数理解析研究所講究録 1207 巻 2001 年 157-164

On a Fuzzy Extension of Stopping Times

北九州市立大学経済学部 吉田祐治 (Yuji YOSHIDA)
千葉大学理学部 安田正實 (Masami YASUDA)
千葉大学理学部 中神潤一 (Jun-ichi NAKAGAMI)
千葉大学教育学部 蔵野正美 (Masami KURANO)

0. Introduction

We have studied fuzzy stopping stopping problems in dynamic fuzzy systems and fuzzy stochastic systems in our papers ([7, 10, 11]). However, in our papers, the problems are defined by α-cuts and the formulations are not given in direct forms. In this talk, we discuss the definition of fuzzy stopping times and formulation of a new fuzzy stopping model from the viewpoint of Zadeh's extension principle.

1. Fuzzy random variables

Fuzzy random variables were first studied by Puri and Ralescu [8] as a fuzzy-valued extension of real random variables, and they have been studied by many authors. In the rest of this section, we give some mathematical notations regarding fuzzy random variables.

Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers, and $\mathcal{B}$ denotes the Borel σ-field of $\mathbb{R}$. Let $\mathcal{C}(\mathbb{R})$ be the set of all non-empty bounded closed intervals, and let δ be the Hausdorff metric on $\mathcal{C}(\mathbb{R})$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support (Kurano et al. [6]). Refer to Zadeh [12] regarding fuzzy set theory. $\mathcal{R}$ denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with its corresponding membership functions. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

Hence we introduce a partial order $\succeq$, so called the *fuzzy max order*, on fuzzy numbers $\mathcal{R}$([2]): Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. Then

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0,1].$$

A fuzzy-number-valued mapping $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* ([8]) if

$$\{(\omega, x) \in \Omega \times \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1]. \tag{1.1}$$

where $\tilde{X}(\omega)$ is a fuzzy number in $\mathcal{R}$ and its α-cut is written as $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ for $\omega \in \Omega$. Therefore, the condition (1.1) is also written as

$$\{(\omega, x) \in \Omega \times \mathbb{R} \mid x \in \tilde{X}_\alpha(\omega)\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1]. \tag{1.2}$$

The following lemma gives a simple characterization of fuzzy random variables.

Lemma 1.1 (Wang and Zhang [9, Theorems 2.1 and 2.2]). *For a mapping $\tilde{X} : \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:*

(i) *$\tilde{X}$ is a fuzzy random variable.*

(ii) *The mappings $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$.*

From [6, Lemma 3] and Lemma 1.1, we obtain the following lemma regarding the α-cuts $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$.

Lemma 1.2.

(i) *Let $\tilde{X}$ be a fuzzy random variable. Then the α-cuts $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)]$, $\omega \in \Omega$, have the following properties (a) – (c):*

(a) $\bigcap_{\alpha' < \alpha} \tilde{X}_{\alpha'}(\omega) = \tilde{X}_\alpha(\omega), \quad \omega \in \Omega$ *if $\alpha > 0$.*

(b) $\mathrm{cl}\left(\bigcup_{\alpha > 0} \tilde{X}_\alpha(\omega)\right) = \tilde{X}_0(\omega), \quad \omega \in \Omega.$

(c) *The mappings $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for $\alpha \in [0,1]$.*

(ii) *Suppose that a family of interval-valued mappings $X_\alpha = [X_\alpha^-, X_\alpha^+] : \Omega \mapsto \mathcal{C}(\mathbb{R})$ ($\alpha \in [0,1]$) satisfies the following conditions (a) – (c):*

(a) $\bigcap_{\alpha' < \alpha} X_{\alpha'}(\omega) = X_\alpha(\omega), \quad \omega \in \Omega$ *if $\alpha > 0$.*

(b) $\mathrm{cl}\left(\bigcup_{\alpha > 0} X_\alpha(\omega)\right) = X_0(\omega), \quad \omega \in \Omega.$

(c) *The mappings $\omega \mapsto X_\alpha^-(\omega)$ and $\omega \mapsto X_\alpha^+(\omega)$ are measurable for $\alpha \in [0,1]$.*

Then, a membership function

$$\tilde{X}(\omega)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{X_\alpha(\omega)}(x)\}, \quad \omega \in \Omega, \ x \in \mathbb{R}, \tag{1.3}$$

gives a fuzzy random variable $\tilde{X}$ and $\tilde{X}_\alpha(\omega) = X_\alpha(\omega)$ for $\omega \in \Omega$ and $\alpha \in [0,1]$.

Next we need to introduce *expectations of fuzzy random variables* in order to describe an optimal stopping model in the next section. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$.

Let $\tilde{X}$ be an integrably bounded fuzzy random variable. By classical expectation, we give closed intervals

$$E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)\right], \quad \alpha \in [0,1]. \tag{1.4}$$

Since the mapping $\alpha \mapsto E(\tilde{X})_\alpha$ is left-continuous by the monotone convergence theorem, from [6, Lemma 3] the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number

$$E(\tilde{X})(x) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}. \tag{1.5}$$

2. A fuzzy stopping model

In this section, we discuss a new fuzzy stopping model for a process of fuzzy random variables. Let a fuzzy stochastic process $\{\tilde{X}_t\}_{t\geq 0}$ be a process of integrably bounded fuzzy random variables such that the mapping $t \mapsto \tilde{X}_t$ is continuous on $[0,\infty)$ almost surely and $E(\sup_{t\geq 0}\tilde{X}_{t,0}^+) < \infty$, where $\tilde{X}_{t,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$. $\{\mathcal{M}_t\}_{t\geq 0}$ denotes a family of the smallest σ-fields on Ω generated by all random variables $\tilde{X}_{s,\alpha}^-$ and $\tilde{X}_{s,\alpha}^+$ ($s \leq t; \alpha \in [0,1]$) such that $t \mapsto \mathcal{M}_t$ is right continuous, and $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{t\geq 0}\mathcal{M}_t$. A mapping $\tau : \Omega \mapsto [0,\infty]$ is said to be a *stopping time* if

$$\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{M}_t \quad \text{for all } t \geq 0. \tag{2.1}$$

Now we introduce *new fuzzy stopping times* by mappings $\tilde{\tau} : [0,\infty)\times\Omega \mapsto [0,1]$ such that $\tilde{\tau}(\cdot,\omega)$ are fuzzy numbers for each $\omega \in \Omega$. The α-cut of the fuzzy number $\tilde{\tau}(\cdot,\omega)$ is given by

$$\tilde{\tau}_\alpha(\omega) := \{t \geq 0 \mid \tilde{\tau}(t,\omega) \geq \alpha\}\ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{\tau}_0(\omega) := \mathrm{cl}\{t \geq 0 \mid \tilde{\tau}(t,\omega) > 0\}, \tag{2.2}$$

where cl denotes the closure of an interval. Then we write the α-cut (2.2) as a bounded closed interval

$$\tilde{\tau}_\alpha(\omega) := [\tilde{\tau}_\alpha^-(\omega), \tilde{\tau}_\alpha^+(\omega)] \quad \text{for } \alpha \in [0,1]. \tag{2.3}$$

Definition 1. (A new fuzzy stopping time). A mapping $\tilde{\tau} : [0,\infty)\times\Omega \mapsto [0,1]$ is called a *fuzzy stopping time* if it satisfies the following (i) and (ii):

(i) $\tilde{\tau}(\cdot,\omega)$ is a fuzzy number for each $\omega \in \Omega$.

(ii) $\tilde{\tau}_\alpha^-$ and $\tilde{\tau}_\alpha^+$ are stopping times for each $\alpha \in [0,1]$, where $\tilde{\tau}_\alpha^\pm$ is defined by (2.3).

Remark. We cannot replace Definition 1(ii) with the following condition (ii'):

(ii') The mapping $\omega \mapsto \tilde{\tau}(t,\omega)$ is $\mathcal{M}_t$-measurable for $t \geq 0$

since it holds that (ii) $\Longrightarrow$ (ii') but the reverse does not hold from

$$\{\omega \mid \tilde{\tau}(t,\omega) \geq \alpha\} = \{\omega \mid \tilde{\tau}_\alpha^-(\omega) \leq t\} \cap \{\omega \mid t \leq \tilde{\tau}_\alpha^+(\omega)\}.$$

Then the following lemma is trivial.

Lemma 2.1.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a mapping $\tilde{\tau}_\alpha : \Omega \mapsto [0,\infty)$ by (2.2) and (2.3). Then, we have:*

- (a) *$\tilde{\tau}_\alpha^-$ and $\tilde{\tau}_\alpha^+$ are stopping times for each $\alpha \in [0,1]$;*
- (b) *$\bigcap_{\alpha'<\alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega), \quad \omega \in \Omega \quad$ if $\alpha > 0$;*
- (c) *$\mathrm{cl}\left(\bigcup_{\alpha>0} \tilde{\tau}_\alpha(\omega)\right) = \tilde{\tau}_0(\omega), \quad \omega \in \Omega$.*

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha\in[0,1]}$ be interval-valued mappings $\tilde{\tau}_\alpha : \Omega \mapsto \mathcal{C}(\mathbb{R})$ satisfying the above (a) – (c). Define a mapping $\tilde{\tau} : [0,\infty) \times \Omega \mapsto [0,1]$ by*

$$\tilde{\tau}(t,\omega) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{\{\omega \mid t\in\tilde{\tau}_\alpha(\omega)\}}(\omega)\} \quad \textit{for } t \geq 0 \textit{ and } \omega \in \Omega. \tag{2.4}$$

Then $\tilde{\tau}$ is a fuzzy stopping time.

We can define the stochastic process stopped at a fuzzy stopping time $\tilde{\tau}$ by

$$\tilde{X}_{\tilde{\tau}}(\omega)(x) = \sup_{t\geq 0} \min\{\tilde{X}_t(\omega)(x), \tilde{\tau}(t,\omega)\}. \tag{2.5}$$

The reason is as follows. Fix $\omega \in \Omega$ and $x \in \mathbb{R}$. By considering a mapping

$$\tilde{X}_\cdot(\omega)(x) : [0,\infty) \mapsto [0,1] \tag{2.6}$$

is a fuzzy set on the time space $[0,\infty)$, the fuzzy random variable (2.5) gives a *type 2 extension* [1] of (2.6), and therefore (2.5) means the stochastic process stopped at a fuzzy stopping time $\tilde{\tau}$.

Since the mapping $t \mapsto \tilde{X}_t$ is continuous on $[0,\infty)$ almost surely, by taking the α-cut in (2.5) we obtain a bounded closed interval

$$\tilde{X}_{\tilde{\tau},\alpha}(\omega) = \bigcup_{t\in\tilde{\tau}_\alpha(\omega)} \tilde{X}_{t,\alpha}(\omega) = \left[\min_{t\in\tilde{\tau}_\alpha(\omega)} \tilde{X}_{t,\alpha}^-(\omega), \max_{t\in\tilde{\tau}_\alpha(\omega)} \tilde{X}_{t,\alpha}^+(\omega)\right], \tag{2.7}$$

[1] *Fuzzy sets of type m* are defined inductively as follows.

$$(\textit{fuzzy sets of type } m) \text{ are mappings} : (\textit{fuzzy sets of type } m-1) \mapsto [0,1].$$

This extension is also called *Zadeh's support fuzzification* ([2]).

where $\tilde{X}_{\tilde{\tau},\alpha}(\omega)$ is the α-cut of $\tilde{X}_{\tilde{\tau}}(\omega)(\cdot)$ and $\tilde{X}_{t,\alpha}(\omega) = [\tilde{X}^-_{t,\alpha}(\omega), \tilde{X}^+_{t,\alpha}(\omega)]$ is the α-cut of $\tilde{X}_t(\omega)(\cdot)$.

Lemma 2.2. Let $\tilde{\tau}$ be a fuzzy stopping time. Then $\tilde{X}_{\tilde{\tau}}$ is a fuzzy random variable.

Proof. $\tilde{X}_{\tilde{\tau},\alpha}(\omega)$ is a bounded closed interval and the mappings $\omega \mapsto \min_{t\in\tilde{\tau}_\alpha(\omega)} \tilde{X}^-_{t,\alpha}(\omega)$ and $\omega \mapsto \max_{t\in\tilde{\tau}_\alpha(\omega)} \tilde{X}^+_{t,\alpha}(\omega)$ are measurable since $t \mapsto \tilde{X}_t$ is continuous on $[0,\infty)$ almost surely. □

Now we introduce an *estimation of the fuzzy random variables* $\tilde{X}_{\tilde{\tau}}$. Fix a parameter $\lambda \in [0,1]$. Put a mapping $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$ by

$$g([a,b]) := \lambda a + (1-\lambda)b, \qquad [a,b] \in \mathcal{C}(\mathbb{R}), \tag{2.8}$$

and give the *ranking method of a fuzzy number* $\tilde{a}$ by (see Campos and Munoz [1])

$$\int_0^1 g(\tilde{a}_\alpha)\,\mathrm{d}\alpha.$$

We note that

$$\int_0^1 g(E(\tilde{X}_{\tilde{\tau}})_\alpha)\,\mathrm{d}\alpha = \int_0^1 E(g(\tilde{X}_{\tilde{\tau},\alpha}))\,\mathrm{d}\alpha \tag{2.9}$$

for fuzzy stopping times $\tilde{\tau}$, and in this paper we discuss the following problem.

Problem 1. Find a fuzzy stopping time $\tilde{\tau}^*$ such that

$$\int_0^1 E(g(\tilde{X}_{\tilde{\tau}^*,\alpha}))\,\mathrm{d}\alpha \geq \int_0^1 E(g(\tilde{X}_{\tilde{\tau},\alpha}))\,\mathrm{d}\alpha \tag{2.10}$$

for all fuzzy stopping times $\tilde{\tau}$, where λ is called a *pessimistic-optimistic index* for decision makers.

Remark. If $E(\tilde{X}_{\tilde{\tau}^*}) \succeq E(\tilde{X}_{\tilde{\tau}})$, then (2.10) holds, where $\succeq$ is the fuzzy max order on $\mathcal{R}$.

3. An optimal fuzzy stopping time

In this section, we consider about the construction of an optimal fuzzy stopping time for Problem 1. For a fuzzy stopping time $\tilde{\tau}$, from (2.7) and (2.8) we have

$$g(\tilde{X}_{\tilde{\tau},\alpha}(\omega)) = \lambda \min_{t\in[\tilde{\tau}^-_\alpha(\omega),\tilde{\tau}^+_\alpha(\omega)]} \tilde{X}^-_{t,\alpha}(\omega) + (1-\lambda) \max_{t\in[\tilde{\tau}^-_\alpha(\omega),\tilde{\tau}^+_\alpha(\omega)]} \tilde{X}^+_{t,\alpha}(\omega), \quad \omega\in\Omega,\ \alpha\in[0,1]. \tag{3.1}$$

Fix $\alpha \in [0,1]$ for simplicity, and define

$$Y_{\tau^-(\omega),\tau^+(\omega)}(\omega) := \lambda \min_{t\in[\tau^-(\omega),\tau^+(\omega)]} \tilde{X}^-_{t,\alpha}(\omega) + (1-\lambda) \max_{t\in[\tau^-(\omega),\tau^+(\omega)]} \tilde{X}^+_{t,\alpha}(\omega), \quad \omega\in\Omega \tag{3.2}$$

for pairs of finite stopping times (τ^-,τ^+) satisfying $0 \leq \tau^- \leq \tau^+$. Then we consider the following optimal stopping problem.

Problem 2. Find a pair of finite stopping times (τ^-, τ^+) satisfying $0 \leq \tau^- \leq \tau^+$ which maximizes the expected value $E(Y_{\tau^-,\tau^+})$.

First we deal with an optimal stopping with respect to τ^+, and next we discuss an optimal stopping with respect to τ^-.

Problem 3. Fix a constant time s_1. Find a finite stopping time τ_1 which maximizes the expected value $E(Y_{s_1,\tau_1})$.

Define Snell's envelope for Problem 3 by

$$U_{s_1,s_2} := \operatorname*{ess\ sup}_{\tau^+ \geq s_2} E(Y_{s_1,\tau^+(\cdot)}(\cdot)|\mathcal{M}_{s_2}) \tag{3.3}$$

for pairs of constant times (s_1, s_2) satisfying $s_2 \geq s_1 \geq 0$. Define

$$Q(s_1,\omega) := \inf\{s_2 \mid s_2 \geq s_1, U_{s_1,s_2}(\omega) = Y_{s_1,s_2}(\omega)\}, \quad \omega \in \Omega, \tag{3.4}$$

where the infimum of the empty set is understood to be $+\infty$. If $Q(s_1, \cdot)$ is finite almost surely, then it is an optimal stopping time for Problem 3.

Next we consider Problem 2. Define Snell's envelope for Problem 2 by

$$L_{s_1} := \operatorname*{ess\ sup}_{\tau^- \geq s_1} E(Y_{\tau^-,Q(\tau^-(\cdot),\cdot)}(\cdot)|\mathcal{M}_{s_1}) \tag{3.5}$$

for constant times s_1. Put a pair of stopping times (σ^-, σ^+) satisfying $0 \leq \sigma^- \leq \sigma^+$ by

$$\sigma^-(\omega) := \inf\{s_1 \mid L_{s_1}(\omega) = E(Y_{s_1,Q(s_1,\cdot)}(\cdot)|\mathcal{M}_{s_1})(\omega)\}, \quad \omega \in \Omega, \tag{3.6}$$

and

$$\sigma^+(\omega) := Q(\sigma^-(\omega), \omega), \quad \omega \in \Omega, \tag{3.7}$$

where the infimum of the empty set is understood to be $+\infty$.

If σ^+ is finite almost surely, then we obtain

$$E(Y_{\sigma^-,\sigma^+}) = E(Y_{\sigma^-,Q(\sigma^-(\cdot),\cdot)}(\cdot)) \geq E(Y_{\tau^-,Q(\tau^-(\cdot),\cdot)}(\cdot)) \geq E(Y_{\tau^-,\tau^+}) \tag{3.8}$$

for all pairs of finite stopping times (τ^-, τ^+) satisfying $0 \leq \tau^- \leq \tau^+$. Therefore (σ^-, σ^+) is optimal for Problem 2.

Theorem 3.1. *Let $\alpha \in [0,1]$. Put a pair of stopping times (σ^-, σ^+) by (3.6) and (3.7). If σ^+ is finite almost surely, then (σ^-, σ^+) is an optimal pair of stopping times for Problem 2.*

Put $\sigma^\pm = \sigma^\pm_\alpha$ for $\alpha \in [0,1]$. In order to construct an optimal fuzzy stopping time from the α-optimal stopping times $\{\sigma^\pm_\alpha\}_{\alpha \in [0,1]}$, we need the following regularity condition.

Assumption A (Regularity). The mapping $\alpha \mapsto \sigma_\alpha^-(\omega)$ is non-decreasing and left continuous and the mapping $\alpha \mapsto \sigma_\alpha^+(\omega)$ is non-increasing and left continuous for almost all $\omega \in \Omega$.

Under Assumption A, we can define a mapping $\tilde{\sigma} : [0,\infty) \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}(t,\omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\omega | \sigma_\alpha^-(\omega) \leq t \leq \sigma_\alpha^+(\omega)\}}(\omega)\} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{3.9}$$

We also write its α-cut of by $\tilde{\sigma}_\alpha(\omega)$ (see (2.2)), then we have $\tilde{\sigma}_\alpha(\omega) = [\sigma_\alpha^-(\omega), \sigma_\alpha^+(\omega)]$.

Theorem 3.2 (Optimal fuzzy stopping time). *Suppose Assumption A holds. If* $P(\sigma_0^+ < \infty) = 1$, *then* $\tilde{\sigma}$ *is an optimal fuzzy stopping time for Problem 1.*

Proof. From Assumption A and Lemma 2.1, $\tilde{\sigma}$ is a fuzzy stopping time. By (3.1), (3.8) and Theorem 3.1, we have

$$E(g(\tilde{X}_{\tilde{\sigma},\alpha})) = E(Y_{\sigma_\alpha^-,\sigma_\alpha^+}) \geq E(Y_{\tilde{\tau}_\alpha^-,\tilde{\tau}_\alpha^+}) = E(g(\tilde{X}_{\tilde{\tau},\alpha}))$$

for all fuzzy stopping times $\tilde{\tau}$ and all $\alpha \in [0,1]$. Therefore, $\tilde{\sigma}$ is optimal for Problem 1. $\square$

The following result implies a comparison between the optimal values of the 'classical' stopping model and the 'fuzzy' stopping model (Problem 1). Then we find that the fuzzy stopping model is more better than the classical one.

Corollary 3.1. *It holds that, under the same assumptions as Theorem 3.2,*

$$\int_0^1 E(g(\tilde{X}_{\tilde{\sigma},\alpha}))\, d\alpha \geq \int_0^1 E(g(\tilde{X}_{\tau^*,\alpha}))\, d\alpha, \tag{3.10}$$

where $\tilde{\sigma}$ *is the optimal fuzzy stopping time and* τ^* *is an optimal stopping time in the class of classical stopping times.*

Proof. Trivial. $\square$

References

[1] L.Campos and A.Munoz, A subjective approach for ranking fuzzy numbers, *Fuzzy Sets and Systems* **29** (1989) 145-153.

[2] D.Dubois and H.Prade, *Fuzzy Sets and Systems, Theory and Applications* (Academic Press, New York, 1980).

[3] A.G.Fakeev, Optimal stopping rules for processes with continuous parameter, *Theory Probab. Appli.* **15** (1970) 324-331.

[4] J.Kacprzyk, Decision making in a fuzzy environment with fuzzy termination time, *Fuzzy Sets and Systems* **1** (1978) 169-179.

[5] J.Kacprzyk, *Multistage Fuzzy Control* (Wiley, Chichester, 1997).

[6] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[7] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, An approach to stopping problems of a dynamic fuzzy system, preprint.

[8] M.L.Puri and D.A.Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114** (1986) 409-422.

[9] G.Wang and Y.Zhang, The theory of fuzzy stochastic processes, *Fuzzy Sets and Systems* **51** (1992) 161-178.

[10] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A monotone fuzzy stopping time for dynamic fuzzy systems, *Bull. Infor. Cyber. Res. Ass. Stat. Sci., Kyushu University* **31** (1999) 91-99.

[11] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, Optimal stopping problems in a stochastic and fuzzy system, *J. Math. Analy. and Appl.* **246** (2000) 135-149.

[12] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.