0 \\ &\vdots \\ X_{2n-4} + X_{2n-3} + X_{2n-2} + X_{2n-1} &\geq 0 \\ X_{2n-2} + X_{2n-1} &\geq 0 \\ X_{2n} = X_{2n+1} &= 1. \end{aligned}$$

(ii) *For* $n = 1, 2, \ldots,$

$$Y_{2n+2} > Z_{2n+1}$$

is equivalent to

$$\begin{aligned} X_3 + X_4 + X_5 + X_6 + \cdots + X_{2n-3} + X_{2n-2} + X_{2n-1} + X_{2n} &\geq 0 \\ X_5 + X_6 + \cdots + X_{2n-3} + X_{2n-2} + X_{2n-1} + X_{2n} &\geq 0 \\ &\vdots \\ X_{2n-3} + X_{2n-2} + X_{2n-1} + X_{2n} &\geq 0 \\ X_{2n-1} + X_{2n} &\geq 0 \\ X_{2n+1} = X_{2n+2} &= 1. \end{aligned} \tag{4}$$

In particular,

$$Y_2 > Z_1$$

is equivalent to

$$X_2 = 1.$$

(iii) *In either case we have*

$$Y_{n+1} - Z_n = 1 \tag{5}$$

$$Y_{n+1}^2 - Z_n^2 = 2Y_n + 1 \qquad n = 0, 1, \ldots . \tag{6}$$

Proof. Since (i) is shown, we first show (ii). We note that the inequality

$$\begin{aligned} Y_{2n+2} &> Z_{2n+1} \\ &= Y_1 \vee Y_2 \vee \cdots \vee Y_{2n-1} \vee Y_{2n} \vee Y_{2n+1} \end{aligned}$$

is equivalent to the system of inequalities

$$\begin{aligned} Y_{2n+2} > Y_{2n+1} &\implies X_{2n+2} > 0 \\ Y_{2n+2} > Y_{2n} &\implies X_{2n+2} + X_{2n+1} > 0 \\ Y_{2n+2} > Y_{2n-1} &\implies X_{2n+2} + X_{2n+1} + X_{2n} > 0 \\ Y_{2n+2} > Y_{2n-2} &\implies X_{2n+2} + X_{2n+1} + X_{2n} + X_{2n-1} > 0 \\ &\vdots \\ Y_{2n+2} > Y_2 &\implies X_{2n+2} + X_{2n+1} + \cdots + X_3 > 0 \\ Y_{2n+2} > Y_1 &\implies X_{2n+2} + X_{2n+1} + \cdots + X_3 + X_2 > 0. \end{aligned}$$

$$\begin{aligned}
X_{2n+2} &= 1 \\
X_{2n+1} &= 1 \\
X_{2n} &> -2 \\
X_{2n-1} + X_{2n} &> -2 \\
X_{2n-2} + X_{2n-1} + X_{2n} &> -2 \\
&\vdots \\
X_3 + X_4 + \cdots + X_{2n-1} + X_{2n} &> -2 \\
X_2 + X_3 + X_4 + \cdots + X_{2n-1} + X_{2n} &> -2.
\end{aligned} \tag{7}$$

Since each X_m takes only two values $-1, 1$, we see that (7) is equivalent to (4). (iii) is shown as follows. First we assume that $Y_{n+1} > Z_n$. Here

$$\begin{aligned}
Y_n &= Y_{n-1} + X_n \\
&> Y_{n-1} \qquad (\text{since } X_n = 1) \\
Y_n &= Y_{n-2} + X_{n-1} + X_n \\
&\geq Y_{n-2} \\
Y_n &= Y_{n-3} + X_{n-2} + X_{n-1} + X_n \\
&> Y_{n-3} \qquad (\text{since } X_{n-2} + X_{n-1} \geq 0) \\
Y_n &= Y_{n-4} + X_{n-3} + X_{n-2} + X_{n-1} + X_n \\
&\geq Y_{n-4} \\
&\vdots
\end{aligned}$$

Then we have $Y_n \geq Y_m$, $m = 1, 2, \ldots, n$, which implies

$$Y_{n+1} - Z_n = Y_{n+1} - Y_n = X_{n+1} = 1.$$

Furthermore, this implies

$$\begin{aligned}
Y_{n+1}^2 - Z_n^2 &= (Y_{n+1} + Z_n)(Y_{n+1} - Z_n) \\
&= Y_{n+1} + Z_n \\
&= 2Y_{n+1} - 1 \\
&= 2Y_n + 1.
\end{aligned}$$

□

Then we have the following recursion formulae :

Lemma 3.2.

$$\begin{aligned}
&(\text{i}) \quad E[Z_{n+1}] = P(Y_{n+1} > Z_n) + E[Z_n] \qquad n = 1, 2, \ldots \\
&\qquad\; E[Z_1] = p - q. \\
&(\text{ii}) \quad E[Z_{n+1}^2] = E[(2Y_n + 1) \cdot 1_{\{Y_{n+1} > Z_n\}}] + E[Z_n^2] \qquad n = 1, 2, \ldots \\
&\qquad\; E[Z_1^2] = 1.
\end{aligned}$$

Proof. First we note the equality

$$1_{\{Y_{n+1}>Z_n\}} + 1_{\{Y_{n+1}\le Z_n\}} = 1$$

where 1_A is the charasteristic function of set A :

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

It holds that

$$\begin{aligned} Z_{n+1} &= Z_n \vee Y_{n+1} \\ &= (Z_n \vee Y_{n+1}) \cdot (1_{\{Y_{n+1}>Z_n\}} + 1_{\{Y_{n+1}\le Z_n\}}) \\ &= Y_{n+1} \cdot 1_{\{Y_{n+1}>Z_n\}} + Z_n \cdot 1_{\{Y_{n+1}\le Z_n\}}) \\ &= (Y_{n+1} - Z_n) \cdot 1_{\{Y_{n+1}>Z_n\}} + Z_n \qquad (8) \\ Z_{n+1}^2 &= (Z_n \vee Y_{n+1})^2 \\ &= (Z_n \vee Y_{n+1})^2 \cdot (1_{\{Y_{n+1}>Z_n\}} + 1_{\{Y_{n+1}\le Z_n\}}) \\ &= Y_{n+1}^2 \cdot 1_{\{Y_{n+1}>Z_n\}} + Z_n^2 \cdot 1_{\{Y_{n+1}\le Z_n\}}) \\ &= (Y_{n+1}^2 - Z_n^2) \cdot 1_{\{Y_{n+1}>Z_n\}} + Z_n^2. \qquad (9) \end{aligned}$$

From (5) we see that

$$\begin{aligned} E[(Y_{n+1} - Z_n) \cdot 1_{\{Y_{n+1}>Z_n\}}] &= E[1_{\{Y_{n+1}>Z_n\}}] \\ &= P(Y_{n+1} > Z_n). \end{aligned}$$

Taking the expectation of both sides in (8), we have

$$\begin{aligned} E[Z_{n+1}] &= E[(Y_{n+1} - Z_n) \cdot 1_{\{Y_{n+1}>Z_n\}}] + E[Z_n] \\ &= P(Y_{n+1} > Z_n) + E[Z_n]. \end{aligned}$$

Similarly, a combination of (6) and (9) yields

$$\begin{aligned} E[Z_{n+1}^2] &= E[(Y_{n+1}^2 - Z_n^2) \cdot 1_{\{Y_{n+1}>Z_n\}}] + E[Z_n^2] \\ &= E[(2Y_n + 1) \cdot 1_{\{Y_{n+1}>Z_n\}}] + E[Z_n^2]. \end{aligned}$$

□

4 Markovization

The profit process $\{Y_n,\ n \ge 0\}$ is a Markov chain, where the state spaces $\{S_n\}$ and the transition probability $p = \{p_n(j \mid i)\}$ are defined by (2) and (3), respectively. Now we consider the maximum profit process $\{Z_n,\ n \ge 1\}$, which is not necessarily Markov. We are interested in some approach which makes the maximum process Markov on a suitable state spaces. In the case, the state spaces are meaningful as far as it is *small.* We call this problem a *Markovization* of process $\{Z_n\}$. In this section the Markovization is performed through an invariant imbedding method [1, p.82], [13, Chap.12].

Let us define the *past-value sets* $\{\Omega_n\}$ as follows :

$$\Omega_n := \{\lambda_n \mid \lambda_n = (-1) \vee y_1 \vee y_2 \vee \cdots \vee y_{n-1},\ y_i \in S_i,\ 1 \le i \le n-1\} \qquad n = 2, 3, \ldots .$$

In particular, we call Ω_n the *maximum-value-set* to date n. Then the past-value sets satisfy the forward recursive formula :

Lemma 4.1.

$$\begin{aligned} \Omega_2 &= \{-1, 1\} \\ \Omega_{n+1} &= \{\lambda \vee y \mid \lambda \in \Omega_n,\ y \in S_n\} \qquad n = 2, 3, \ldots . \end{aligned}$$

In fact, we have

$$\begin{aligned} \Omega_2 &= \{-1, 1\} \\ \Omega_n &= \{-1,\ 0,\ 1, \ldots,\ n-2,\ n-1\} \qquad n = 3, 4, \ldots . \end{aligned}$$

Then Z_n takes values on Ω_{n+1} for $n = 1, 2, \ldots$ (Figure 2).

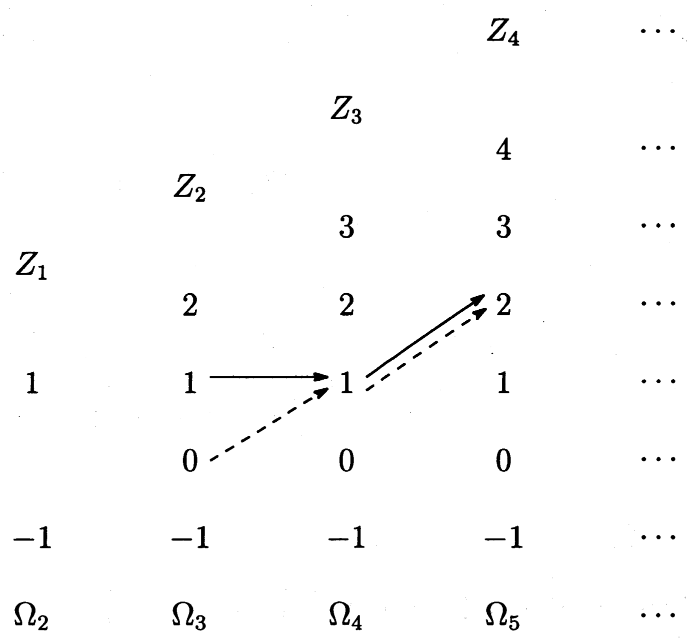

Figure 2: Maximum profit process $Z = \{Z_n\}_{n \ge 1}$

Lemma 4.2. *However, the maximum profit process* $Z = \{Z_n,\ n \ge 1\}$ *is not Markov on the state spaces* $\{\Omega_{n+1}\}$*. In fact, both transition probabilities*

$$P(Z_4 = 2 \mid Z_2 = 0, Z_3 = 1) = p$$

and

$$P(Z_4 = 2 \mid Z_2 = 1, Z_3 = 1) = p^2$$

depend on the yesterday's state and are not equal for $0 < p < 1$.

Proof. The following four equivalences are easily shown.

$$Z_2 = 0, Z_3 = 1 \iff X_1 = -1, X_2 = 1, X_3 = 1$$
$$Z_2 = 0, Z_3 = 1, Z_4 = 2 \iff X_1 = -1, X_2 = 1, X_3 = 1, X_4 = 1$$

$$Z_2 = 1, Z_3 = 1 \iff X_1 = 1, X_2 = -1$$
$$Z_2 = 1, Z_3 = 1, Z_4 = 2 \iff X_1 = 1, X_2 = -1, X_3 = 1, X_4 = 1.$$

The first two and the last two yield the former transition probability and the latter, respectively. This completes the proof. □

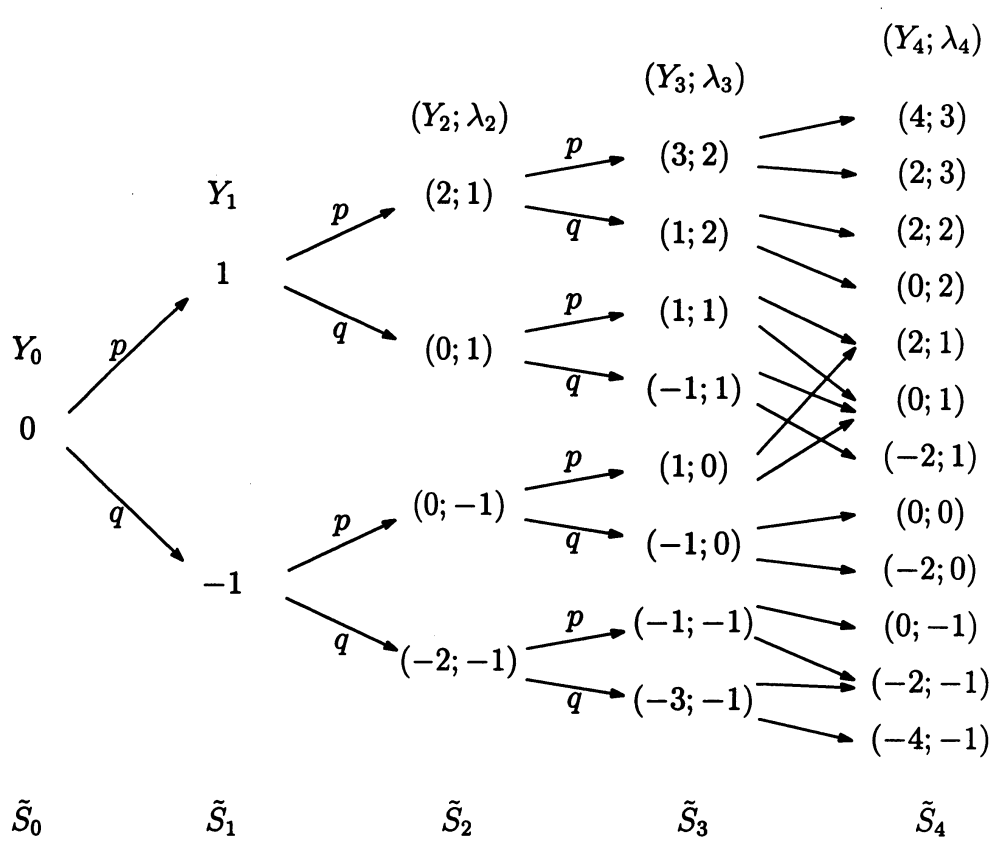

Figure 3: Expanded profit process $\tilde{Y} = \{\tilde{Y}_n,\ n \geq 0\}$

First, by attaching Ω_n, we expand the state space S_n to *augmented state spaces* $\{\tilde{S}_n\}$ as follows :

$$\tilde{S}_n := S_n, \qquad n = 0, 1$$

$$\tilde{S}_n := \left\{ (y_n; \lambda_n) \;\middle|\; \begin{array}{l} \lambda_n = y_1 \vee y_2 \vee \cdots \vee y_{n-1} \\ y_i = x_1 + x_2 + \cdots + x_i \\ x_i \in \{-1, 1\},\ 1 \leq i \leq n \end{array} \right\} \subset S_n \times \Omega_n \qquad n = 2, 3, \ldots, N.$$

Then the expanded state spaces $\{\tilde{S}_n\}$ are forwardly generated as follows.

Lemma 4.3.

$$\tilde{S}_2 = \{\,(2;1),\ (0;1),\ (0;-1),\ (-2;-1)\,\}$$
$$\tilde{S}_{n+1} = \{\,(y+x;\lambda \vee y)\mid (y;\lambda)\in \tilde{S}_n,\ x\in\{-1,1\}\,\} \qquad n=2,3,\dots,N.$$

Second we define a *new transition law* $q = \{q_n\}$ there by

$$q_0(j\mid 0) := \begin{cases} p & \text{if } j=1 \\ q & \text{if } j=-1 \end{cases}$$

$$q_1((j;i)\mid i) := \begin{cases} p & \text{if } j=i+1 \\ q & \text{if } j=i-1 \end{cases}$$

$$q_n((j;\mu)\mid (i;\lambda)) := \begin{cases} p_n(j\mid i) & \text{if } \lambda \vee i = \mu \\ 0 & \text{otherwise} \end{cases} \qquad n=2,3,\dots .$$

Finally we define a sequence of random variables $\{\tilde{Y}_n; n=0,1,\dots\}$ by

$$\tilde{Y}_n := Y_n \quad n=0,1; \qquad \tilde{Y}_n := (Y_n;\lambda_n) \quad n=2,3,\dots .$$

We call $\tilde{Y} = \{\tilde{Y}_n\}$ an *expanded profit process.* Thus Y_n represents today's profit, which behaves stochastically. And $\lambda_n = (-1)\vee y_1 \vee y_2 \vee \cdots \vee y_{n-1} = z_{n-1}$ denotes the *maximum-to-yesterday*, which has been already determined. Thus $\{\lambda_n\}$ is deterministic. Every time Y_n realizes a value y_n, the resulting pair $(y_n;\lambda_n)$ yields the *today's maximum-value*

$$\lambda_n \vee y_n = y_1 \vee \cdots \vee y_{n-1} \vee y_n = z_n$$

under *maximum* operation $\vee$. In other words, the expanded profit process $\tilde{Y} = \{\tilde{Y}_n\}$ generates the maximum process $Z = \{Z_n\}$ under the binary operation. Furthermore, we have the following desired property.

Lemma 4.4. *The expanded profit process $\{\tilde{Y}_n,\ n\geq 0\}$ is a Markov chain on the state spaces $\{\tilde{S}_n\}$ with the transition probability $q=\{q_n\}$ (Figure 3).*

We define a mapping $T_n : \tilde{S}_n \to R^1$ by

$$T_n(\tilde{y}_n) := \lambda_n \vee y_n \qquad n=2,3,\dots .$$

Then the sequence of mappings $\{T_n\}$ transforms process $\{\tilde{Y}_n\}$ into $\{Z_n\}$ in the following sense :

$$T_n(\tilde{Y}_n) = Z_{n-1} \vee Y_n = Z_n \qquad n=2,3,\dots .$$

Thus the process $\{\tilde{Y}_n\}$ is a Markovization of $\{Z_n\}$. Thus the Markovization is a stochastic version of the final state model [1, p.82], [13, p.71].

4.1 Recursion for $E[Z_N]$

Now let us take a positive integer $N(\geq 2)$. We derive a recursive formula for the expected value of the maximum random variable Z_N. We define a *terminal function* $T : \tilde{S}_N \to R^1$ by

$$T(\tilde{y}_N) := \lambda_N \vee y_N.$$

Lemma 4.5. *It holds that*

$$E[Z_N] = \tilde{E}_{\tilde{y}_0}[T(\tilde{Y}_N)]$$

where $\tilde{E}_{\tilde{y}_0}$ is the expectation operator induced from the transition law q and the initial state $\tilde{y}_0 = y_0 = 0$.

Let us define the sequence of expected value functions $\{u_n\}$ as follows :

$$\begin{aligned} u_n(\tilde{y}_n) &:= \tilde{E}_{\tilde{y}_n}[T(\tilde{Y}_N)] \qquad n = 0, 1, \ldots, N-1 \\ u_N(\tilde{y}_N) &:= \lambda_N \vee y_N \end{aligned}$$

where $\tilde{E}_{\tilde{y}_n}$ is the expectation operator induced from the transition law $\{q_n, q_{n+1}, \cdots, q_N\}$ and an initial state $\tilde{y}_n \in \tilde{S}_n$. We remark that $\tilde{y}_n$ takes the following values :

$$\tilde{y}_0 = y_0 = 0, \quad \tilde{y}_1 = y_1 = -1 \text{ or } 1, \quad \tilde{y}_n = (y_n; \lambda_n) \qquad n = 2, 3, \ldots, N.$$

Then we have the backward recursive formula :

Theorem 4.1.

$$\begin{aligned} u_0(0) &= \sum_{j \in S_1} u_1(j) q_0(j \mid 0) \\ u_1(i) &= \sum_{j \in S_2} u_2(j; i) q_1(j \mid i) \qquad i \in S_1 \\ u_n(i; \lambda) &= \sum_{j \in S_{n+1}} u_{n+1}(j; \lambda \vee i) q_n(j \mid i) \qquad (i; \lambda) \in \tilde{S}_n, \; n = 2, 3, \ldots, N-1 \qquad (10) \\ u_N(i; \lambda) &= \lambda \vee i \qquad (i; \lambda) \in \tilde{S}_N. \end{aligned}$$

Thus, successively solving (10), we have the desired expected value of the maximum profit Z_N over N-stage problem :

$$u_0(0) = E[Z_N].$$

Actually the recursive formula reduces as follows :

Corollary 4.1.

$$\begin{aligned} u_0(0) &= p u_1(1) + q u_1(-1) \\ u_1(i) &= p u_2(i+1; i) + q u_2(i-1; i) \qquad i = -1, \; 1 \\ u_n(i; \lambda) &= p u_{n+1}(i+1; \lambda \vee i) + q u_{n+1}(i-1; \lambda \vee i) \\ & \qquad\qquad (i; \lambda) \in \tilde{S}_n, \; n = 2, 3, \ldots, N-1 \\ u_N(i; \lambda) &= \lambda \vee i \qquad (i; \lambda) \in \tilde{S}_N. \end{aligned}$$

4.2 Recursion for $E[Z_N^2]$

Now let us consider the expected value of the *squared* random variable Z_N^2. Here in place of the terminal function T we introduce the *squared terminal function* $T^2 : \tilde{S}_N \to R^1$ by

$$T^2(\tilde{y}_N) := (\lambda_N \vee y_N)^2.$$

Then we have

Lemma 4.6. *It holds that*

$$E[Z_N^2] = \tilde{E}_{\tilde{y}_0}[T^2(\tilde{Y}_N)].$$

Let us define $\{v_n\}$ as follows :

$$\begin{aligned} v_n(\tilde{y}_n) &:= \tilde{E}_{\tilde{y}_n}[T^2(\tilde{Y}_N)] \qquad n = 0, 1, \ldots, N-1 \\ v_N(\tilde{y}_N) &:= (\lambda_N \vee y_N)^2. \end{aligned}$$

Then we have

Theorem 4.2.

$$\begin{aligned} v_0(0) &= pv_1(1) + qv_1(-1) \\ v_1(i) &= pv_2(i+1; i) + qv_2(i-1; i) \qquad i = -1,\ 1 \\ v_n(i;\lambda) &= pv_{n+1}(i+1; \lambda \vee i) + qv_{n+1}(i-1; \lambda \vee i) \\ &\qquad\qquad (i;\lambda) \in \tilde{S}_n,\ n = 2, 3, \ldots, N-1 \\ v_N(i;\lambda) &= (\lambda \vee i)^2 \qquad (i;\lambda) \in \tilde{S}_N. \end{aligned} \tag{11}$$

Thus, the recursive computation of (11) yields the desired expected value :

$$v_0(0) = E[Z_N^2].$$

4.3 Recursion for $P(Z_N = k)$

Now let us consider the probability distribution of Z_N. For each fixed $k \in \Omega_{n+1}$ the probability it takes value k is calculated through invariant imbedding. We define the terminal function $T : \tilde{S}_N \to R^1$ by

$$T(\tilde{y}_N) := \psi(\lambda_N \vee y_N)$$

where $\psi(\cdot)$ is the indicator function of a one-point set $\{k\}$:

$$\psi(z) := 1_{\{k\}}(z) := \begin{cases} 1 & \text{if } z = k \\ 0 & \text{otherwise} \end{cases}$$

Then

Lemma 4.7. *It holds that*

$$P(Z_N = k) = \tilde{E}_{\tilde{y}_0}[T(\tilde{Y}_N)].$$

Further we define $\{w_n\}$ as follows :

$$w_n(\tilde{y}_n) := \tilde{E}_{\tilde{y}_n}[T(\tilde{Y}_N)] \qquad n = 0, 1, \ldots, N-1$$
$$w_N(\tilde{y}_N) := \psi(\lambda_N \vee y_N).$$

Then

Theorem 4.3.

$$\begin{aligned} w_0(0) &= pw_1(1) + qw_1(-1) \\ w_1(i) &= pw_2(i+1; i) + qw_2(i-1; i) \qquad i = -1,\ 1 \\ w_n(i;\lambda) &= pw_{n+1}(i+1; \lambda \vee i) + qw_{n+1}(i-1; \lambda \vee i) \\ &\qquad\qquad (i;\lambda) \in \tilde{S}_n,\ n = 2, 3, \ldots, N-1 \\ w_N(i;\lambda) &= \psi(\lambda \vee i) \qquad (i;\lambda) \in \tilde{S}_N. \end{aligned} \tag{12}$$

Thus, the recursion (12) yields the desired probability :

$$w_0(0) = P(Z_N = k).$$

References

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