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EIGENVALUE PROBLEMS ON DOMAINS WITH CRACKS

Kazushi Yoshitomi (吉冨和志)
Graduate School of Mathematics, Kyushu University,
Hakozaki, Fukuoka 812-8581, Japan
(yositomi@math.kyushu-u.ac.jp)

0. *Introduction*

Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and let

$$\gamma:\ [0,t_0]\ni t\mapsto\gamma(t)\in\mathbb{R}^2\quad(t_0>0)$$

be a smooth curve without self-intersection. We impose the following hypotheses on Ω and γ.

(H.1) $0\in\partial\Omega,\quad\gamma(0)=0,\quad\gamma(t_0)\in\partial\Omega,\quad$ and $\quad\gamma((0,t_0))\subset\Omega$.

(H.2) There exists $r_0\in(0,t_0)$ such that

$$\gamma(t)=(t,0)\quad\text{on}\quad[0,r_0],$$
$$\Omega\cap\{x\in\mathbb{R}^2;\quad|x|<r_0\}=\{x=(x_1,x_2)\in\mathbb{R}^2;\quad x_1>0,\quad|x|<r_0\}.$$

(H.3) The curve γ intersects $\partial\Omega$ transversely at $\gamma(t_0)$.

The set $\Omega\backslash\gamma((0,t_0))$ consists of two connected components. Let $\Omega_\pm$ be the connected component of $\Omega\backslash\gamma((0,t_0))$ which contains $(r_0/2,\pm r_0/2)$.

For $\epsilon\in(0,t_0)$, we set

$$\Omega_\epsilon=\Omega\backslash\gamma([\epsilon,t_0)),$$
$$Q_\epsilon=\{u\in H^1(\Omega_\epsilon);\quad u=0\quad\text{on}\quad\partial\Omega\},$$
$$q_\epsilon(u,v)=(\nabla u,\nabla v)_{L^2(\Omega)},\quad u,v\in Q_\epsilon.$$

Let L_ϵ be the self-adjoint operator associated with the form q_ϵ. The operator L_ϵ is the negative Laplacian on Ω_ϵ with Dirichlet's boundary condition on $\partial\Omega$ and Neumann's boundary condition on $\gamma((\epsilon,t_0))$. We denote by $\lambda_j(\epsilon)$ the j-th eigenvalue of L_ϵ counted with multiplicity. The function $\lambda_j(\cdot)$ is monotone non-decreasing. We study the asymptotic behavior of $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ as ϵ tends to 0. We set

$$Q_0^\pm=\{u\in H^1(\Omega_\pm);\quad u=0\quad\text{on}\quad\partial\Omega\cap\partial\Omega_\pm\},$$
$$q_0^\pm(u,v)=(\nabla u,\nabla v)_{L^2(\Omega_\pm)},\quad u,v\in Q_0^\pm.$$

Let $L_0^\pm$ be the self-adjoint operator associated with the form $q_0^\pm$. Let $\lambda_1^\pm<\lambda_2^\pm\leqslant\cdots$ be the eigenvalues of $L_0^\pm$ repeated according to multiplicity. We further assume that

(H.4) $\quad\lambda_1^+=\lambda_1^-$.

Let $\varphi_1^\pm$ be the eigenvector of $L_0^\pm$ associated with the eigenvalue $\lambda_1^\pm$ which is normalized by the conditions $\varphi_1^\pm>0$ on $\Omega_\pm$ and $\|\varphi_1^\pm\|_{L^2(\Omega_\pm)}=1$. We claim that $\varphi_1^\pm$ is real analytic in a neighborhood of 0 and $\varphi_1^\pm$ is given by a convergent power series expansion:

$$\varphi_1^\pm(x)=\sum_{j=1}^{\infty}\sum_{k=1}^{j}C_{j,k}^\pm r^{2j-1}\cos(2k-1)\theta\quad\text{in a neighborhood of}\quad 0\quad\text{with}\quad C_{1,1}^\pm>0,\tag{1}$$

where (r,θ) is the polar coordinate of x centered at 0. Let

$$K=\left\{j\geqslant 2;\quad\frac{C_{j,j}^+}{C_{1,1}^+}\neq\frac{C_{j,j}^-}{C_{1,1}^-}\right\}.$$

We define $\nu=\min K$ if and only if $K\neq\emptyset$. We set $\lambda_0=\lambda_1^+(=\lambda_1^-)$. Our main result is the following.

THEOREM 0.1. *The function $\lambda_2(\epsilon)$ admits the asymptotic expansion:*

$$\lambda_2(\epsilon) \sim \lambda_0 + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \lambda_{m,n}\, \epsilon^{2m} (\log \epsilon)^n \quad as \quad \epsilon \to 0, \quad \lambda_{1,0} > 0. \tag{2}$$

If $K \neq \emptyset$ then the function $\lambda_1(\epsilon)$ admits the asymptotic expansion:

$$\lambda_1(\epsilon) \sim \lambda_0 + \sum_{i=2\nu-1}^{\infty} \sum_{j=0}^{[(i-2\nu+1)/3]} \mu_{i,j}\, \epsilon^{2i} (\log \epsilon)^j \quad as \quad \epsilon \to 0, \quad \mu_{2\nu-1,0} > 0. \tag{3}$$

If $K = \emptyset$ then $\lambda_1(\epsilon) = \lambda_0$ for sufficiently small $\epsilon > 0$.

This work is inspired and motivated by that of Dauge and Helffer [**2**]. In a more general setting, they proved that $\lim_{\epsilon\to 0} \lambda_j(\epsilon) = \kappa_j$ for all $j \in \mathbb{N}$, where $\kappa_1 \leqslant \kappa_2 \leqslant \cdots$ are the rearrangement of $\{\lambda_j^+\}_{j=1}^{\infty} \cup \{\lambda_j^-\}_{j=1}^{\infty}$ repeated according to multiplicity. This result interests us in the asymptotic behavior of $\lambda_j(\epsilon)$ as ϵ tends to 0. They also suggested the following problem called *geometric tunneling*. The problem they suggested is to estimate $\lambda_2(\epsilon) - \lambda_1(\epsilon)$ in the case that Ω is symmetric with respect to the x_1-axis and γ is a line segment on the x_1-axis. Our Theorem 0.1 also solves this problem.

There are many works on asymptotics of eigenvalues of elliptic differential equations on singularly perturbed domains. R. Gadyl'shin and A. M. Il'n [**6**] considered a domain with a narrow slit. S. Ozawa [**9**] and V. Maz'ya, S. Nazarov, and B. Plamenevskii [**8**] considered a domain with a small hole. S. Jimbo [**7**], R. Gadyl'shin [**5**], and J. Arrieta, J. Hale, and Q. Han [**1**] considered a dumbbel domain with a shrinking handle. R. Gadyl'shin [**3, 4**] considered a problem to change the boundary condition on a small part of the boundary. In [**3, 4, 5, 6**], R. Gadyl'shin and A. M. Il'n used the method of *matched expansion*: they decomposed the region into overlapping subregions and constructed asymptotic solutions on the respective subregions such that the solutions are asymptotically same on the intersection of the subregions. In [**8**], V. Maz'ya, S. Nazarov, and B. Plamenevskii used the method of compound expansion which is somewhat similar to the method of matched expansion.

We use the method of matched expansion to construct an approximate solution in the proof of Theorem 0.1. Our procedure to construct the approximate solution is somewhat similar to the procedure used in [**6**]. But the form of the approximate solution in this paper differs from that in [**3, 4, 5, 6, 8**]. In the present paper, we prove only (2) for simplicity.

1. *Eigenvector of the limit problem*

We begin with introducing some notations and conventions which we use throughout this paper. For $x \in \mathbb{R}^2$ and $r > 0$, we denote by $D(x, r)$ the open disk of radius r centered at x. For $r > 0$, we define

$$D(r) = D(0,r), \quad D_{\pm}(r) = \{(x_1,x_2) \in D(r); \quad \pm x_1 > 0\}, \quad D_{+\pm}(r) = \{(x_1,x_2) \in D_+(r); \quad \pm x_2 > 0\}.$$

Let $f, g \in \cap_{\delta\in(0,r_0)} L^2(\Omega_\pm \backslash D_{+\pm}(\delta))$. If the principal value $\lim_{\delta\to+0}(f,g)_{L^2(\Omega_\pm\backslash D_{+\pm}(\delta))}$ exists, then we denote it by $(f,g)_{\Omega_\pm}$. To avoid cumbersome classification, we use the following conventions about summations and sequences: if $p > q$, we define $\sum_{i=p}^{q} a_i = 0$ and $\{a_i\}_{i=p}^{q} = \emptyset$. We regard an undefined term as 0 in formulae. For example, if b_1 and b_3 are defined and b_2 is not defined, then the formula $b_1 + b_2 = b_3$ means $b_1 = b_3$.

In order to construct the asymptotic expansion of $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$, we first analyze the asymptotic behavior of the eigenvector $\varphi_1^{\pm}$ near 0. Let us show the following.

PROPOSITION 1.1. *The function $\varphi_1^{\pm}$ is real analytic in a neighborhood of* 0 *and $\varphi_1^{\pm}$ is given by a convergent power-series expansion:*

$$\varphi_1^{\pm}(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} C_{j,k}^{\pm}\, r^{2j-1} \cos(2k-1)\theta \quad in\ a\ neighborhood\ of \quad 0 \quad with \quad C_{1,1}^{\pm} > 0.$$

Proof. We prove the assertion only for φ_1^+ because that for φ_1^- is similar. In the proof, we use the method of reflection. We set

$$\psi(x_1,x_2)=\begin{cases}\varphi_1^+(x_1,x_2) & \text{for } \ x\in D_{++}(r_0),\\ \varphi_1^+(x_1,-x_2) & \text{for } \ x\in D_{+-}(r_0).\end{cases}$$

The function $\psi(x)$ is even with respect to x_2. Because $\varphi_1^+\in Q_0^+$, we have

$$\psi\in H^1(D_+(r_0)),\qquad \psi=0 \quad \text{on} \quad \{0\}\times(-r_0,r_0). \tag{4}$$

Moreover we get

$$-\Delta\psi=\lambda_0\psi \quad \text{in the sense of distribution on} \quad D_+(r_0). \tag{5}$$

Using (4), (5), and the regularity estimate for elliptic differential equations, we obtain $\psi\in H^2(D_+(r_0/2))$. We set

$$\tilde\psi(x_1,x_2)=\begin{cases}\psi(x_1,x_2) & \text{for } \ x\in D_+(r_0/2),\\ -\psi(-x_1,x_2) & \text{for } \ x\in D_-(r_0/2).\end{cases}$$

The function $\tilde\psi(x)$ is even with respect to x_2 and odd with respect to x_1. From (4), we have $\tilde\psi\in H^1(D(r_0/2))$. Besides we get

$$-\Delta\tilde\psi=\lambda_0\tilde\psi \quad \text{in the sense of distribution on} \quad D(r_0/2). \tag{6}$$

Thus the regularity estimate for elliptic differential equations implies that $\tilde\psi\in C^\infty(D(r_0/2))$. Moreover, the analytic hypoellipticity for elliptic differential equations with analytic coefficients implies that $\tilde\psi$ is real analytic in a neighborhood of 0. Combining this with the fact that $\tilde\psi(x_1,x_2)=-\tilde\psi(-x_1,x_2)=\tilde\psi(x_1,-x_2)$ on $D(r_0/2)$, we obtain $\tilde\psi(x)=\sum_{j=1}^\infty\sum_{k=0}^\infty d_{j,k}\,x_1^{2j-1}x_2^{2k}$ in a neighborhood of 0. Rewriting this in the polar coordinate, we infer that there exists $\tilde r_0\in(0,r_0/2)$ such that

$$\tilde\psi(x)=\sum_{j=1}^\infty\sum_{k=1}^j C_{j,k}^+\,r^{2j-1}\cos(2k-1)\theta \quad \text{on} \quad D(\tilde r_0). \tag{7}$$

Let us show that $C_{1,1}^+>0$ by a contradiction. Note that $\tilde\psi=\varphi_1^+$ in an Ω_+-neighborhood of 0. Because $\varphi_1^+>0$ in Ω_+ and $\cos\theta>0$ on $(0,\pi/2)$, we have $C_{1,1}^+\geqslant 0$. We assume that $C_{1,1}^+=0$. Using (6), (7), and the analiticity of $\tilde\psi$ near 0, we obtain

$$C_{j+1,k}^+=-\frac{\lambda_0}{4(j+k)(j+1-k)}C_{j,k}^+ \tag{8}$$

for $j\geqslant 1$ and $1\leqslant k\leqslant j$. Let us show that for all $j\in\mathbb{N}$,

$$C_{j,k}^+=0 \quad \text{for} \quad 1\leqslant k\leqslant j \tag{9}$$

by using induction on j. (9) is valid for $j=1$ by the assumption. Let $m\in\mathbb{N}$. Suppose that (9) is valid for $j\leqslant m$. Then (8) implies that $C_{m+1,k}^+=0 \quad \text{for} \quad 1\leqslant k\leqslant m$. So, (7) implies that

$$\tilde\psi(x)=C_{m+1,m+1}^+r^{2m+1}\cos(2m+1)\theta+O(r^{2m+3})$$

as $r\to 0$. Because $\varphi_1^+>0$ in Ω_+ and $\cos(2m+1)\theta$ takes both positive and negative sign on $(0,\pi/2)$, we get $C_{m+1,m+1}^+=0$. Hence (9) is valid for $j\leqslant m+1$. Thus (9) holds for all $j\in\mathbb{N}$. This together with the analiticity of $\tilde\psi$ at 0 implies that $\tilde\psi=0$ in a neighborhood of 0. But this violates the fact that $\varphi_1^+>0$ in Ω_+. Therefore we get $C_{1,1}^+>0$. This completes the proof of Proposition 1.1. $\square$

2. *Preliminaries for the construction of the asymptotic expansion of the second eigenvalue*

In order to analyze the asmptotic behaviour of the eigenvalues, we use the method of matched expansion (see [**6**]). We define

$$\xi = \epsilon^{-1} x.$$

The variable ξ is called the *inner variable*. We denote by (ρ, θ) the polar coordinate of ξ centered at 0. We seek the asymptotic expansions of the second eigenvalue of L_ϵ and the associated eigenvector in the following form.

$$\lambda_\epsilon := \lambda_0 + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \epsilon^{2m} (\log \epsilon)^n \lambda_{m,n}, \tag{10}$$

$$\varphi_\epsilon^{\mathrm{out},\pm}(x) := \varphi_{0,0}^{\pm}(x) + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \epsilon^{2i} (\log \epsilon)^j \varphi_{i,j}^{\pm}(x) \quad \text{on} \quad \Omega_\pm \backslash D_{+\pm}(\epsilon^{1/2}), \tag{11$_\pm$}$$

$$\varphi_\epsilon^{\mathrm{in}}(x) := \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \epsilon^{2k-1} (\log \epsilon)^l v_{k,l}(\xi) \quad \text{on} \quad D_+(2\epsilon^{1/2}) \cap \Omega_\epsilon. \tag{12}$$

We shall construct $\varphi_\epsilon^{\mathrm{in}}$ and $\varphi_\epsilon^{\mathrm{out},\pm}$ such that $\varphi_\epsilon^{\mathrm{in}}$ asymptotically coincides with $\varphi_\epsilon^{\mathrm{out},\pm}$ in the intermediate region $D_+(2\epsilon^{1/2}) \backslash D_{+\pm}(\epsilon^{1/2})$. Inserting (10) and (11)$_\pm$ into the equation $-\Delta \varphi_\epsilon^{\mathrm{out},\pm} = \lambda_\epsilon \varphi_\epsilon^{\mathrm{out},\pm}$ and identifying the powers of ϵ and $\log \epsilon$, we get the *outer equation*:

$$\left.\begin{aligned} &(\Delta + \lambda_0) \varphi_{p,q}^{\pm} = -\sum_{m=1}^{p} \sum_{j=\max\{0,q-m+1\}}^{\min\{p-m,q\}} \lambda_{m,q-j}\, \varphi_{p-m,j}^{\pm} \quad \text{on} \quad \Omega_\pm, \\ &\varphi_{p,q}^{\pm} = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_\pm, \qquad \frac{\partial}{\partial n} \varphi_{p,q}^{\pm} = 0 \quad \text{on} \quad \Gamma := \gamma((0, t_0)). \end{aligned}\right\} \tag{OUT$^\pm_{p,q}$}$$

We put

$$\Pi = \{(\xi_1, \xi_2) \in \mathbb{R}^2; \quad \xi_1 > 0\} \backslash ([1, \infty) \times \{0\}).$$

Inserting (10) and (12) into the equation $-\Delta \varphi_\epsilon^{\mathrm{in}} = \lambda_\epsilon \varphi_\epsilon^{\mathrm{in}}$ and identifying the powers of ϵ and $\log \epsilon$, we get the *inner equation*:

$$\left.\begin{aligned} &\Delta v_{p+1,q} = -\lambda_0 v_{p,q} - \sum_{m=1}^{p-1} \sum_{l=\max\{0,q-m+1\}}^{\min\{q,p-m-1\}} \lambda_{m,q-l}\, v_{p-m,l} \quad \text{on} \quad \Pi, \\ &v_{p+1,q} = 0 \quad \text{on} \quad \{0\} \times \mathbb{R}, \qquad \frac{\partial}{\partial \xi_2} v_{p+1,q}(\cdot, \pm 0) = 0 \quad \text{on} \quad (1, \infty). \end{aligned}\right\} \tag{IN$_{p,q}$}$$

We introduce some notations and function spaces. We put

$$P = (0, \infty) \times (0, \infty), \quad P_1^\delta = P \backslash (D(0, \delta) \cup D((1,0), \delta)), \quad \delta \in (0, \frac{1}{2}).$$

Let $(\tilde{r}, s)$ be the polar coordinate centered at $(1, 0)$. Let $\mathcal{A}$ denote the class of function $f \in C^\infty(P)$ satisfying the following (i) and (ii).

(**i**) $f \in C^1(\overline{D_{++}(\frac{1}{2})}) \cap \bigcap_{\delta \in (0,1/2)} C^\infty(\overline{P_1^\delta})$.

(**ii**) There exists a constant $C > 0$ such that

$$\tilde{r}^{-1/2} |f(\tilde{r}, s)| + \tilde{r}^{-1/2} \left| \frac{\partial}{\partial s} f(\tilde{r}, s) \right| + \tilde{r}^{1/2} \left| \frac{\partial}{\partial \tilde{r}} f(\tilde{r}, s) \right| \leqslant C \quad \text{on} \quad D((1,0), \frac{1}{2}) \cap P.$$

We identify $\mathbb{R}^2$ with $\mathbb{C}$ by the map $\mathbb{R}^2 \ni (y_1, y_2) \mapsto y_1 + \sqrt{-1}y_2 \in \mathbb{C}$. For $\eta \in \mathbb{C}$, we put

$$\xi(\eta) = \sqrt{\eta^2 + 1},$$

where the branch cut of the square root is the positive real axis. Let $\mathcal{B}$ denote the class of function $g \in C^\infty(P)$ such that there exists a function $h \in C^\infty(\mathbb{R}^2 \backslash \{\sqrt{-1}, \ -\sqrt{-1}\})$ satisfying the following (i), (ii), and (iii).

(i) $h(\eta) = \frac{|\eta|^2}{|\eta^2+1|} g(\xi(\eta))$ on P.

(ii) $h(\eta_1, \eta_2) = h(\eta_1, -\eta_2) = -h(-\eta_1, \eta_2)$ on $\mathbb{R}^2 \backslash \{\sqrt{-1}, \ -\sqrt{-1}\}$.

(iii) The function h is bounded near $\{\sqrt{-1}\}$ and $\{-\sqrt{-1}\}$.

For $i \geqslant 0$ and $r > 0$, we define

$$J_i(r) = \{f \in C^i(D(r)); \quad f(x_1, x_2) = f(x_1, -x_2) = -f(-x_1, x_2) \quad \text{on} \quad D(r)\}.$$

We choose $\chi \in C^\infty([0, \infty))$ such that

$$\chi(r) = 1 \quad \text{for} \quad r \leqslant 1 \quad \text{and} \quad \chi(r) = 0 \quad \text{for} \quad r \geqslant 2.$$

Instead of giving a complete proof of (2), we prove the following lemma which plays a central role in the proof of (2).

LEMMA 2.1. *There exist* $\{\varphi^{\pm}_{i,0}\}_{i\geqslant 0} \cup \{\varphi^{\pm}_{N+i+1,N}\}_{N\geqslant 1, i\geqslant 0}$, $\{v_{N+i+1,N}\}_{N\geqslant 0, i\geqslant 0}$, *and* $\{\lambda_{N+i,N}\}_{N\geqslant 0, i\geqslant 1}$ *satisfying* $\lambda_{1,0} > 0$, (IN), (OUT)$^{\pm}$, *and the following* $(P1)$–$(P9)$.

$(P1)_{i,N}$

$$v_{N+i+1,N}(\xi) = v^+_{N+i+1,N}(\xi) + v^-_{N+i+1,N}(\xi), \quad \xi \in \Pi.$$

$(P2)_{i,N}$

$$v^+_{N+i+1,N}(\xi) = \sum_{j=1}^{i+1}\sum_{k=1}^{j} C_{i,j,k,N}\rho^{2j-1}\cos(2k-1)\theta.$$

$(P3)_{i,N}$ *The function* $v^-_{N+i+1,N}(\xi_1, \cdot)$ *is odd, and* $v^-_{N+i+1,N}(\xi) \in \mathcal{A} \cap \mathcal{B}$.

$(P4)_{i,N}$ *The function* $v^-_{N+i+1,N}$ *has the following asymptotic expansion which can be differentiated term by term two times.*

$$\begin{aligned} v^-_{N+i+1,N}(\xi) \sim &\Big(\sum_{l=0}^{[(i+1)/2]}\sum_{j=l-i}^{0} + \sum_{l=[(i+1)/2]+1}^{i}\sum_{j=1}^{l} + \sum_{l=i+1}^{\infty}\sum_{j=l-i}^{l}\Big) C^-_{i,l,j,N}\rho^{2i-2l+1}\cos(2j-1)\theta \\ &+ \sum_{j=1}^{[(i+1)/2]}\sum_{l=0}^{i-2j+1} d_{i,j,l,N}\rho^{2j+2l-1}\log\rho\cos(2j-1)\theta \end{aligned}$$

as $\rho \to \infty$, $\xi \in P$.

$(P5)_{p,q}$ *The function* $\varphi^{\pm}_{q+p,q} \in C^\infty(\Omega_\pm)$ *has the following asymptotic expansion which can be differentiated term by term infinitely many times.*

$$\begin{aligned} \varphi^{\pm}_{q+p,q}(x) \sim &\Big(\sum_{s=0}^{p-1}\sum_{j=p-s}^{p} + \sum_{s=p}^{2p-2}\sum_{j=1}^{p} + \sum_{s=2p-1}^{\infty}\sum_{j=p-s}^{0}\Big) k^{\pm}_{s,p,j,q} r^{2s-2p+1}\cos(2j-1)\theta \\ &+ \sum_{s=p}^{\infty}\sum_{j=1}^{\min\{p,s-p+1,[(s+1)/2]\}} l^{\pm}_{s,j,s-p-j+1,q} r^{2s-2p+1}\log r\cos(2j-1)\theta \end{aligned}$$

as $r \to 0$, $x \in \Omega_\pm$. *Besides,* $(\varphi^{\pm}_{q+p,q}(x), \varphi^{\pm}_1)_{\Omega_\pm}$ *is well defined.*

$(P6)_{p,q}$ *The function*

$$\varphi_{q+p,q}(x) := \begin{cases} \varphi^+_{q+p,q}(x) & \text{for} \quad x \in \Omega_+, \\ \varphi^-_{q+p,q}(x) & \text{for} \quad x \in \Omega_- \end{cases}$$

satisfies

$$(1 - \chi(\epsilon^{-1/2}r))\varphi_{q+p,q}(x) \in \mathcal{D}(L_\epsilon)$$

for sufficiently small $\epsilon > 0$.

We rewrite the asymptotic expansions of $v_{N+i+1,N}$ *and* $\varphi^{\pm}_{N+i,N}$ *as follows.*

$$v_{N+i+1,N}(\xi) \sim \sum_{l=0}^{\infty} h^{\pm}_{i,l,N}(\rho,\theta) + \sum_{l=1}^{i} m^{\pm}_{i,l,N}(\rho,\theta)\log\rho \quad (\rho\to\infty,\quad \xi\in\Pi,\quad \pm\xi_2>0), \tag{13$_\pm$}$$

$$\varphi^{\pm}_{N+i,N}(x) \sim \sum_{s=0}^{\infty} t^{\pm}_{s,i,N}(r,\theta) + \sum_{s=i}^{\infty} u^{\pm}_{s,i,N}(r,\theta)\log r \quad (r\to 0,\ x\in\Omega_\pm), \tag{14$_\pm$}$$

where $h^{\pm}_{s,l,q}(\cdot,\theta)$, $m^{\pm}_{s,l,q}(\cdot,\theta)$, $t^{\pm}_{s,l,q}(\cdot,\theta)$ *and* $u^{\pm}_{s,l,q}(\cdot,\theta)$ *are homogeneous polynomials of degree* $2s-2l+1$.

$(P7)_{p,q}$ *For* $p\geqslant 1$ *and* $q\geqslant 0$, *there exists* $r_{p,q}>0$ *such that for all* $M\geqslant 2p-1$, *the function*

$$\varphi^{\pm}_{q+p,q}(x) - \Big(\sum_{s=0}^{M} t^{\pm}_{s,p,q}(r,\theta) + \sum_{s=i}^{M} u^{\pm}_{s,p,q}(r,\theta)\log r\Big)$$

can be extended to a function which belongs to $J_{2M-2p+3}(r_{p,q})$.

$(P8)$ *The following 'matching conditions' are satisfied. For* $0\leqslant q$, $1\leqslant l$, *and* $l\leqslant s$, *we have*

$$u^{\pm}_{s,l,q}(\cdot,\theta) = m^{\pm}_{s,l,q}(\cdot,\theta). \tag{$M.1)^{\pm}_{s,l,q}$}$$

For $0\leqslant q$, $0\leqslant s$, *and* $0\leqslant l$, *we have*

$$t^{\pm}_{s,l,q}(\cdot,\theta) = h^{\pm}_{s,l,q}(\cdot,\theta) - m^{\pm}_{s+1,l+1,q-1}(\cdot,\theta). \tag{$M.2)^{\pm}_{s,l,q}$}$$

$(P9)$ *For* $N\geqslant 0$ *and* $M\geqslant 1$, *the following 'compatibility condition' holds.*

$$\begin{aligned}
&(C^{+}_{1,1})^{-1}\Bigg\{-\sum_{s=0}^{M-1}\frac{\pi}{2}(2M-2s-1)C^{-}_{s,M,M-s,N}C^{+}_{M-s,M-s}\\
&\qquad -\sum_{s=1}^{N+M-1}\ \sum_{j=N-s+1}^{N+M-s}\lambda_{s,N-j}(\varphi^{+}_{N+M-s,j},\varphi^{+}_{1})_{\Omega_+}\Bigg\}\\
&= -(C^{-}_{1,1})^{-1}\Bigg\{\sum_{s=0}^{M-1}\frac{\pi}{2}(2M-2s-1)C^{-}_{s,M,M-s,N}C^{-}_{M-s,M-s}\\
&\qquad -\sum_{s=1}^{N+M-1}\ \sum_{j=N-s+1}^{N+M-s}\lambda_{s,N-j}(\varphi^{-}_{N+M-s,j},\varphi^{-}_{1})_{\Omega_-}\Bigg\}.
\end{aligned} \tag{$(C)_{M,N}$}$$

Remark. The number $\lambda_{M+N,N}$ is equal to the both sides of $(C)_{M,N}$. Besides, it can be proved that $\lambda_{N+1,N} = 2^{-2N-3}\pi{\lambda_0}^{N}\{(C^{+}_{1,1})^2+(C^{-}_{1,1})^2\}$ for $N\geqslant 0$.

Remark. The matching conditions in $(P8)$ are derived as follows. Inserting $(14)_+$ into $(11)_+$, we obtain a formal power series of ϵ and $\log\epsilon$ whose coefficients are polynomials of r. Inserting $(13)_+$ into (12) and using the coordinate change $\xi=\epsilon^{-1}x$, we also obtain a formal power series of ϵ and $\log\epsilon$ whose coefficients are polynomials of r. Identifying the powers of ϵ and $\log\epsilon$ of these two formal power series, we get $(M.1)^{+}_{s,l,q}$ and $(M.1)^{+}_{s,l,q}$. Similarly, we obtain $(M.1)^{-}_{s,l,q}$ and $(M.1)^{-}_{s,l,q}$ by using $(11)_-$, (12), $(13)_-$, and $(14)_-$.

Lemma 2.1 can be proved by induction. For simplicity, we give a construction only for the first two terms of $(11)_\pm$ and (12). We first give the precise form of $\varphi^{\pm}_{0,0}$ and $v_{1,0}$, which are the coefficients of the leading terms of $(11)_\pm$ and (12) respectively. To this end, we introduce some harmonic functions. For $j\in\mathbf{N}$, we define

$$U_j(\xi) = \Re((\sqrt{\xi^2-1})^{2j-1}),\quad \xi\in\Pi,$$

where the branch cut of the square root is the positive real axis. We claim that U_j is harmonic on Π, $U_j \in \mathcal{A} \cap \mathcal{B}$, and $U_j(\xi_1, \cdot)$ is an odd function. Moreover, U_j has the following asymptotic expansion which can be differentiated term by term infinitely many times.

$$U_j(\xi) \sim \pm \sum_{l=0}^{\infty} \tau_{j,l} \rho^{2j-2l-1} \cos(2j-2l-1)\theta \quad \text{as} \quad \rho \to \infty, \quad \xi \in \Pi, \quad \pm\xi_2 > 0, \tag{15}$$

where $\tau_{1,1} = -\frac{1}{2}$, $\tau_{2,1} = -\frac{3}{2}$, and $\tau_{j,0} = 1$ for $j \in \mathbb{N}$. We recall Proposition 1.1. We put

$$\varphi_{0,0}^{\pm} = \pm C_{1,1}^{\pm} \varphi_1^{\pm}.$$

Then we get

$$\varphi_{0,0}^{\pm}(x) = \pm (C_{1,1}^{\pm})^2 r \cos\theta + \mathcal{O}(r^3) \quad \text{as} \quad r \to 0, \quad x \in \Omega_{\pm}. \tag{16}$$

We claim that $\varphi_{0,0}^{\pm}$ satisfies $(\mathrm{OUT})_{0,0}^{\pm}$:

$$(\Delta + \lambda_0)\varphi_{0,0}^{\pm} = 0 \quad \text{on} \quad \Omega_{\pm},$$

$$\varphi_{0,0}^{\pm} = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_{\pm}, \qquad \frac{\partial}{\partial n}\varphi_{0,0}^{\pm} = 0 \quad \text{on} \quad \Gamma.$$

We put

$$v_{1,0}^{+}(\xi) = \frac{1}{2}((C_{1,1}^{+})^2 - (C_{1,1}^{-})^2)\rho\cos\theta, \quad v_{1,0}^{-}(\xi) = \frac{1}{2}((C_{1,1}^{+})^2 + (C_{1,1}^{-})^2)U_1(\xi), \quad v_{1,0}(\xi) = v_{1,0}^{+}(\xi) + v_{1,0}^{-}(\xi).$$

The functions $v_{1,0}^{+}(\xi_1, \cdot)$ and $v_{1,0}^{-}(\xi_1, \cdot)$ are even and odd respectively. We claim that $v_{1,0}$ satisfies $(\mathrm{IN})_{1,0}$:

$$\Delta v_{1,0} = 0 \quad \text{on} \quad \Pi,$$

$$v_{1,0} = 0 \quad \text{on} \quad \{0\} \times \mathbb{R}, \qquad \frac{\partial}{\partial \xi_2} v_{1,0}(\cdot, \pm 0) = 0 \quad \text{on} \quad (1, \infty).$$

Hence, the functions $\varphi_{0,0}^{\pm}$ and $v_{1,0}$ satisfy $(P1)_{0,0}$–$(P7)_{0,0}$. From (15), we obtain

$$v_{1,0}(\xi) = \pm (C_{1,1}^{\pm})^2 \rho\cos\theta + \mathcal{O}(\rho^{-1}) \quad \text{as} \quad \rho \to \infty, \quad \xi \in \Pi, \quad \pm\xi_2 > 0. \tag{17}$$

(1) and (15) allow us to write the asymptotic expansions of $\varphi_{0,0}^{\pm}$ and $v_{1,0}$ as follows.

$$\varphi_{0,0}^{\pm}(x) \sim \sum_{s=0}^{\infty} t_{s,0,0}^{\pm}(r,\theta) \quad \text{as} \quad r \to 0, \quad x \in \Omega_{\pm},$$

$$v_{1,0}(\xi) \sim \sum_{l=0}^{\infty} h_{0,l,0}^{\pm}(\rho,\theta) \quad \text{as} \quad \rho \to \infty, \quad \xi \in \Pi, \quad \pm\xi_2 > 0,$$

where $t_{s,0,0}^{\pm}(\cdot,\theta)$ and $h_{0,l,0}^{\pm}(\cdot,\theta)$ are homogeneous polynomials of degree $(2s+1)$ and $(-2l+1)$ respectively. It follows from (16) and (17) that

$$t_{0,0,0}^{\pm}(\cdot,\theta) = h_{0,0,0}^{\pm}(\cdot,\theta).$$

So $(M.2)_{0,0,0}^{\pm}$ holds. Moreover, $(C)_{1,0}$ holds.

Let us construct $\varphi_{1,0}^{\pm}$, $v_{2,0}$, and $\lambda_{1,0}$. We shall prove the following.

LEMMA 2.2. *There exist $\varphi_{1,0}^{\pm}$, $v_{2,0}$, and $\lambda_{1,0} > 0$ satisfying $(IN)_{1,0}$, $(OUT)_{1,0}^{\pm}$, $(P1)_{1,0} - (P7)_{1,0}$, $(M.1)_{1,1,0}^{\pm}$; $(M.2)_{s,l,0}^{\pm}$ for $0 \leqslant s \leqslant 1$, $0 \leqslant l \leqslant 1$; $(C)_{2,0}$.*

We need the following proposition in the proof of Lemma 2.2.

PROPOSITION 2.3. *Suppose that $g \in \mathcal{B}$ has the following asymptotic expansion which can be differentiated term by term two times.*

$$g(\xi) \sim \sum_{l=0}^{\infty} a_{0,l,l}\rho^{-2l+1}\cos(2l-1)\theta \quad as \quad \rho \to \infty, \quad \xi \in P. \tag{18}$$

Then the problem

$$-\Delta v = g \quad on \quad P, \quad \frac{\partial}{\partial \xi_2} v = 0 \quad on \quad \Gamma_1 := (1,\infty)\times\{0\}, \quad v = 0 \quad on \quad \Gamma_2 := \partial P \backslash \Gamma_1 \tag{19}$$

has a solution $v \in \mathcal{A} \cap \mathcal{B}$ having the following asymptotic expansion which can be differentiated term by term two times.

$$v(\xi) \sim (\sum_{l=0}^{1}\sum_{j=l-1}^{0} + \sum_{l=2}^{\infty}\sum_{j=l-1}^{l}) a_{1,l,j}\rho^{-2l+3}\cos(2j-1)\theta + b_{1,0,0}\rho\log\rho\cos\theta \quad as \quad \rho \to \infty, \quad \xi \in P.$$

To prove this proposition, we need the following.

PROPOSITION 2.4. *Suppose that the function f satisfies the following* (**i**)–(**iv**).

(**i**) $f \in C^\infty(\mathbb{R}^2 \backslash \{\sqrt{-1},\ -\sqrt{-1}\})$.

(**ii**) *The function f is bounded near $\{\sqrt{-1}\}$ and $\{-\sqrt{-1}\}$.*

(**iii**) $f(y_1, y_2) = f(y_1, -y_2) = -f(-y_1, y_2)$ *on* $\mathbb{R}^2 \backslash \{\sqrt{-1},\ -\sqrt{-1}\}$.

(**iv**) *There exists an integer $N \geqslant 10$ such that for any multi-index α satisfying $|\alpha| \leqslant 2$, there exists $C_\alpha > 0$ such that $|\partial^\alpha f(y)| \leqslant C_\alpha |y|^{-N-|\alpha|}$ for $|y| \geqslant 2$.*

We put

$$u(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|\eta - y| f(y)\, dy.$$

Then we have the following.

$$u \in C^\infty(\mathbb{R}^2 \backslash \{\sqrt{-1},\ -\sqrt{-1}\}) \cap C^1(\mathbb{R}^2).$$

$$-\Delta u = f \quad on \quad \mathbb{R}^2 \backslash \{\sqrt{-1},\ -\sqrt{-1}\}.$$

$$u(\eta_1, \eta_2) = u(\eta_1, -\eta_2) = -u(-\eta_1, \eta_2) \quad on \quad \mathbb{R}^2.$$

Moreover, u has the following asymptotic expansion which can be differentiated term by term two times.

$$u(\eta) = \sum_{\substack{1 \leqslant j \leqslant [N/2]-4 \\ j \equiv 1 \bmod 2}} C_j \tilde{\rho}^{-j}\cos j\omega + \mathcal{O}(\tilde{\rho}^{-[N/2]+3})$$

as $\tilde{\rho} \to \infty$, where $(\tilde{\rho}, \omega)$ is the polar coordinate of η.

Since this proposition is derived by a straightforward computation, we omit the proof. By using this proposition, we shall prove the following.

PROPOSITION 2.5. *Let $g(\xi) \in \mathcal{B}$. Suppose that there exists an integer $N \geqslant 10$ such that for any multi-index α satisfying $|\alpha| \leqslant 2$, there exists $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha g(\xi)| \leqslant C_\alpha |\xi|^{-N-|\alpha|} \quad for \quad \xi \in P, \quad |\xi| \geqslant 2. \tag{20}$$

Then the boundary value problem

$$\begin{gathered} -\Delta v(\xi) = g(\xi) \quad on \quad P, \\ \frac{\partial}{\partial \xi_2} v = 0 \quad on \quad \Gamma_1, \qquad v = 0 \quad on \quad \Gamma_2 \end{gathered} \tag{21}$$

has a unique solution $v \in \mathcal{A} \cap \mathcal{B}$ having the following asymptotic expansion which can be differentiated term by term two times.

$$v(\xi) = \sum_{\substack{1 \leqslant j \leqslant [N/2]-4 \\ j \equiv 1 \bmod 2}} C_j \rho^{-j} \cos j\theta + O(\rho^{-[N/2]+3}) \quad as \quad \rho = |\xi| \to \infty. \tag{22}$$

Proof. It is easy to see that $\xi = \xi(\eta)$ is a conformal map from P onto P and that ξ maps $\{0\} \times [0, \infty)$ and $(0, \infty) \times \{0\}$ onto Γ_2 and Γ_1 respectively. Thus the problem (21) is equivalent to the problem

$$-\Delta u(\eta) = \frac{|\eta|^2}{|\eta^2 + 1|} g(\xi(\eta)) \quad \text{in} \quad P,$$

$$\frac{\partial}{\partial \eta_2} u = 0 \quad \text{on} \quad (0, \infty) \times \{0\}, \qquad u = 0 \quad \text{on} \quad \{0\} \times (0, \infty).$$

We set $f(\eta) = \frac{|\eta|^2}{|\eta^2+1|} g(\xi(\eta))$. From (20), we claim that for any multi-index α satisfying $|\alpha| \leqslant 2$, there exists $C_\alpha > 0$ such that $|\partial_\eta^\alpha f(\eta)| \leqslant C_\alpha |\eta|^{-N-|\alpha|}$ for $\eta \in P$, $|\eta| \geqslant 2$. Because $g \in \mathcal{B}$, the function f satisfies the assumptions **(i)**–**(iii)** in Proposition 2.4. From Proposition 2.4, we claim that there exists $u \in C^\infty(\mathbb{R}^2 \backslash \{\sqrt{-1}, -\sqrt{-1}\}) \cap C^1(\mathbb{R}^2)$ such that

$$-\Delta u = f \quad \text{on} \quad \mathbb{R}^2 \backslash \{\sqrt{-1}, -\sqrt{-1}\},$$

$$u(\eta_1, \eta_2) = u(\eta_1, -\eta_2) = -u(-\eta_1, \eta_2) \quad \text{on} \quad \mathbb{R}^2.$$

Moreover, the function u has the following asymptotic expansion.

$$\partial_{\tilde\rho}^m \partial_\omega^n (u(\eta) - \sum_{\substack{1 \leqslant j \leqslant [N/2]-4 \\ j \equiv 1 \bmod 2}} C_j \tilde\rho^{-j} \cos j\omega) = O(\tilde\rho^{-[N/2]+3-m}) \quad \text{as} \quad \tilde\rho \to \infty \tag{23}$$

for $m+n \leqslant 2$, $(\tilde\rho, \omega)$ being the polar coodinate of η. We put $\eta(\xi) = \sqrt{\xi^2 - 1}$, where the branch cut of the square root is the positive real axis. This function is the inverse function of $\xi(\eta)$. We set $v(\xi) = u(\eta(\xi))$. Then we infer that v is a solution of (21). Besides, it can be verified that $v \in \mathcal{A} \cap \mathcal{B}$. Let (ρ, θ) be the polar coordinate of ξ. Then we have

$$\tilde\rho^{-j} \cos j\omega = \Re(\eta(\xi)^{-j}) = \sum_{k=0}^\infty \tau_{j,k} \rho^{-j-2k} \cos(j - 2k)\theta \tag{24}$$

for $\rho \geqslant 2$, where the above summation can be differentiated term by term infinitely many times. By a direct computation, we infer that there exists $C > 0$ such that

$$|\eta(\xi)| \leqslant C|\xi|, \quad \left|\frac{d}{d\xi}\eta(\xi)\right| \leqslant C, \quad \text{and} \quad \left|\frac{d^2}{d\xi^2}\eta(\xi)\right| \leqslant C|\xi|^{-3} \quad \text{for} \quad |\xi| \geqslant 2, \tag{25}$$

where $\frac{d}{d\xi}$ stands for the differentiation by the complex variable ξ. Using (23), (24), and (25), we obtain (22).

Let us show the uniqueness of the solution of (21) which has the asymptotic expansion of the form (22). Let v_1, $v_2 \in \mathcal{A}$ be the solutions of (21) which have the asymptotic expansion of the form (22). We set $h = v_1 - v_2$. Then we get

$$-\Delta h = 0 \quad \text{on} \quad P, \quad \frac{\partial}{\partial \xi_2} h = 0 \quad \text{on} \quad \Gamma_2, \quad h = 0 \quad \text{on} \quad \Gamma_1. \tag{26}$$

From (22), we infer that for any multi-index α satisfying $|\alpha| \leqslant 2$, there exists $C_\alpha > 0$ such that

$$|\partial_\xi^\alpha h(\xi)| \leqslant C_\alpha |\xi|^{-1-|\alpha|} \quad \text{for} \quad |\xi| \geqslant 2. \tag{27}$$

For $\delta \in (0, \frac{1}{2})$ and $R > 2$, we put

$$Q_{\delta,R} = D_{++}(R) \backslash \overline{D(1,\delta)}, \quad \Lambda_\delta = \partial D(1,\delta) \cap \partial Q_{\delta,R}, \quad \Lambda^R = \partial D(R) \cap \partial Q_{\delta,R}.$$

We have $h \in C^\infty(Q_{\delta,R}) \cap C^1(\overline{Q_{\delta,R}})$. Let $(\tilde{r}, s)$ be the polar coodinate centered at $(1,0)$. Using (26) and Green's formula, we obtain

$$0 = \int_{Q_{\delta,R}} (-\Delta h) h \, d\xi = \int_{Q_{\delta,R}} |\nabla h|^2 \, d\xi + \int_{\Lambda_\delta} \frac{\partial}{\partial \tilde{r}} h \cdot h \, dS + \int_{\Lambda^R} \frac{\partial}{\partial \rho} h \cdot h \, dS. \tag{28}$$

From (27), we get $| \int_{\Lambda^R} \frac{\partial}{\partial \rho} h \cdot h \, dS| \leqslant CR^{-2} \to 0$ as $R \to \infty$. Since $h \in \mathcal{A}$, we get $| \int_{\Lambda_\delta} \frac{\partial}{\partial \tilde{r}} h \cdot h \, dS| \leqslant C\delta \to 0$ as $\delta \to 0$. Taking the limit $\delta \to 0$ and $R \to \infty$ in (28), we get $\int_P |\nabla h|^2 \, d\xi = 0$. Because $h = 0$ on Γ_1, we have $h = 0$ on P. This completes the proof of Proposition 2.5. □

Proof of Proposition 2.3. By using formulae

$$\Delta(\rho^{2j+2m-1} \cos(2j-1)\theta) = 4m(2j+m-1)\rho^{2j+2m-3} \cos(2j-1)\theta, \tag{29}$$

$$\begin{aligned} \Delta(\rho^{2j+2m-1} \log \rho \cos(2j-1)\theta) =& 4m(2j+m-1)\rho^{2j+2m-3} \log \rho \cos(2j-1)\theta \\ &+ 2(2j+2m-1)\rho^{2j+2m-3} \cos(2j-1)\theta, \quad j, m \in \mathbb{Z}, \end{aligned} \tag{30}$$

we claim that there exists a formal power series

$$\Big(\sum_{l=0}^{1} \sum_{j=l-1}^{0} + \sum_{l=2}^{\infty} \sum_{j=l-1}^{l}\Big) \tilde{a}_{1,l,j} \rho^{-2l+3} \cos(2j-1)\theta + \tilde{b}_{1,0,0} \rho \log \rho \cos\theta$$

such that the function

$$v_N(\xi) := \Big(\sum_{l=0}^{1} \sum_{j=l-1}^{0} + \sum_{l=2}^{N} \sum_{j=l-1}^{l}\Big) \tilde{a}_{1,l,j} \rho^{-2l+3} \cos(2j-1)\theta + \tilde{b}_{1,0,0} \rho \log \rho \cos\theta$$

satisfies

$$-\Delta v_N(\xi) = \sum_{l=0}^{N} a_{0,l,l} \rho^{-2l+1} \cos(2l-1)\theta =: g_N(\xi)$$

for all $N \geqslant 2$.

Let N be an arbitrary integer satisfying $-2N - 1 \leqslant -10$. From (18), we have $\partial_\xi^\alpha (g(\xi) - g_N(\xi)) = O(\rho^{2s-2N-1-|\alpha|})$ as $\rho \to \infty$ for $|\alpha| \leqslant 2$. We choose $\tilde{\chi}_1 \in C^\infty([0,\infty))$ such that $\tilde{\chi}_1(s) = 0$ for $s \leqslant 2$, $\tilde{\chi}_1(s) = 1$ for $s \geqslant 3$. We put $\chi_1(\xi) = \tilde{\chi}_1(|\eta(\xi)|)$. We seek a solution of (19) which takes the form

$$v = \tilde{v}_N + \chi_1 v_N. \tag{31}$$

Inserting (31) into (19), we derive the equation for $\tilde{v}_N$:

$$\begin{aligned} -\Delta \tilde{v}_N = (1-\chi_1) g + \chi_1 (g - g_N) + 2\nabla \chi_1 \cdot \nabla v_N + v_N \Delta \chi_1 =: h_N \quad \text{on} \quad P, \\ \frac{\partial}{\partial \xi_2} \tilde{v}_N = 0 \quad \text{on} \quad \Gamma_1, \qquad \tilde{v}_N = 0 \quad \text{on} \quad \Gamma_2. \end{aligned} \tag{32}$$

We have $h_N \in \mathcal{B}$ and $\partial_\xi^\alpha h_N(\xi) = O(\rho^{-2N-1-|\alpha|})$ as $\rho \to \infty$ for $|\alpha| \leqslant 2$. Thus, Proposition 2.5 implies that the problem (32) has a solution $\tilde{v}_N \in \mathcal{A} \cap \mathcal{B}$ having the following asymptotic expansion which can be differentiated term by term two times.

$$\tilde{v}_N = \sum_{\substack{j=1 \\ j \equiv 1 \bmod 2}}^{N-4} c_{N,j} \rho^{-j} \cos j\theta + O(\rho^{-N+s+3}) \quad \text{as} \quad \rho \to \infty. \tag{33}$$

We show that the right side of (31) is independent of the choice of N. Let N_1 and N_2 be arbitrary integers satisfying $-2N_j - 1 \leqslant -10$ $(j = 1, 2)$. We set $w = (\tilde{v}_{N_1} + \chi_1 v_{N_1}) - (\tilde{v}_{N_2} + \chi_1 v_{N_2}) (\in \mathcal{A} \cap \mathcal{B})$. From the definition of v_N and (33), we get $\partial_\xi^\alpha w = O(\rho^{-1-|\alpha|})$ as $\rho \to \infty$ for $|\alpha| \leqslant 2$. As in the proof of Proposition 2.5, we obtain $w = 0$. This completes the proof of Proposition 2.3. □

3. *Construction of the asymptotic expansion of the second eigenvalue*

In this section, we complete the proof of Lemma 2.2.

PROOF OF LEMMA 2.2. Let us construct $\varphi_{1,0}^{\pm}$. By using formulae (29) and (30), one can construct a formal solution $\breve{\varphi}_{1,0}^{\pm}$ of $(\Delta+\lambda_0)\varphi=0$:

$$\begin{aligned}\breve{\varphi}_{1,0}^{\pm}(x) &= \sum_{s=0}^{\infty} \tilde{h}_{s,1,0}^{\pm}(r,\theta) + \sum_{s=1}^{\infty} \tilde{m}_{s,1,0}^{\pm}(t,\theta)\log r \\ &= \tilde{k}_{0,1,1,0}^{\pm} r^{-1}\cos\theta + \sum_{s=1}^{\infty}\sum_{j=1-s}^{0} \tilde{k}_{s,1,j,0}^{\pm} r^{2s-1}\cos(2j-1)\theta + \sum_{s=1}^{\infty} \tilde{l}_{s,1,s-1,0}^{\pm} r^{2s-1}\log r\cos\theta \end{aligned} \tag{34}$$

satisfying

$$\tilde{h}_{0,1,0}^{\pm}(r,\theta) = h_{0,1,0}^{\pm}(r,\theta), \tag{35}$$

where $\tilde{h}_{s,1,0}^{\pm}(\cdot,\theta)$ and $\tilde{m}_{s,1,0}^{\pm}(\cdot,\theta)$ are homogeneous polynomials of degree $2s-1$. For $N\in\mathbb{N}$, we put

$$\breve{\varphi}_{1,0}^{N\pm}(x) = \sum_{s=0}^{N} \tilde{h}_{s,1,0}^{\pm}(r,\theta) + \sum_{s=1}^{N} \tilde{m}_{s,1,0}^{\pm}(r,\theta)\log r. \tag{36}$$

We choose $\chi_0\in C^\infty([0,\infty))$ such that $\chi_0=1$ on $[0,\tilde{r}_0/4]$ and $\chi_0=0$ on $[\tilde{r}_0/2,\infty)$. We first seek a solution $\breve{\varphi}_{1,0}^{N\pm}$ of $(\mathrm{OUT})_{1,0}^{\pm}$ which takes the form

$$\breve{\varphi}_{1,0}^{N\pm}(x) = \chi_0(r)\varphi_{1,0}^{N\pm}(x) + \tilde{\varphi}_{1,0}^{N\pm}(x), \quad \tilde{\varphi}_{1,0}^{N\pm}\in\mathcal{D}(L_0^{\pm}), \quad (\breve{\varphi}_{1,0}^{N\pm},\varphi_1^{\pm})_{\Omega_\pm}=0. \tag{37}$$

We have

$$(\Delta+\lambda_0)\breve{\varphi}_{1,0}^{N\pm} = (\Delta+\lambda_0)\tilde{\varphi}_{1,0}^{N\pm} + \chi_0(r)(\Delta+\lambda_0)\varphi_{1,0}^{N\pm} + \varphi_{1,0}^{N\pm}\Delta\chi_0(r) + 2\nabla\chi_0(r)\cdot\nabla\varphi_{1,0}^{N\pm}. \tag{38}$$

Substituting $\breve{\varphi}_{1,0}^{N\pm}$ for $\varphi_{1,0}^{\pm}$ in $(\mathrm{OUT})_{1,0}^{\pm}$ and using (37), (38), and the fact that $\breve{\varphi}_{1,0}^{\pm}$ is a formal solution of $(\Delta+\lambda_0)\varphi=0$, we get the equations for $\tilde{\varphi}_{1,0}^{N\pm}$:

$$\begin{gathered}(\Delta+\lambda_0)\tilde{\varphi}_{1,0}^{N\pm} = f_{N,1}^{\pm} - \lambda_{1,0}\varphi_{0,0}^{\pm} \quad\text{on}\quad \Omega_\pm, \\ \tilde{\varphi}_{1,0}^{N\pm}=0 \quad\text{on}\quad \partial\Omega\cap\partial\Omega_\pm, \qquad \frac{\partial}{\partial n}\tilde{\varphi}_{1,0}^{N\pm}=0 \quad\text{on}\quad \Gamma, \quad (\tilde{\varphi}_{1,0}^{N\pm},\varphi_1^{\pm})_{L^2(\Omega_\pm)} = -(\chi_0(r)\varphi_{1,0}^{N\pm},\varphi_1^{\pm})_{\Omega_\pm},\end{gathered} \tag{39}$$

where

$$f_{N,1}^{\pm} := -\varphi_{1,0}^{N\pm}\Delta\chi_0 - 2\nabla\chi_0\cdot\nabla\varphi_{1,0}^{N\pm} - \chi_0(r)\lambda_0(\tilde{h}_{N,1,0}^{\pm}(r,\theta) + \tilde{m}_{N,1,0}^{\pm}(r,\theta)\log r).$$

Let $N\geqslant 2$. Let us show that $f_{N,1}^{\pm}\in L^2(\Omega_\pm)$ and that the function $f_{N,1}^{\pm}$ can be extended to a function belonging to $J_{2N-3}(\tilde{r}_0/4)$. Since

$$\tilde{h}_{N,1,0}^{\pm}(r,\theta) = \sum_{j=1-N}^{0} \tilde{k}_{N,1,j,0}^{\pm} r^{2s-1}\cos(2j-1)\theta \quad\text{and}\quad \tilde{m}_{N,1,0}^{\pm}(r,\theta)\log r = \tilde{l}_{N,1,N-1,0}^{\pm} r^{2N-1}\log r\cos\theta,$$

we have

$$\tilde{h}_{N,1,0}^{\pm}(r,\theta)\in\cap_{i=0}^{\infty}\cap_{r>0}J_i(r) \quad\text{and}\quad \tilde{m}_{N,1,0}^{\pm}(r,\theta)\log r\in\cap_{r>0}J_{2N-3}(r). \tag{40}$$

Because $\Delta\chi_0=0$ and $\nabla\chi_0=0$ in a neighborhood of 0, we have

$$-\varphi_{1,0}^{N\pm}\Delta\chi_0 - 2\nabla\chi_0\cdot\nabla\varphi_{1,0}^{N\pm}\in\cap_{i=0}^{\infty}J_i(\tilde{r}_0/4).$$

Thus we obtain

$$f_{N,1}^{\pm}\in J_{2N-3}(\tilde{r}_0/4). \tag{41}$$

Moreover we obtain

$$-\varphi_{1,0}^{N\pm}\Delta\chi_0 - 2\nabla\chi_0\cdot\nabla\varphi_{1,0}^{N\pm} - \chi_0(r)\lambda_0(\tilde{h}_{N,1,0}^{\pm}(r,\theta) + \tilde{m}_{N,1,0}^{\pm}(r,\theta)\log r) \in C^\infty(\mathbb{R}^2\backslash D(\tilde{r}_0/16)).$$

Thus, we get

$$f_{N,1}^{\pm} \in L^2(\Omega_\pm\backslash D(\tilde{r}_0/8)). \tag{42}$$

Using (41) and (42), we get $f_{N,1}^{\pm} \in L^2(\Omega_\pm)$.

Let us consider the following equations derived by substituting $\lambda_{1,0}^{N\pm}$ for $\lambda_{1,0}$ in (39).

$$\begin{gathered}(\Delta+\lambda_0)\tilde{\varphi}_{1,0}^{N\pm} = f_{N,1}^{\pm} - \lambda_{1,0}^{N\pm}\varphi_{0,0}^{\pm} \quad \text{on} \quad \Omega_\pm, \\ \tilde{\varphi}_{1,0}^{N\pm} = 0 \quad \text{on} \quad \partial\Omega\cap\partial\Omega_\pm, \qquad \frac{\partial}{\partial n}\tilde{\varphi}_{1,0}^{N\pm} = 0 \quad \text{on} \quad \Gamma, \quad (\tilde{\varphi}_{1,0}^{N\pm}, \varphi_1^{\pm})_{L^2(\Omega_\pm)} = -(\chi_0(r)\varphi_{1,0}^{N\pm}, \varphi_1^{\pm})_{\Omega_\pm}.\end{gathered} \tag{43}$$

Because λ_0 is a simple eigenvalue of $L_0^{\pm}$, the equation (43) has a solution $\tilde{\varphi}_{1,0}^{N\pm} \in \mathcal{D}(L_0^{\pm})$ if and only if

$$\lambda_{1,0}^{N\pm} = \frac{(f_{N,1}^{\pm}, \varphi_1^{\pm})_{L^2(\Omega_\pm)}}{(\varphi_{0,0}^{\pm}, \varphi_1^{\pm})_{L^2(\Omega_\pm)}}. \tag{44}$$

We define $\lambda_{1,0}^{\pm}$ by (44). Then (43) has a unique solution which belongs to $\mathcal{D}(L_0^{\pm})$. We denote it by $\tilde{\varphi}_{1,0}^{N\pm}$.

Now we compute the asymptotic expansion of $\tilde{\varphi}_{1,0}^{N\pm}(x)$ as $x \to 0$, $x \in \Omega_\pm$. For this purpose, we first compute the asymptotic expansion of $\tilde{\varphi}_{1,0}^{N\pm}$ by the reflection argument used in the proof of Proposition 1.1. From (41), we claim that there exists $\tilde{f}_{N,1}^{+} \in C^{2N-3}(D(\tilde{r}_0/4))$ such that

$$\tilde{f}_{N,1}^{+} = f_{N,1}^{+} \quad \text{on} \quad D_{++}(\tilde{r}_0/4), \quad \tilde{f}_{N,1}^{+}(x_1,x_2) = \tilde{f}_{N,1}^{+}(x_1,-x_2) = -\tilde{f}_{N,1}^{+}(-x_1,x_2) \quad \text{on} \quad D(\tilde{r}_0/4).$$

Let $\Psi_{1,0}^{N+}(x)$ be such that

$$\Psi_{1,0}^{N+}(x) = \tilde{\varphi}_{1,0}^{N+}(x) \text{ on } D_{++}(\tilde{r}_0/8), \quad \Psi_{1,0}^{N+}(x_1,x_2) = \Psi_{1,0}^{N+}(x_1,-x_2) = -\Psi_{1,0}^{N+}(-x_1,-x_2) \text{ on } D(\tilde{r}_0/8).$$

As in the proof of Proposition 1.1, we obtain

$$\Psi_{1,0}^{N+} \in H^1(D(\tilde{r}_0/8)),$$

$$(\Delta+\lambda_0)\Psi_{1,0}^{N+} = \tilde{f}_{N,1}^{+} - \lambda_{1,0}^{N+}\varphi_{0,0}^{+} \quad \text{on} \quad D(\tilde{r}_0/8). \tag{45}$$

From (41), we get

$$\tilde{f}_{N,1}^{+} - \lambda_{1,0}^{N+}\varphi_{0,0}^{+} \in C^{2N-3}(D(\tilde{r}_0/8)) \subset H^{2N-3}(D(\tilde{r}_0/16)). \tag{46}$$

By using (45), (46), the regularity estimate for elliptic differential equations, and Sobolev's imbedding theorem, we obtain

$$\Psi_{1,0}^{N+} \in H^{2N-1}(D(\tilde{r}_0/32)) \subset C^{2N-3}(D(\tilde{r}_0/32)).$$

Hence we have

$$\partial_x^\beta(\Psi_{1,0}^{N+}(x) - \sum_{|\alpha|\leqslant 2N-4}\frac{1}{\alpha!}\partial_x^\alpha\Psi_{1,0}^{N+}(0)x^\alpha) = \mathcal{O}(r^{2N-3-|\beta|})$$

for $|\beta| \leqslant 2N-4$. Because $\Psi_{1,0}^{N+}(x)$ is even with respect to x_2 and odd with respect to x_1, we get

$$\sum_{|\alpha|\leqslant 2N-4}\frac{1}{\alpha!}\partial_x^\alpha\Psi_{1,0}^{N+}(0)x^\alpha = \sum_{j=1}^{N-2}\sum_{k=1}^{j}\gamma_{j,k,N}^{+}r^{2j-1}\cos(2k-1)\theta.$$

Thus, we claim that $\check{\varphi}_{1,0}^{N+}$ has the following asymptotic expansion which can be differentiated term by term $(2N-4)$ times.

$$\check{\varphi}_{1,0}^{N+}(x) = \varphi_{1,0}^{N+}(x) + \sum_{j=1}^{N-2}\sum_{k=1}^{j} \gamma_{j,k,N}^{+} r^{2j-1}\cos(2k-1)\theta + \mathcal{O}(r^{2N-3}) \quad \text{as} \quad r \to 0, \quad x \in \Omega_+. \tag{47}$$

In a similar fashion, $\check{\varphi}_{1,0}^{N-}$ has the following asymptotic expansion which can be differentiated $(2N-4)$ times.

$$\check{\varphi}_{1,0}^{N-}(x) = \varphi_{1,0}^{N-}(x) + \sum_{j=1}^{N-2}\sum_{k=1}^{j} \gamma_{j,k,N}^{-} r^{2j-1}\cos(2k-1)\theta + \mathcal{O}(r^{2N-3}) \quad \text{as} \quad r \to 0, \quad x \in \Omega_-. \tag{48}$$

Next we show that $\check{\varphi}_{1,0}^{N\pm} = \chi_0(r)\varphi_{1,0}^{N\pm} + \tilde{\varphi}_{1,0}^{N\pm}$ and $\lambda_{1,0}^{N\pm}$ are independent of the choice of $N \geqslant 2$. We set $N_1 = 2$. Let $N_2 > N_1$. Because $\check{\varphi}_{1,0}^{N_1\pm}$ and $\check{\varphi}_{1,0}^{N_2\pm}$ are the solutions of $(\mathrm{OUT})_{1,0}^{\pm}$, we get

$$(\Delta + \lambda_0)\check{\varphi}_{1,0}^{N_j\pm} = -\lambda_{1,0}^{N_j\pm}\varphi_{0,0}^{\pm} \quad \text{on} \quad \Omega_\pm \quad \text{for} \quad j = 1, 2,$$

$$(\check{\varphi}_{1,0}^{N_1\pm}, \varphi_1^{\pm})_{\Omega_\pm} = (\check{\varphi}_{1,0}^{N_2\pm}, \varphi_1^{\pm})_{\Omega_\pm} = 0.$$

Thus we obtain

$$(\Delta + \lambda_0)(\check{\varphi}_{1,0}^{N_1\pm} - \check{\varphi}_{1,0}^{N_2\pm}) = (\lambda_{1,0}^{N_2\pm} - \lambda_{1,0}^{N_1\pm})\varphi_{0,0}^{\pm}, \tag{49}$$

$$(\check{\varphi}_{1,0}^{N_1\pm} - \check{\varphi}_{1,0}^{N_2\pm}, \varphi_1^{\pm})_{\Omega_\pm} = 0. \tag{50}$$

Note that

$$\check{\varphi}_{1,0}^{N_1\pm} - \check{\varphi}_{1,0}^{N_2\pm} = \chi_0(r)(\varphi_{1,0}^{N_1\pm} - \varphi_{1,0}^{N_2\pm}) + (\tilde{\varphi}_{1,0}^{N_1\pm} - \tilde{\varphi}_{1,0}^{N_2\pm}). \tag{51}$$

Because $\tilde{\varphi}_{1,0}^{N_1\pm}, \tilde{\varphi}_{1,0}^{N_2\pm} \in \mathcal{D}(L_0^{\pm})$, we have $\tilde{\varphi}_{1,0}^{N_1\pm} - \tilde{\varphi}_{1,0}^{N_2\pm} \in \mathcal{D}(L_0^{\pm})$. From (40), we have

$$\chi_0(r)(\varphi_{1,0}^{N_1\pm} - \varphi_{1,0}^{N_2\pm}) \in J_3(\tilde{r}_0/2) \quad \text{and} \quad \operatorname{supp}(\chi_0(r)(\varphi_{1,0}^{N_1\pm} - \varphi_{1,0}^{N_2\pm})) \subset \overline{D_{+\pm}(\tilde{r}_0/2)}. \tag{52}$$

Note that for $u \in Q_0^{\pm}$, we have $u \in \mathcal{D}(L_0^{\pm})$ if and only if there exists $w \in L^2(\Omega_\pm)$ such that $(\nabla u, \nabla v)_{L^2(\Omega_\pm)} = (w, v)_{L^2(\Omega_\pm)}$ for all $v \in Q_0^{\pm}$. Combining (52) with Green's identity:

$$(\nabla u, \nabla v)_{L^2(\Omega_\pm)} = \int_{\partial\Omega_\pm} \frac{\partial}{\partial n} u \cdot v \, dS - (\Delta u, v)_{L^2(\Omega_\pm)}, \quad u \in H^2(\Omega_\pm), \quad v \in H^1(\Omega_\pm),$$

we have $\chi_0(r)(\varphi_{1,0}^{N_1\pm} - \varphi_{1,0}^{N_2\pm}) \in \mathcal{D}(L_0^{\pm})$. So (51) implies that

$$\check{\varphi}_{1,0}^{N_1\pm} - \check{\varphi}_{1,0}^{N_2\pm} \in \mathcal{D}(L_0^{\pm}). \tag{53}$$

Note that $\varphi_1^{\pm}$ is the eigenvector of $L_0^{\pm}$ associated with the simple eigenvalue λ_0. Thus (49) and (53) imply that the right side of (49) is orthogonal to $\varphi_1^{\pm}$ in $L^2(\Omega_\pm)$. So we have $\lambda_{1,0}^{N_1\pm} = \lambda_{1,0}^{N_2\pm}$. This together with (49), (50), and (53) imply that $\check{\varphi}_{1,0}^{N_1\pm} = \check{\varphi}_{1,0}^{N_2\pm}$. Thus the function $\check{\varphi}_{1,0}^{N\pm}$ and the number $\lambda_{1,0}^{N\pm}$ are independent of the choice of $N \geqslant 2$, which we denote by $\check{\varphi}_{1,0}^{\pm}$ and $\lambda_{1,0}^{\pm}$ respectively.

Let us show that $\lambda_{1,0}^{+}$ and $\lambda_{1,0}^{-}$ are given by the left side and the right side of $(C)_{1,0}$ respectively. Note that

$$(\Delta + \lambda_0)\check{\varphi}_{1,0}^{\pm} = -\lambda_{1,0}^{\pm}\varphi_{0,0}^{\pm} \quad \text{on} \quad \Omega_\pm. \tag{54}$$

Multiplying (54) by $\varphi_1^{\pm}$, and integrating over $\Omega_\pm$, we obtain

$$((\Delta + \lambda_0)\check{\varphi}_{1,0}^{\pm}, \varphi_1^{\pm})_{\Omega_\pm} = -\lambda_{1,0}^{\pm}(\varphi_{0,0}^{\pm}, \varphi_1^{\pm})_{L^2(\Omega_\pm)}. \tag{55}$$

From (37), we get

$$((\Delta + \lambda_0)\check{\varphi}_{1,0}^{+}, \varphi_1^{+})_{\Omega_+} = ((\Delta + \lambda_0)(\chi_0(r)\varphi_{1,0}^{N_1+}), \varphi_1^{+})_{\Omega_+} + ((\Delta + \lambda_0)\tilde{\varphi}_{1,0}^{N_1+}, \varphi_1^{+})_{\Omega_+}. \tag{56}$$

Because $\tilde{\varphi}_{1,0}^{N_1+}$, $\varphi_1^+ \in \mathcal{D}(L_0^+)$ and $(\Delta+\lambda_0)\varphi_1^+ = 0$, we have

$$((\Delta+\lambda_0)\tilde{\varphi}_{1,0}^{N_1+}, \varphi_1^+)_{\Omega_+} = (\tilde{\varphi}_{1,0}^{N_1+}, (\Delta+\lambda_0)\varphi_1^+)_{\Omega_+} = 0. \tag{57}$$

Let $\delta \in (0, \tilde{r}_0/4)$. We put

$$g(r,\theta) = \chi_0(r)\varphi_{1,0}^{N_1+}, \quad S_\delta = D_{++}(\tilde{r}_0/2)\backslash D_{++}(\delta), \quad w_1 = \partial D_{++}(\tilde{r}_0/2) \cap \partial D(\tilde{r}_0/2),$$

$$w_{2,\delta} = \partial D_{++}(\delta) \cap \partial D(\delta), \quad w_{3,\delta} = [\delta, \tilde{r}_0/2] \times \{0\}, \quad \text{and} \quad w_{4,\delta} = \{0\} \times [\delta, \tilde{r}_0/2].$$

We have $\partial S_\delta = w_1 \cup w_{2,\delta} \cup w_{3,\delta} \cup w_{4,\delta}$. Using (34) and (36), we get

$$g \in C^\infty(\overline{S_\delta}), \quad g(r,\theta) = 0 \quad \text{for} \quad r \geqslant \tilde{r}_0/2, \quad g(\delta,\theta) = \varphi_{1,0}^{N_1+}(\delta,\theta) \quad \text{on} \quad w_{2,\delta},$$

$$g = \frac{\partial}{\partial n} g = 0 \quad \text{on} \quad w_1, \quad g = 0 \quad \text{on} \quad w_{4,\delta}, \quad \text{and} \quad \frac{\partial}{\partial n} g = 0 \quad \text{on} \quad w_{3,\delta}.$$

Notice that $\frac{\partial}{\partial n}\varphi_1^+ = 0$ on $w_{3,\delta}$ and $\varphi_1^+ = 0$ on $w_{4,\delta}$. Thus we obtain

$$\begin{aligned}
&((\Delta+\lambda_0)(\chi_0(r)\varphi_{1,0}^{N_1+}), \varphi_1^+)_{\Omega_+} \\
&= \lim_{\delta\to+0} \int_{\Omega_+\backslash D_+(0,\delta)} \varphi_1^+ (\Delta+\lambda_0) g(r,\theta)\, dx \\
&= \lim_{\delta\to+0} \int_{S_\delta} \varphi_1^+ (\Delta+\lambda_0) g(r,\theta)\, dx \\
&= \lim_{\delta\to+0} \left\{ \int_{\partial S_\delta} (\varphi_1^+ \frac{\partial}{\partial n} g - g \frac{\partial}{\partial n}\varphi_1^+)\, dS - \int_{S_\delta} g(\Delta+\lambda_0)\varphi_1^+\, dx \right\} \\
&= \lim_{\delta\to+0} \int_0^{\pi/2} \left(-\frac{\partial}{\partial r}\varphi_{1,0}^{N_1+}(\delta,\theta)\varphi_1^+(\delta,\theta) + \varphi_{1,0}^{N_1+}(\delta,\theta)\frac{\partial}{\partial r}\varphi_1^+(\delta,\theta) \right) \delta\, d\theta.
\end{aligned} \tag{58}$$

From (34) and (36), we obtain

$$\varphi_{1,0}^{N_1\pm}(x) = \tilde{k}_{0,1,1,0}^{\pm} r^{-1}\cos\theta + \mathcal{O}(r|\log r|) \quad \text{as} \quad r \to 0. \tag{59}$$

From (1), we have

$$\varphi_1^\pm(x) = C_{1,1}^\pm r\cos\theta + \mathcal{O}(r^3) \quad \text{as} \quad r \to 0. \tag{60}$$

Using (58)–(60), we get

$$((\Delta+\lambda_0)(\chi_0(r)\varphi_{1,0}^{N_1+}), \varphi_1^+)_{\Omega_+} = \frac{\pi}{2}\tilde{k}_{0,1,1,0}^{+} C_{1,1}^+. \tag{61}$$

Using (55)–(57) and (61), we get $\lambda_{1,0}^+ = -\frac{\pi}{2}\tilde{k}_{0,1,1,0}^+$. In a similar fashion, we obtain $\lambda_{1,0}^- = \frac{\pi}{2}\tilde{k}_{0,1,1,0}^-$. From (35), we have $\tilde{k}_{0,1,1,0}^\pm = \mp\frac{1}{4}((C_{1,1}^+)^2 + (C_{1,1}^-)^2)$. So we get $\lambda_{1,0}^+ = \lambda_{1,0}^- = \frac{\pi}{8}((C_{1,1}^+)^2 + (C_{1,1}^-)^2)$.

We set $\lambda_{1,0} = \lambda_{1,0}^+ (= \lambda_{1,0}^-)$. From (34), (36), (37), (47), and (48), we claim that the function $\check{\varphi}_{1,0}^\pm$, which is a solution of $(\mathrm{OUT})_{1,0}^\pm$, has the following asymptotic expansion which can be differentiated term by term infinitely many times.

$$\check{\varphi}_{1,0}^\pm(x) \sim \check{k}_{0,1,1,0} r^{-1}\cos\theta + \sum_{s=1}^{\infty}\sum_{j=1-s}^{0} \check{k}_{s,1,j,0}^\pm r^{2s-1}\cos(2j-1)\theta + \sum_{s=1}^{\infty} \check{l}_{s,1,s-1,0}^\pm r^{2s-1}\log r\cos(2j-1)\theta$$

as $r \to 0$, $x \in \Omega_\pm$. We rewrite this as

$$\check{\varphi}_{1,0}^\pm(x) \sim \sum_{s=0}^{\infty} \check{t}_{s,1,0}^\pm(r,\theta) + \sum_{s=1}^{\infty} \check{u}_{s,1,0}^\pm(r,\theta)\log r \quad (r \to 0,\ x \in \Omega_\pm), \tag{62}$$

where $\check{t}^{\pm}_{s,1,0}(\cdot,\theta)$ and $\check{u}^{\pm}_{s,1,0}(r,\theta)$ are homogeneous polynomials of degree $2s-1$. From (34)–(36), (47) and (48), we have

$$\check{t}^{\pm}_{0,1,0}(\cdot,\theta) = \tilde{h}^{\pm}_{0,1,0}(\cdot,\theta) = h^{\pm}_{0,1,0}(\cdot,\theta). \tag{63}$$

Let us show that there exists a solution $\check{v}_{2,0}$ of $(\mathrm{IN})_{1,0}$ satisfying the following (a)–(e).

(a) $\check{v}_{2,0}(\xi) = \check{v}^{+}_{2,0}(\xi) + \check{v}^{-}_{2,0}(\xi)$ for $\xi \in \Pi$.

(b) $\check{v}^{+}_{2,0}(\xi) = \sum_{j=1}^{2}\sum_{k=1}^{j} \check{C}^{+}_{2,j,k,0}\rho^{2j-1}\cos(2k-1)\theta$.

(c) The function $\check{v}^{-}_{2,0}(\xi_1,\cdot)$ is odd, and $\check{v}^{-}_{2,0}(\xi) \in \mathcal{A}\cap\mathcal{B}$.

(d) The function $\check{v}^{-}_{2,0}(\xi)$ has the following asymptotic expansion which can be differentiated term by term two times.

$$\check{v}^{-}_{2,0}(\xi) \sim (\sum_{l=0}^{1}\sum_{j=l-1}^{0} + \sum_{l=2}^{\infty}\sum_{j=l-1}^{l})\check{C}_{1,l,j,0}\rho^{2M-2l+1}\cos(2j-1)\theta + \check{d}_{1,1,0,0}\rho\log\rho\cos\theta \quad \text{as} \quad \rho\to\infty, \quad \xi\in P.$$

The conditions (a)–(d) allow us to write the asymptotic expansion of $\check{v}_{2,0}$ as follows.

$$\check{v}_{2,0}(\xi) \sim \sum_{l=0}^{\infty} \check{h}^{\pm}_{1,l,0}(\rho,\theta) + \check{m}^{\pm}_{1,1,0}(\rho,\theta)\log\rho \quad \text{as} \quad \rho\to\infty,\ \xi\in\Pi,\ \pm\xi_2>0, \tag{64}$$

where $\check{h}^{\pm}_{1,l,0}(\cdot,\theta)$ and $\check{m}^{\pm}_{1,l,0}(\cdot,\theta)$ are homogeneous polynomials of degree $3-2l$.

(e) We have

$$\check{h}^{\pm}_{1,1,0}(\cdot,\theta) = \check{t}^{\pm}_{1,1,0}(\cdot,\theta), \tag{65}$$

$$\check{h}^{\pm}_{1,0,0}(\cdot,\theta) = t^{\pm}_{1,0,0}(\cdot,\theta), \tag{66}$$

$$\check{m}^{\pm}_{1,1,0}(\cdot,\theta) = \check{u}^{\pm}_{1,1,0}(\cdot,\theta), \tag{67}$$

We seek a solution of $(\mathrm{IN})_{1,0}$ which takes the form

$$\hat{v}_{2,0}(\xi) = \hat{v}^{+}_{2,0}(\xi) + \hat{v}^{-}_{2,0}(\xi), \quad \xi\in\Pi, \tag{68}$$

where $\hat{v}^{+}_{2,0}(\xi_1,\cdot)$ and $\hat{v}^{-}_{2,0}(\xi_1,\cdot)$ are even and odd respectively. Inserting (68) into $(\mathrm{IN})_{1,0}$ and identifying the terms of the both sides which are even with respect to ξ_2, we get

$$\begin{gathered} \Delta\hat{v}^{+}_{2,0} = \lambda_0\tilde{v}^{+}_{1,0} \quad \text{on} \quad P, \\ \hat{v}^{+}_{2,0} = 0 \quad \text{on} \quad \{0\}\times(0,\infty), \qquad \frac{\partial}{\partial\xi_2}\hat{v}^{+}_{2,0} = 0 \quad \text{on} \quad (0,\infty)\times\{0\}. \end{gathered} \tag{69}$$

Inserting (68) into $(\mathrm{IN})_{1,0}$ and identifying the terms of the both sides which are odd with respect to ξ_2, we get

$$\begin{gathered} \Delta\hat{v}^{-}_{2,0} = \lambda_0\tilde{v}^{-}_{1,0} \quad \text{on} \quad P, \\ \hat{v}^{-}_{2,0} = 0 \quad \text{on} \quad \Gamma_2, \qquad \frac{\partial}{\partial\xi_2}\hat{v}^{-}_{2,0} = 0 \quad \text{on} \quad \Gamma_1. \end{gathered} \tag{70}$$

First, we consider the equation (69). Using (29), we infer that there exists a solution of (69) which takes the form

$$\hat{v}^{+}_{2,0}(\xi) = \hat{C}^{+}_{1,2,1}\rho^{3}\cos\theta. \tag{71}$$

Next, we consider the equation (70). From Proposition 2.3, we claim that there exists a solution $\hat{v}^{-}_{2,0} \in \mathcal{A}\cap\mathcal{B}$ of (70) having the following asymptotic expansion which can be differentiated term by term two times.

$$\hat{v}^{-}_{2,0}(\xi) \sim (\sum_{l=0}^{1}\sum_{j=l-1}^{0} + \sum_{l=2}^{\infty}\sum_{j=l-1}^{l})\hat{C}^{-}_{1,l,j}\rho^{3-2l}\cos(2j-1)\theta + \hat{d}_{1,1,0}\rho\log\rho\cos\theta \tag{72}$$

as $\rho \to \infty$, $\xi \in P$. We extend $\hat{v}_{2,0}^-(\xi)$ to an odd function with respect to ξ_2, which we denote by $\hat{v}_{2,0}^-(\xi)$ again. We define $\hat{v}_{2,0}(\xi)$ by (68), (71), and (72). Then we can write

$$\hat{v}_{2,0}(\xi) \sim \sum_{l=0}^{\infty} \hat{h}_{1,l,0}^{\pm}(\rho,\theta) + \hat{m}_{1,1,0}^{\pm}(\rho,\theta)\log\rho \quad \text{as} \quad \rho \to \infty, \quad \xi \in \Pi, \quad \pm\xi_2 > 0, \tag{73}$$

where $\hat{h}_{M,l,0}^{\pm}(\cdot,\theta)$ and $\hat{m}_{M,l,0}^{\pm}(\cdot,\theta)$ are homogeneous polynomials of degree $2M-2l+1$. By adding some harmonic function to $\hat{v}_{M+1,0}$, we shall construct $\tilde{v}_{2,0}$ satisfying (e) as well as (a)–(d). Let us show that

$$\hat{m}_{1,1,0}^{\pm}(r,\theta) = \check{u}_{1,1,0}^{\pm}(r,\theta), \tag{74}$$

$$\hat{h}_{1,0,0}^{\pm}(r,\theta) - t_{1,0,0}^{\pm}(r,\theta) = \alpha_1^{\pm} r^3 \cos 3\theta, \tag{75}$$

$$\hat{h}_{1,1,0}^{\pm}(r,\theta) - \check{t}_{1,1,0}^{\pm}(r,\theta) = \alpha_2^{\pm} r \cos\theta, \tag{76}$$

where $\alpha_1^{\pm}$ and $\alpha_2^{\pm}$ are some constant. From $(\mathrm{IN})_{1,0}$, (29), (30), and (71)–(73), we get

$$\Delta(\hat{m}_{1,1,0}^{\pm}(r,\theta)\log r) = -\lambda_0 h_{0,1,0}^{\pm}, \quad \Delta\hat{h}_{1,0,0}^{\pm} = -\lambda_0 h_{0,0,0}^{\pm},$$

$$\hat{h}_{1,1,0}^{\pm}(r,\theta) = \pm\hat{C}_{1,1,0} r\cos\theta. \tag{77}$$

Since $(\Delta+\lambda_0)\check{\varphi}_{1,0}^{\pm} = -\lambda_{1,0}\varphi_{0,0}^{\pm}$ on $\Omega_{\pm}$, we get $\Delta(\check{u}_{1,1,0}^{\pm}(r,\theta)\log r) = -\lambda_0 \check{t}_{0,1,0}^{\pm}(r,\theta)$. Since $(\Delta+\lambda_0)\varphi_{0,0}^{\pm} = 0$ on $\Omega_{\pm}$, we have $\Delta t_{1,0,0}^{\pm} = -\lambda_0 t_{0,0,0}^{\pm}$. So, we obtain

$$\Delta\{(\hat{m}_{1,1,0}^{\pm}(r,\theta) - \check{u}_{1,1,0}^{\pm}(r,\theta))\log r\} = -\lambda_0(h_{0,1,0}^{\pm} - \check{t}_{0,1,0}^{\pm}),$$

$$\Delta(\hat{h}_{1,0,0}^{\pm} - t_{1,0,0}^{\pm}) = -\lambda_0(h_{0,0,0}^{\pm} - t_{0,0,0}^{\pm}).$$

From $(M.2)_{0,0,0}^{\pm}$ and (63), we get

$$\Delta\{\hat{m}_{1,1,0}^{\pm}(r,\theta) - \check{u}_{1,1,0}^{\pm}(r,\theta))\log r\} = 0, \tag{78}$$

$$\Delta(\hat{h}_{1,0,0}^{\pm} - t_{1,0,0}^{\pm}) = 0. \tag{79}$$

Note that

$$\hat{m}_{1,1,0}^{\pm}(r,\theta) - \check{u}_{1,1,0}^{\pm}(r,\theta) = \pm\hat{d}_{1,1,0} r\cos\theta - \check{l}_{1,1,0,0}^{\pm} r\cos\theta, \tag{80}$$

$$\hat{h}_{1,0,0}^{\pm}(r,\theta) - t_{1,0,0}^{\pm}(r,\theta) = \hat{C}_{1,2,1}^{+} r^3\cos\theta \pm \sum_{j=-1}^{0} \hat{C}_{1,0,j}^{-} r^3 \cos(2j-1)\theta \mp C_{1,1}^{\pm} \sum_{k=1}^{2} C_{2,k}^{\pm} r^3 \cos(2k-1)\theta, \tag{81}$$

$$\check{t}_{1,1,0}^{\pm}(r,\theta) = \check{k}_{1,1,0,0}^{\pm} r\cos\theta. \tag{82}$$

From (77) and (82), we get (76). (75) follows from (79) and (81). We have (74) from (78) and (80). We put

$$\begin{aligned}\check{v}_{2,0}(\xi) =& \hat{v}_{2,0}(\xi) - \frac{1}{2}(\alpha_1^+ + \alpha_1^-)\rho^3\cos 3\theta - \frac{1}{2}(\alpha_1^+ - \alpha_1^-)U_2(\xi) \\ & - \frac{1}{2}(\alpha_2^+ + \alpha_2^-)\rho\cos\theta - \{\frac{3}{4}(\alpha_1^+ - \alpha_1^-) + \frac{1}{2}(\alpha_2^+ - \alpha_2^-)\}U_1(\xi).\end{aligned}$$

Then it follow from (15) and (74)–(76) that the function $\check{v}_{2,0}$ satisfies (a)–(e).

We modify $\check{\varphi}_{1,0}^{\pm}$ and $\check{v}_{2,0}$ such that the resulting functions satisfy the compatibility condition as well as the matching condition. We put

$$\varphi_{1,0}^{\pm} = \kappa^{\pm}\varphi_1^{\pm} + \check{\varphi}_{1,0}^{\pm}, \tag{83}$$

$$v_{2,0}(\xi) = \check{v}_{2,0}(\xi) + \frac{1}{2}(\kappa^+ C_{1,1}^+ + \kappa^- C_{1,1}^-)\rho\cos\theta + \frac{1}{2}(\kappa^+ C_{1,1}^+ - \kappa^- C_{1,1}^-)U_1(\xi), \tag{84}$$

$\kappa^{\pm}$ being a constant which we shall determine later. From (62), (83), and Proposition 1.1, we claim that the function $\varphi^{\pm}_{1,0}$, which is a solution of $(\mathrm{OUT})^{\pm}_{1,0}$ satisfying $(P5)_{1,0}$ and $(P7)_{1,0}$, has the asymptotic expansion:

$$\varphi^{\pm}_{1,0}(x) \sim \sum_{s=0}^{\infty} t^{\pm}_{s,1,0}(r,\theta) + \sum_{s=1}^{\infty} u^{\pm}_{s,1,0}(r,\theta)\log r \quad (r \to 0,\ x \in \Omega_{\pm}),$$

where $t^{\pm}_{s,1,0}(\cdot,\theta)$ and $u^{\pm}_{s,1,0}(\cdot,\theta)$ are homogeneous polynomials of degree $2s-1$. Besides we have

$$t^{\pm}_{0,1,0}(r,\theta) = \check{t}^{\pm}_{0,1,0}(r,\theta), \tag{85}$$

$$t^{\pm}_{1,1,0}(r,\theta) = \check{t}^{\pm}_{1,1,0}(r,\theta) + \kappa^{\pm} C^{\pm}_{1,1} r\cos\theta, \tag{86}$$

$$u^{\pm}_{1,1,0}(r,\theta) = \check{u}^{\pm}_{1,1,0}(r,\theta). \tag{87}$$

From (15), (64), and (84), we claim that the function $v_{2,0}$, which is a solution of $(\mathrm{IN})_{1,0}$ satisfying $(P1)_{1,0}$–$(P4)_{1,0}$, has the asymptotic expansion:

$$v_{2,0}(\xi) \sim \sum_{l=0}^{\infty} h^{\pm}_{1,l,0}(\rho,\theta) + m^{\pm}_{1,1,0}(\rho,\theta)\log\rho \quad \text{as} \quad \rho \to \infty,\ \pm\xi_2 > 0,$$

where $h^{\pm}_{1,l,0}(\cdot,\theta)$, and $m^{\pm}_{1,l,0}(\cdot,\theta)$ are homogeneous polynomials of degree $3-2l$. Besides we get

$$h^{\pm}_{1,0,0}(\rho,\theta) = \check{h}^{\pm}_{1,0,0}(\rho,\theta), \tag{88}$$

$$h^{\pm}_{1,1,0}(\rho,\theta) = \check{h}^{\pm}_{1,1,0}(\rho,\theta) + \kappa^{\pm} C^{\pm}_{1,1}\rho\cos\theta, \tag{89}$$

$$m^{\pm}_{1,1,0}(\rho,\theta) = \check{m}^{\pm}_{1,1,0}(\rho,\theta). \tag{90}$$

From (67), (87), and (90), we have

$$u^{\pm}_{1,1,0}(\cdot,\theta) = m^{\pm}_{1,1,0}(\cdot,\theta). \tag{91}$$

It follows from (63) and (85) that

$$t^{\pm}_{0,1,0}(\cdot,\theta) = h^{\pm}_{0,1,0}(\cdot,\theta). \tag{92}$$

From (65), (66), (86), (88), and (89), we get

$$t^{\pm}_{1,l,0}(\cdot,\theta) = h^{\pm}_{1,l,0}(\cdot,\theta) \quad \text{for} \quad 0 \leqslant l \leqslant 1. \tag{93}$$

Note that $(M.2)^{\pm}_{0,0,0}$ holds by the assumption. Combining this with (91), (92), and (93), we claim that $(M.1)^{\pm}_{1,1,0}$ holds and $(M.2)^{\pm}_{s,l,0}$ holds for $0 \leqslant s \leqslant 1$, $0 \leqslant l \leqslant 1$. Besides $(P1)_{1,0}$–$(P7)_{1,0}$ hold. Let us show that there exist κ^{+} and κ^{-} satisfying $(C)_{2,0}$. Notice that $(C)_{2,0}$ is equivalent to

$$-\frac{3}{2}\pi(C^{+}_{1,1})^{-1}C^{-}_{0,2,2,0}C^{+}_{2,2} + \frac{3}{2}\pi(C^{-}_{1,1})^{-1}C^{-}_{0,2,2,0}C^{-}_{2,2} = (C^{+}_{1,1})^{-1}\lambda_{1,0}(\varphi^{+}_{1,0},\varphi^{+}_{1})_{\Omega_{+}} + (C^{-}_{1,1})^{-1}\lambda_{1,0}(\varphi^{-}_{1,0},\varphi^{-}_{1})_{\Omega_{-}},$$

and the left side of the above equality is independent of κ^{+} and κ^{-}. The right side of the above equality is equal to $\lambda_{1,0}((C^{+}_{1,1})^{-1}\kappa_{+} + (C^{-}_{1,1})^{-1}\kappa_{-})$. Because $\lambda_{1,0} > 0$, there exist κ^{+} and κ^{-} satisfying $(C)_{2,0}$. □

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