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BC-method and Stability of Gel'fand inverse spectral problem

Atsushi KATSUDA
Department of Mathematics, Okayama University

This note is based on joint works with Y.V.Kurylev (Loughborough Univ. UK) and M. Lassas (Helsinki Univ. Finland) [4,5].

1 Introduction

A well known problem in geometry is the title of M. Kac's celebrated paper,

'Can one hear the shape of a drum?'

This is the question whether the spectrum of the Laplacian determine the geometry of the underlying manifold. It is known that in general the answer is negative even in the 2-dimensional case and the possible answer at present is essentially one-dimensional. From this situation, there seems to be a consideration as follows; The information on the spectrum is on real line, i.e. essentially one dimensional and thus, to obtain information in multidimensional case we need to have more spectral information. The rigorous definition of this type is given by Gel'fand. Its original form is the detemination of the potential of the Schrödinger operator on a bounded domain in $\mathbf{R}^n$. In the case of a Riemannian manifold with boundary $M = (M, \partial M, g)$, it is modified to the following;

Problem (Generalized Gel'fand inverse spectral problem): Let Boundary spectral data (BSD)

$$\{\partial M, \lambda_j, \phi_j|_{\partial M},\ j = 1, 2, \ldots\}$$

be given where λ_j and ϕ_j are the eigenvalues and the $L^2(M)$-orthonormal eigenfunctions of the Neumann Laplacian $-\Delta_g$. Do these data determine (M, g)?

The answer of this question is already known.

Theorem 1 *(Belishev-Kurylev[1] + Tataru[7]) There exists a reconstruction method of M and g from BSD.*

The method of proof is so called 'Boundary Control (BC) method', which is invented by Belishev and developped in the paper of Belishev and Kurylev. Their paper state in the real analytic category. They need to use Holmgrem-John unique continuation theorem. In later, this can be subtituted by Tataru's result to the case of smooth manifold. Concerning to our result, it should be remarked that their method need to all BSD even in approximation of manifold.

The natural question that rises in the study of the Gel'fand problem is the stability.

Problem 1: If BSD's of M and M' are close, then are M and M' themselves close?
Problem 2: Let the Finite part of Boundary Spectral Data (FBSD)

$$\{\partial M, \lambda_j, \phi_j|_{\partial M},\ j = 1, \ldots, N\}$$

be approximately given. Do these data determine an approximation of (M, g) in a stable way?

However, it is well known that inverse problems are generally ill-posed. In our case this implies that even the topological class of the manifold can not be stably reconstructed for FBSD without some additional assumption. For example, adding a small handle essentially does not change the small eigenvalues or eigenfunctions at the boundary. Thus we need to consider the conditional stability, which is a restriction of a class of manifolds in our case. Our results are answering to the above questions, which are roughly stated as follows;

Theorem 2 *(i) There exist a class $\mathbf{M}$ of manifolds such that if $M, M' \in \mathbf{M}$ have a large number of similar eigenvalues and similar boundary restrictions of eigenfunctions, then M, M' are diffeomorphic and have similar Riemannian metrics.*

(ii) If we know sufficiently large number of BSD $\{(\lambda_i, \phi|_{\partial M})\}$ of $M \in \mathbf{M}$ with sufficiently small error, then there exists a discrete metric space approximating M.

(iii) Under more restriction of a class, explicit estimate in approximation is possible.

Note that our class **M** is in some sense "compact" and it is known that a kind of compactness argument and uniqueness implies a stability. However, there are two defects; NO ALGORITHMS and NO ESTIMATES. We answer these points here.

More precise form of our results are given by introducing several notions. Our class $\mathbf{M} = \mathbf{M}(\partial M; m, \Lambda, D, i_0)$ is usually called the class of bouned geometry. It consists of m-dimensional manifolds M with fixed boundary ∂M satisfying

$$|K_M| < \Lambda, \ |k_{\partial M}| < \Lambda, \ \ \text{diam } (M) < D, \ i_M > i_0$$

where K_M is the sectional curvature, $k_{\partial M}$ is the principal curvature of ∂M, diam (M) is the diameter and i_M is the minimum of the injectivity radiuses and the boundary injectivity radius. Here the injectivity radius (resp. the boundary injectivity radius) is the largest radius s of neighborhood such that for any point $p \in M$ (resp. ∂M), the exponential map (resp. the boundary exponential map) is diffeomorphism to its image.

Topology on **M** is defined by an approximation of discrete metric spaces (nets), which is induced from the following Gromov-Hausdorff distance d_{GH}. For $M, M' \in \mathbf{M}$, $d_{GH}(M, M') < \delta$ if and only if there exist δ_1-nets $\{m_i\} \subset M$, $\{m_i'\} \subset M'$ and $\delta_2 > 0$ satisfying

$$\frac{1}{1+\delta_2} < \frac{d(m_i', m_j')}{d(m_i, m_j)} < 1 + \delta_2,$$

where $\delta = \min(\delta_1, \delta_2)$.

Our result based on the "Gromov compactness" theorem. The following version is due to Kodani.

Theorem 3 *(Kodani) (i)* **M** *is precompact with respect to* d_{GH}.

(ii) If $M, M' \in \mathbf{M}$ *is sufficiently close in Gromov-Hausdorff topology, thenthey are diffeomorphic and the Lipshitz constants between them are sufficiently close to one.*

Moreover, we need a stricter result that the closure of **M** is compact in $C^{1,\alpha}$-topology, which is proven by elliptic thoery. In fact, the arguments in [2] can be extended to the case of manifolds with boundary.

We also need to introduce topology of the set of BSD.

Definition 1 *A collection $\{\mu_j, \psi_j|_{\partial M}\}$ and the collection $\{\lambda_j, \phi_j|_{\partial M}\}$ satisfy*

$$d_{BSD}(\{(\mu_j, \psi_j)\}, \{(\lambda_j, \phi_j)\}) < \delta$$

if there exist disjoint intervals $I_p \subset [0, \delta^{-1}]$, $p = 1, \ldots, P$ such that

i. All $\lambda_j, \mu_j < \delta^{-1} - \delta$ belong to $\bigcup_{p=1}^{P} I_p$

ii. The length $|I_p|$ satisfy $|I_p| < \delta$

iii. $d(I_p, I_q) > \delta^b$ where $b = \frac{m}{2} + 2$. ($m = dimM$)

vi. On interval I_p the number n_p of the points μ_j is equal to the number of points λ_j.

v. There are unitary matrices $(a_{jk}) \in \mathbf{C}^{n_p \times n_p}$ such that

$$\|\sum_{\lambda_j, \lambda_k \in I_p} a_{jk}\phi_k - \psi_j\|_{H^{1/2}(\partial M)} < \delta.$$

The proof of Theorem 2 consists of analytic part and geometric part. Analytic part is based on BC method. Geometric part is an argument in Riemannian geometry.

2 Anaytic part

The BC-Method gives the information the distance function $r_p(y) := d(p, y)$ from points $p \in M$ to each points y on the boundary, which we call the boundary distance function. We devide into two steps.

2.1 Recongnization of the domain of influence

Main point here is to obtain the information of the domain of influence of the following wave equation from BSD. Consider the hyperbolic initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 \text{ in } M \times [0, T] \\ \partial_\nu u &= f \text{ on } \partial M \times [0, T] \\ u|_{t=0} &= u_t|_{t=0} = 0 \end{aligned}$$

for $f \in H^1([0,T], L^2(\partial M))$. We will denote its solution by $u^f(x,t)$ and define

$$W_T : H^1([0,T], L^2(\partial M)) \to L^2(M)$$

by $W_T(f) = u^f(\cdot, T)$. Take the eigenfunction expansions

$$u^f(x,t) = \sum_{j=0}^{\infty} u_j^f(t)\phi_j(x)$$

where

$$u_j^f(t) = (u^f, \phi_j) := \int_{M\times\{t\}} u^f(x,t)\phi_j(x)dx.$$

Note that the Stokes theorem implies the following.

Proposition 1

$$(u^f, \phi_j) = (f, S_j^t) := \int_0^t \left(\int_{\partial M} S_j^t(y,x)\, f(y,s)\, dS_y \right) ds$$

where

$$S_j^t(y,s) = \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} \phi_j(y).$$

Take an open subset $\Gamma \subset \partial M$ and consider the domain of influence

$$M(\Gamma, t) = \{x \in M = M \times \{T\} | d(x,\Gamma) < t\}.$$

The finite propagation property of the wave equation implies

$$W_T(H^1([T-t,T],\Gamma)) \subset M(\Gamma, t),$$

where

$$H^1([T-t,T],\Gamma) = \{f \in H^1([0,T], L^2(\partial M)) | \mathrm{supp} f \subset [T-t,T] \times \Gamma\}.$$

Moreover the following holds.

Theorem 4 *(Tataru[7])*

$$Cl_{L^2}(W_T(H^1([T-t,T],\Gamma))) = L^2(M(\Gamma,t)),$$

where we identify $L^2(M(\Gamma,t))$ with the set of fuctions in $L^2(M)$ whose support are contained in $M(\Gamma,t)$.

The above property is called approximate controllability. We can read Proposition 1 and Theorem 4 as follows; If we know all information of BSD, then we know all coefficients of u in the dense subset $W_T(H^1([T-t,T],\Gamma))$ of $L^2(M(\Gamma,t))$. Then we can recongnize $M(\Gamma,t)$. For example, the emptiness of the set

$$M(\Gamma_1,t_1,t_2)\cap M(\Gamma_2,t_3,t_4),$$

with

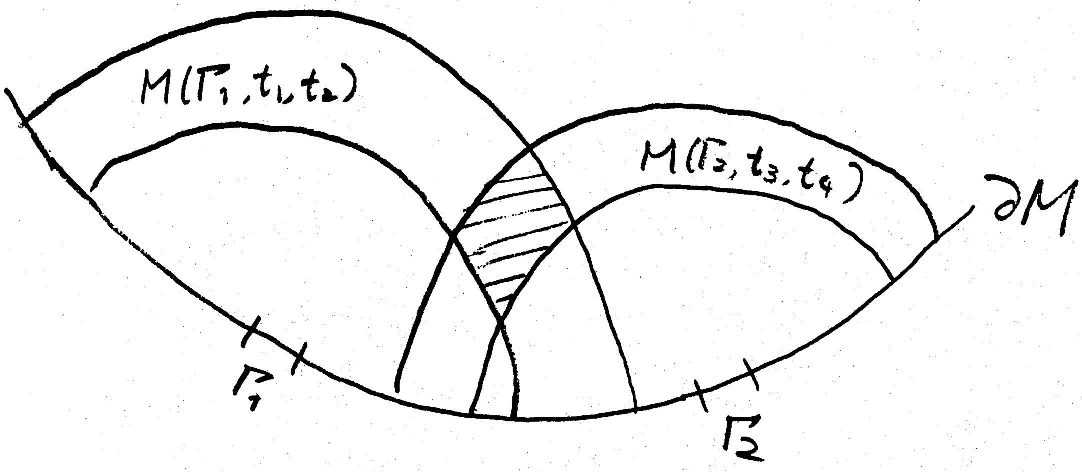
$$M(\Gamma_1,t_1,t_2)=M(\Gamma_1,t_2)\setminus M(\Gamma_1,t_1)$$

is nothing but that of the intersection of sets

$$\{u_j^f | f\in (H^1([T-t_2,T],\Gamma_1)\setminus H^1([T-t_1,T],\Gamma_1), j=1,2,\cdots\}$$

and

$$\{u_j^f | f\in (H^1([T-t_4,T],\Gamma_2)\setminus H^1([T-t_3,T],\Gamma_2), j=1,2,\cdots\}.$$



In our case, we only know the finite information of BSD containing some errors. We have the following two results in this moment;

(a) A result under less assumption using compactness arguments (no estimate); First, we note that the wave operator and BSD are continuous with respect to the Gromov-Hausdorff distance d_{GH} in M. In fact, if $d_{GH}(M,M')$

is small, then, by Theorem 3, M and M' are diffeomorphic and these Riemannian metrics are similar and thus we consider quantities on fixed manifold, Then, the usual perterbation arguments can be applied to imply the continuity.

Put

$$S^T_{\Gamma,j}(y,t) = \mathcal{X}_\Gamma(y) S^T_j(y,t)$$

for the characteristic function $\mathcal{X}_\Gamma$ of $\Gamma \subset \partial M$.

Proposition 2 *Given $m, \Lambda, D, i_0 > 0$ and $\Gamma \subset \partial M, t, \varepsilon > 0$, there exist $\delta > 0$ and a_j $(j = 0, 1, \cdots j_0 = [1/\delta - \delta])$ such that if $M \in \mathbf{M}(\partial M; m, \Lambda, D, i_0)$ with BSD $\{(\mu_j, \psi_j|_{\partial M})\}$ and $d_{BSD}(\{(\mu_j, \psi_j|_{\partial M})\}, \{(\lambda_j, \phi_j|_{\partial M}\}) < \delta$, then*

$$\|\mathcal{X}_{M(\Gamma,t)}\phi_0 - W_T(\sum_{j=0}^{j_0} a_j S^T_{\Gamma,j})\|_{L^2(M)} < \varepsilon,$$

where $\mathcal{X}_{M(\Gamma,t)}$ is the characteristic function of $M(\Gamma, t)$. Moreover, we have an algorithm to find $\{a_j\}$.

Key point here is uniformity of δ in the class $\mathbf{M}$ and computability of $\{a_j\}$. Since the proof of this propositon is rather technical, we present here only typical arguments used in the proof. For fixed $M \in \mathbf{M}$ and $t > 0$, there exists a function f such that

$$\|\mathcal{X}_{M(\Gamma,t)}\phi_0 - W_T(f)\| < \varepsilon/10$$

by Theorem 4 and thus, there exist $j_1 > 0$ and b_j such that

$$\|\mathcal{X}_{M(\Gamma,t)}\phi_0 - W_T(\sum_{j=0}^{j=j_1} b_j S^T_{\Gamma,j}\| < \varepsilon/10.$$

by the density of $S^T_{\Gamma,j}$. Then, by the continuty of d_{GH}, we may assume the above j_1 can be chosen uniformly in some neighborhood of M and thus, uniformly in whole $\mathbf{M}$ by the compactness in Theorem 3.

To obtain an algorithm, we need to transform the problem of finding $\{a_j\}$ into the minimizing problem of a nonlinear functional on a finite dimensional space. This is done by approximating the space spanned by the all functions appearing the above inequalities by the boundary objects like $S^T_{\Gamma,j}$ etc. and reduce everything into the computaion on the boundary. But we omit it here.

(b) A result with explicit estimate; Here we recongnize $M(\Gamma, t)$ by the following stabiliy estimate. Assume tha $s_1 > s_0 > 0$.

Proposition 3 *Given $\Gamma \subset \partial M, t, c, \varepsilon_0, \varepsilon_1 > 0$, there exists $\delta > 0$ such that if $u \in Domain(\Delta)$ satisfies*

$$\|u\|_{L^2(M)} < 1, \|u\|_{H^{s_1}(M)} < c$$

and the solution $U(x,t)$ of the following equation

$$\begin{aligned} U_{tt} - \Delta U &= 0 \\ U|_{\partial M \times \mathbf{R}} &= 0 \\ U_t|t = 0 = 0 \qquad & U|_{t=0} = u \end{aligned}$$

satisfies

$$\|\partial_\nu U\|_{H^{s_0-1}(\Gamma\times[-t,t])} < \delta,$$

then

$$\|U|_{M(\Gamma,t-\varepsilon_0)\times t}\| < \varepsilon_1$$

This propostion is used to relate approximation errors of functions M with support $M(\Gamma, t)$ by $\phi_j|_{M(\Gamma,t)}$ and functions in $\partial M \times \mathbf{R}$ in the support $\Gamma \times [0, t]$ by $S^T_{\Gamma,j}$, from which we obtain the information on $M(\Gamma, t)$ approximately. We also omit the details.

Theorem 4 and Propsition 2 are obtained from the Carleman estimate due to Tataru and the similar stability estimate is written in Tataru's lecture note in his homepage. But we believe that our proof, although similar to Tataru, clarifies the dependence of geometric quantities more explicitly.

2.2 Reconstruction of the boundary distance.

We explain reconstruction of the boundary distance r_p in the situation (a). First we note that the volume of $M(\Gamma, t)$ is computable approximately. Put $\tilde{u} = \sum_{j=0}^{j_0} a_j S^T_{\Gamma,j}$ in Proposition 2. Then we have

$$\mathrm{Vol}(M(\Gamma, t)) = \mathrm{Vol}(M)(\mathcal{X}_{M(\Gamma,t)}\phi_0, \phi_0) \approx (W^T(\tilde{u}), \phi_0) = (\tilde{u}, S^t_0)$$

Here we call the right hand side the approximate volume $v^{\mathrm{app}}(M(\Gamma, t))$ of $M(\Gamma, t)$.

Next, take a partition $\{\Gamma_j\}$ of ∂M satisfying

$$\bigcup_{j=1}^{L} \Gamma_j = \partial M$$

and diam $(\Gamma_j) < \sigma$ and the partition $\{t_j = j\sigma\}$ be of interval $[0, T]$. Let α be a multi-indexes

$$\alpha = (\alpha_1, \ldots, \alpha_L), \ \alpha_j \in \mathbf{Z}.$$

We need to analyze the sets

$$I_\alpha = \bigcap_{j=1}^{L} \{x \in M : \ d(x, \Gamma_j) \in](\alpha_j - 2)\sigma, (\alpha_j + 2)\sigma]\}.$$

The set I_α is either a small 'cube' in M or an empty set. In the case where I_α is not empty, it contains a point x for which the corresponding boundary distance function r_p has approximately value $\alpha_j\sigma$ on the intervals Γ_j. To distinguish I_α is empty or not, we need to compute the approximate volume of I_α. This is done by repetition of the following type of arguments.

$$\begin{aligned} v^{\mathrm{app}}(M(\Gamma_1, t_1, t_2) \cap M(\Gamma_2, t_3, t_4)) \ &\approx \ v^{\mathrm{app}}(M(\Gamma_1, t_1, t_2)) + v^{\mathrm{app}}(M(\Gamma_2, t_3, t_4)) \\ &- \ v^{\mathrm{app}}(M(\Gamma_1, t_1, t_2) \cap M(\Gamma_2, t_3, t_4)) \end{aligned}$$

and

$$v^{\mathrm{app}}(M(\Gamma_1, t_1, t_2) \approx v^{\mathrm{app}}(M(\Gamma_1, t_2)) - v^{\mathrm{app}}(M(\Gamma_1, t_1)).$$

3 Geometric part

There are two methods to reconstruct the interior distance. It is obtained only from the boundary distance by the first method. The second method uses reconstruction of the heat kernel.

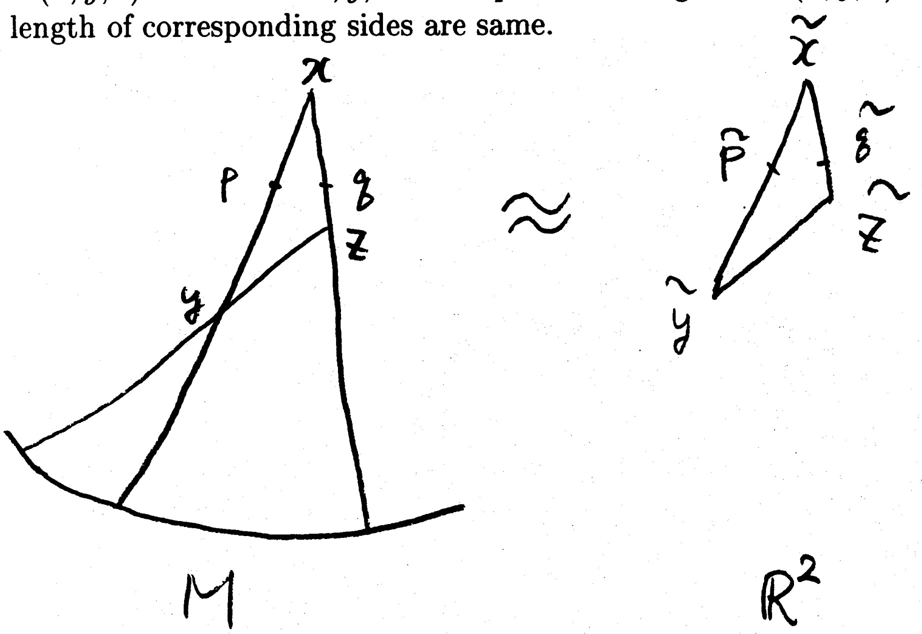
3.1 Reconstruction of the interior distance from the boundary distance

The interior distance are recongnized in the following order.

(1) If there is a geodesics from p to q can be extended minimally to the boundary point $y \in \partial M$, then we have done by

$$d(p,q) = r_p(y) - r_q(y)$$

(2) When this is not the case, we try to find a triangle, whose sides are parts of minimal geodesics to the boundary, which is called "good" triangle, such that points p, q are on its sides. Then, since the length of sides are obtained from arguments in (1), we can estimate $d(p,q)$ using comparison triangle on the euclidean plane. Here, we say that an Euclidian triangle $\triangle(\tilde{x},\tilde{y},\tilde{z})$ of verteces $\tilde{x},\tilde{y},\tilde{z}$ is çomparison triangle of $\triangle(x,y,z)$ in M when length of corresponding sides are same.



(3) The real problem is finding "good" triangles. If we assume the extra condition on the bound of the curvature derivative, we can find "good" triangles satisfying the situation (2) for any sufficiently close points $p, q \in M$. Namely, we can recongnize the approximate interior distance directly. In our case, we replace it by constructing a net such that distances between its elements are recongnizable approximately. It is done by arguments essentially based on the measure theory.

3.2 Reconstruction of the interior distance from the heat kernel

First we note that, similarly to the previous section, eigenfunctions ϕ_j can also be reconstructed from BSD. If we know full data of BSD, then the interior distance can be reconstructed from the following well known equalities. We denote $k(t, x, y)$ is the heat kernel.

$$k(t, x, y) = \sum_{i=0}^{\infty} e^{\lambda_i t} \phi_(x) \phi_(y),$$

and

$$\lim_{t \to 0} k(t.x.y) = \frac{1}{4} d^2(x, y).$$

To proceed this kind of arguments under the knowledge on FBSD, upper and lower estimates of $k(t, x, y)$ are neccessary. Available estimates now hold under the assumption on the Ricci curvature (weaker than assumptions on M) but in the case of convex boundary (stronger than assumptions on M) .

Last but not least, the boundary spectral distance d_{BSD} can be expressed bya boundary version of spectral distance defined by Kasue-Kumura [3] using heat kernel. It should be intersting to investigate to inverse problem connected with spectral convergence in their sense; e.g. What conditions assure that the boundary spectral convergence imply the spectral convergence of manifolds themselves?

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There is survey article of BC method by Belishev in Inverse Problem **13**(1997)R1-R45. Moreover, a Book on BC-method is in progress of writing by Katchalov, Kurylev and Lassas. The method in this book will be explained by using the Gaussian beam in their book, which is another version of BC-method. This is a little bit different to here.