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<td>Kadowaki, Mitsuteru</td>
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ON EXISTENCE OF SCATTERING SOLUTIONS FOR DISSIPATIVE SYSTEMS

MITSUTERU KADOWAKI (門脇 光輝)

In this report we shall give two frameworks (Theorem 1 and 3) for the existence of scattering solutions of dissipative systems and apply these to some dissipative wave equations.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. This norm is denoted by $\| \cdot \|_{\mathcal{H}}$. Let $\{V(t)\}_{t \geq 0}$ and $\{U_0(t)\}_{t \in \mathbb{R}}$ be a contraction semigroup and a unitary group in $\mathcal{H}$, respectively. We denote these generators by $A$ and $A_0$ ($V(t) = e^{-itA}$ and $U_0(t) = e^{-itA_0}$). We make the following assumptions on $A$ and $A_0$.

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R}$ or $[0, \infty)$.

(A2) $(A-i)^{-1} - (A_0-i)^{-1}$ defined as a form is extended to a compact operator $K$ in $\mathcal{H}$.

(A3) There exist non-zero projection operators $P_+$ and $P_-$ such that $P_+ + P_- = I_d$ and

(A3.1) $\| KU_0(t) \psi(A_0) P_+ \| \in L^1(\mathbb{R}_+)$,

(A3.2) $\| K^*U_0(t) \psi(A_0) P_+ \| \in L^1(\mathbb{R}_+)$,

(A3.3) $\| K^*U_0(-t) \psi(A_0) P_- \| \in L^1(\mathbb{R}_+)$,

(A3.4) $\lim_{t \to -\infty} U_0(-t) \psi(A_0) P_- f_t = 0$,

for each $\psi \in C_0^\infty(\mathbb{R}\backslash 0)$ and $\{f_t\}_{t \in \mathbb{R}}$ satisfying $\sup_{t \in \mathbb{R}} \| f_t \|_{\mathcal{H}} < \infty$, where $\| \cdot \|$ is the operator norm of bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

Let $\mathcal{H}_b$ be the space generated by the eigenvectors of $A$ with real eigenvalues.

**Theorem 1.** Assume that (A1) $\sim$ (A3). For any $f \in \mathcal{H}_b^\perp$, the wave operator $Wf = \lim_{t \to -\infty} U_0(-t)V(t)f$ exists. Moreover $W$ is not zero as an operator in $\mathcal{H}$.

To prove Theorem 1 we shall use the following facts (see [17] and [14]):

(F1) $\{(A-i)^{-2}Af \in \mathcal{H} : f \in D(A) \cap \mathcal{H}_b^\perp\}$ is dense in $\mathcal{H}_b^\perp$.

(F2) There exists a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} t_n = \infty$$

and

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w - \lim_{n \to \infty} V(t_n)f = 0, \text{ for any } f \in \mathcal{H}^k_\nu.

Theorem 1 implies that there exists scattering states of $\frac{dV(t)f}{dt} = -iAV(t)f, f \in D(A)$ as follows:

**Corollary 2.** Assume that (A1) \sim (A3). Then there exist non-trivial initial data $f \in \mathcal{H}$ and $f_+ \in \mathcal{H}$ such that for any $k = 0, 1, 2, \cdots$, and $\zeta_0 \in \mathbb{C}$ satisfying $\Re \zeta > 0$

$$\lim_{t \to \infty} \| V(t)(A - \zeta_0)^{-k}f - U_0(t)(A_0 - \zeta_0)^{-k}f_+ \|_{\mathcal{H}} = 0.$$  

Theorem 1 is proven by using Enss's approach [3] and [17]. Examples of Theorem 1 contain scattering problem for elastic wave equation with dissipative boundary condition in a half space of $\mathbb{R}^3$ (cf. [2]). To show (A3) we use the Mellin transformation (cf.[13]). Theorem 1 is not applied to acoustic wave equations with dissipative terms in stratified media(cf. [19]). Since generalized eigenfunctions of acoustic wave propagation in stratified media are not smooth at thresholds, the key estimates (A3.1)\sim (A3.3) have not been obtained in the neighborhood of each threshold. So we consider the following assumptions to deal with such equations.

Let $B_0$ be non-negative operator.

(A4) $B_0$ is $A_0$-compact.

(A5) Let $\zeta$ belong to $\mathbb{C} \backslash \mathbb{R}$. $\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}$ can be extended to a bounded operator $Q(\zeta)$ which satisfies that for any $\beta > \alpha > 0$, there exist positive constants $C_{\alpha, \beta}$ and $e$ such that

$$\sup_{\alpha \leq |Re\zeta| \leq \beta, 0 < |Im\zeta| < \eta} \| Q(\zeta) \| \leq C_{\alpha, \beta}.$$  

We reset $A = A_0 - iB_0$, $D(A) = D(A_0)$. Then [15] (see Theorem X-50) implies that $A$ generates a contraction semi-group, $\{V(t)\}_{t \geq 0}(V(t) = e^{-itA})$.

We have the following theorem.

**Theorem 3.** Assume that (A1), (A4) and (A5). Then

1. $A$ has no real eigenvalues.
2. The wave operator

$$W = s - \lim_{t \to \infty} U_0(-t)V(t)$$

exists. Moreover $W$ is not zero as an operator in $\mathcal{H}$.

**Corollary 4.** Assume that (A1), (A4) and (A5). Then we have the same conclusion of Corollary 2.

To prove Theorem 3 we shall used Mochizuki's idea [12] due to Kato's smooth perturbation theory [8].

In §4 we shall apply our frameworks to elastic wave equation with dissipative boundary condition in a half space of $\mathbb{R}^3$ and acoustic wave equation with dissipative term in stratified media. It seems that there is little literature concerning such dissipative systems (cf. [7]).
2. Proof of Theorem 1 and Corollary 2.

In this section we deal the case \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \) only. The another case can be dealt in the same way. We set \( F(\lambda) = (\lambda - i)^{-2}\lambda \) and \( W(t) = U_0(-t)V(t) \).

In this section \( C \) is used as positive constants.

Below we shall give the proof of Theorem 1. First we prove the existence of \( W \) by refering to [3], [17], [10], [13], [4], [18] and [14]. But we sometimes omit to note the above references.

**proof of the existence of \( W \).** For any \( f \in \mathcal{H} \cap D(A) \) and \( t, s > t_n \), note (F1) and

\[
\|(W(t) - W(s))F(A)^2f\|_{\mathcal{H}} \\
\leq ||(W(t) - W(t_n))F(A)^2f||_{\mathcal{H}} + ||(W(s) - W(t_n))F(A)^2f||_{\mathcal{H}}.
\]

Thus in order to prove the existence of \( W \), it is sufficient to show

\[
(2.1) \quad \lim_{n \to \infty} \lim_{t \to \infty} ||(W(t) - W(t_n))F(A)^2f||_{\mathcal{H}} = 0
\]

(cf. [4])

We estimate \( ||(W(t) - W(t_n))F(A)^2f||_{\mathcal{H}} \) as follows (cf. [17]):

\[
||(W(t) - W(t_n))F(A)^2f||_{\mathcal{H}} = ||U_0(-t)(V(t-t_n) - U_0(t-t_n))F(A)^2V(t_n)f||_{\mathcal{H}}
\]

\[
\leq \sum_{j=1}^{5} ||T_j||_{\mathcal{H}},
\]

where

\[
T_1 = (V(t-t_n) - U_0(t-t_n))(F(A)^2 - F(A_0)^2)V(t_n)f,  \\
T_2 = (V(t-t_n) - U_0(t-t_n))(I_d - \psi_M(A_0))F(A_0)^2V(t_n)f,  \\
T_3 = (V(t-t_n) - U_0(t-t_n))(\psi_M(A_0)P_+(A_0)P^-F(A_0)V(t_n)f,  \\
T_4 = (V(t-t_n) - U_0(t-t_n))(\psi_M(A_0)P_+(A_0)P^-F(A_0)(I_d - \psi_M(A_0))V(t_n)f,  \\
T_5 = (V(t-t_n) - U_0(t-t_n))(\psi_M(A_0)P_+(A_0)P^-F(A_0)V(t_n)f
\]

and \( \psi_M(\lambda) \in C_0^\infty(\mathbb{R}) \) satisfies \( 0 \leq \psi_M(\lambda) \leq 1, \psi_M(\lambda) = 0(|\lambda| < 1/2M, |\lambda| > 2M) \) and \( \psi_M(\lambda) = 1(1/M < |\lambda| < M) \).

First, we note that for any \( \epsilon \), there exists \( M > 0 \) such that

\[
||T_j||_{\mathcal{H}} \leq C||(1 - \psi_M)F||_{L^\infty(\mathbb{R})} < \epsilon \quad (j = 2, 4)
\]

Therefore once the limits

\[
(2.2) \quad \lim_{n \to \infty} \lim_{t \to \infty} ||T_j||_{\mathcal{H}} = 0, \quad (j = 1, 3, 5)
\]

are proved, we obtain (2.1). Below we shall show (2.2). For \( j = 1 \) (A2) implies that \( F(A)^2 - F(A_0)^2 \) is a compact operator in \( \mathcal{H} \). Using (F2) we have

\[
||T_1||_{\mathcal{H}} \leq C||(F(A)^2 - F(A_0)^2)V(t_n)f||_{\mathcal{H}} \to 0 \quad (n \to \infty)
\]
For $j = 3$, we decompose $T_3$ as follows

$$T_3 = T_{31} + T_{32} + T_{33},$$

where

$$T_{31} = V(t - t_n)(F(A_0) - F(A))(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$

$$T_{32} = (F(A) - F(A_0))U_0(t - t_n)(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$

$$T_{33} = F(A)(V(t - t_n) - U_0(t - t_n))\psi_M(A_0)P_+F(A_0)V(t_n)f$$

The same argument as in the proof of $T_1$ implies

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_{31}\|_\mathcal{H} = 0.$$

We have by (A1)

$$w - \lim_{t \to \infty} U_0(t - t_n)f = 0.$$

Thus (A2) implies

$$\lim_{t \to \infty} \|T_{32}\|_\mathcal{H} = 0.$$

To estimate $T_{33}$, we use Cook-Kuroda method. We have by (A2)

$$\langle T_{33}, g \rangle_\mathcal{H}$$

$$= -i \int_0^{t-t_n} \langle V(t-t_n-s)A(A-i)^{-1}KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n, g \rangle_{\mathcal{H}} ds,$$

where $g \in \mathcal{H}$, $f_n = V(t_n)f$ and $\tilde{\psi}_M(\lambda) = (\lambda - i)\psi_M(\lambda)$.

Therefore we obtain

$$\|T_{33}\|_\mathcal{H} \leq C \int_0^\infty \|KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n\| ds.$$  

For each $s \geq 0$ we have by (F2) and (A2),

$$\lim_{n \to \infty} \|KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n\|_\mathcal{H} = 0.$$

Therefore (A3.1) and Lebesgue's theorem imply

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_{33}\|_\mathcal{H} = 0.$$

Now we obtain

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_3\|_\mathcal{H} = 0.$$

We estimate $T_5$ as follows:

$$\|T_5\|_\mathcal{H}^2 \leq C \|P_- (F\psi_M)(A_0)V(t_n)f\|_\mathcal{H}^2$$

$$= C \sum_{j=1}^{3} T_{5j},$$
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where

\[ T_{51} = (\psi_M(A_0)P_- h_n, (F(A_0) - F(A))V(t_n)f)_\mathcal{H} \]
\[ T_{52} = (\psi_M(A_0)P_- h_n, (V(t_n) - U_0(t_n))F(A)f)_\mathcal{H} \]
\[ T_{53} = (U_0(-t_n)\psi_M(A_0)P_- h_n, F(A)f)_\mathcal{H} \]

and \( h_n = (F\psi_M)(A_0)V(t_n)f \).

(A2) and (F2) imply

\[ \lim_{n \to \infty} T_{51} = 0. \]

(A3.4) implies

\[ \lim_{n \to \infty} T_{53} = 0. \]

To estimate \( T_{52} \), again we use Cook-Kuroda method. Note that

\[ |T_{52}| \leq C ||f||_\mathcal{H} \int_0^\infty \|K^* U_0(-s)\tilde{\psi}_M(A_0)P_- l_{l_n}\|_\mathcal{H} ds. \]

Using (A2), (F2) and (A3.2) we have by Lebesgue’s theorem

\[ \lim_{n \to \infty} T_{52} = 0. \]

Now we obtain

\[ \lim_{n \to \infty} \lim_{t \to \infty} ||T_5||_\mathcal{H} = 0. \]

Therefore the proof of the existence of \( W \) is completed. \qed

To show \( W \not\equiv 0 \), we introduce a subspace of \( \mathcal{H} \), \( D \), as follows:

\[ D = \{ f \in \mathcal{H} : \lim_{t \to \infty} V(t)f = 0 \}. \]

Since

\[ Af = \lambda f, \lambda \in \mathbb{R}, f \in \mathcal{H} \implies A^* f = \lambda f \]

(see Lemma 1.1.5 of [14]), we can easily show

\[ D \subset \mathcal{H}_b^\perp. \]

We prepare the following proposition without the proof.

**Proposition 2.1.** Assume that

\[ \mathcal{H}_b^\perp \cap D = \{0\}. \]

Then one has

\[ w - \lim_{t \to \infty} U_0(-t)V(t)f = 0 \]

(2.3)
for any \( f \in \mathcal{H} \).

Below we shall show \( W \not\equiv 0 \) (cf. [12]§3).

**Proof of \( W \not\equiv 0 \)**. For any \( f \in \mathcal{H} \) and \( g \in \mathcal{H} \), note that

\[
\langle U_0(-t)V(t)(A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}}
= \langle (A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}} + i \int_0^t \langle V(\tau)f, K^*U_0(\tau)g, \rangle_{\mathcal{H}}d\tau.
\]

We assume that \( W \equiv 0 \), i.e., for any \( f \in \mathcal{H} \)

\[
\|Wf\|_{\mathcal{H}} = \lim_{t \to \infty} \|V(t)f\|_{\mathcal{H}} = 0.
\]

(2.7) means

\[\mathcal{H}_b^\perp \ominus D = \{0\}.\]

Hence Proposition 2.1 and (2.6) imply

\[
\langle (A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}} = -i \int_0^\infty \langle V(\tau)f, K^*U_0(\tau)g, \rangle_{\mathcal{H}}d\tau.
\]

Putting

\[f = (A_0 - i)U_0(s)\psi_M(A_0)P_+h \quad \text{and} \quad g = (A_0 + i)U_0(s)\psi_M(A_0)P_+h\]
for any \( h \in \mathcal{H} \), we have

\[
\|\psi_M(A_0)P_+h\|_{\mathcal{H}}^2 \leq \|h\|_{\mathcal{H}} \|(A - i)^{-1} - (A_0 - i)^{-1}\|U_0(s)\tilde{\psi}_M(A_0){P_+h}\|_{\mathcal{H}}
+ C_M \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}d\tau.)
\]

(A1) and (A2) imply

\[
\lim_{s \to \infty} \|(A - i)^{-1} - (A_0 - i)^{-1}\|U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} = 0
\]
and (A3.2) implies

\[
\lim_{s \to \infty} \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}d\tau = 0.
\]

Therefore we have

(2.8) \[\|\psi_M(A_0)P_+h\|_{\mathcal{H}} = 0,\]
for any \( h \in \mathcal{H}_0 \) and any \( M > 0 \).

(2.8) means \( P_+ \equiv 0 \). This is a contradiction with (A3). Now we complete the proof of \( W \not\equiv 0 \). \(\square\)

We give a brief sketch of the proof of Corollary 2.

**Proof of Corollary 2**. Noting that \( U_0(t) \) is unitary in \( \mathcal{H} \) we have the case \( k = 0 \) by Theorem 1. It follows from the case \( k = 0 \) and (A1) that the case \( k = 1 \).

We can show the cases \( k = 2, 3, 4, \cdots \) by the induction. \(\square\)
For the sake of simplicity, we shall also restrict ourselves to the case \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \) only.

Let \( E(\lambda) \) be the spectral family of \( A_0 \). Then we have

\[
A_0 = \int_{-\infty}^{\infty} \lambda dE(\lambda).
\]

For \( \beta > \alpha > 0 \), we denote \( E((-\beta, -\alpha) \cup (\alpha, \beta)) \) by \( E_{\alpha, \beta}(A_0) \).

(A3) means that \( \sqrt{B_0}E_{\alpha, \beta}(A_0) \) is \( A_0 \)-smooth, i.e. for any \( g \in \mathcal{H} \)

\[
\int_{-\infty}^{\infty} \|\sqrt{B_0}U_0(t)E_{\alpha, \beta}(A_0)g\|_\mathcal{H}^2 dt \leq \tilde{C}_{\alpha, \beta}\|g\|_\mathcal{H}^2
\]

(cf. [8] or [16]), where \( \tilde{C}_{\alpha, \beta} \) is a positive constant which depends on \( \alpha \) and \( \beta \) only. Moreover we note the following identity of \( V(t)f, f \in D(A) \) :

\[
\|V(t)f\|_\mathcal{H}^2 + 2\int_0^t \|\sqrt{B_0}V(\tau)f\|_\mathcal{H}^2 d\tau = \|f\|_\mathcal{H}^2,
\]

Using (3.1) and (3.2) we prove the following lemma.

**Lemma 3.1.** Let \( \beta > \alpha > 0 \). Then for any \( f \in D(A) \) one has

\[
\lim_{t, s \to \infty} \|E_{\alpha, \beta}(A_0)(U_0(-t)V(t)-U_0(-s)V(s))f\|_\mathcal{H} = 0.
\]

**proof.** See [12] §3.

By Lemma 3.1 and (A1) we have the following lemma.

**Lemma 3.2.** One has

\[
w - \lim_{t \to \infty} V(t) = 0.
\]

Using Lemma 3.2 we prove Theorem 3(1) as follows.

**Proof of Theorem 3(1).** Assume that there exists \( f \in D(A), \lambda \in \mathbb{R} \) such that \( Af = \lambda f \). Then we have

\[
\langle V(t)f, f \rangle_\mathcal{H} = e^{-it\lambda}\|f\|_\mathcal{H}^2
\]

This yields a contradiction with Lemma 3.2. \( \square \)

Theorem 3(1) and (F1) imply that

\[
\{(A - i)^{-2}Af \in \mathcal{H} : f \in D(A)\}
\]

is dense in \( \mathcal{H} \).

Below we prove Theorem 3(2).
proof of Theorem 3(2). First we show the existence of \( W \). Set \( F(\lambda) = (\lambda - i) \)
By (2.6) it is sufficient to show that \( \{ U_0(-t)V(t)F(A)f \}_{t \geq 0} \) is Cauchy in \( t \to \infty \), where \( f \in D(A) \). We estimate as follows (cf. [17]):

\[
\|(U_0(-t)V(t) - U_0(-s)V(s))F(A)f\|_{\mathcal{H}} \leq \sum_{j=1}^{4} \|T_j\|_{\mathcal{H}},
\]

where

\[
\begin{align*}
T_1 &= U_0(-t)(F(A) - F(A_0))V(t)f \\
T_2 &= U_0(-s)(F(A) - F(A_0))V(s)f \\
T_3 &= F(A_0)(I_d - E_{1/M,M}(A_0))(U_0(-t)V(t) - U_0(-s)V(s))f \\
T_4 &= F(A_0)E_{1/M,M}(A_0)(U_0(-t)V(t) - U_0(-s)V(s))f.
\end{align*}
\]

We note that for any \( \varepsilon \), there exists \( M > 1 \) such that

\[
\|(1 - \chi_{(-M,-1/M) \cup (1/M,M)})F\|_{L^\infty(\mathbb{R})} < \varepsilon.
\]

Thus we have

(3.5) \[
\|T_3\|_{\mathcal{H}} < \varepsilon \|f\|_{\mathcal{H}}.
\]

By (A4), \( F'(A) - F(A_0) \) is a compact operator. Hence Lemma 3.2 implies

(3.6) \[
\lim_{t \to \infty} \|T_1\|_{\mathcal{H}} = \lim_{s \to \infty} \|T_2\|_{\mathcal{H}} = 0.
\]

Lemma 3.1 implies

(3.7) \[
\lim_{t,s \to \infty} \|T_4\|_{\mathcal{H}} = 0.
\]

(3.5), (3.6) and (3.7) imply the existence of \( W \).

Next we prove \( W \equiv 0 \) (cf. [12]§3). Assume that \( W \equiv 0 \) i.e. for any \( f \in \mathcal{H} \)

(3.8) \[
\lim_{t \to \infty} \|V(t)f\|_{\mathcal{H}} = 0.
\]

We set \( G(\lambda) = (\lambda - i)^{-1} \). Then noting

\[
\langle U_0(-t)V(t)G(A)f, G(A_0)f \rangle_{\mathcal{H}} = \langle G(A)f, G(A_0)f \rangle_{\mathcal{H}} - \int_{0}^{t} \langle U_0(-\tau)BV(\tau)G(A)f, G(A_0)f \rangle_{\mathcal{H}} d\tau,
\]

we have by (3.8) and Schwartz inequality

(3.9) \[
|\langle G(A)f, G(A_0)f \rangle_{\mathcal{H}}| \leq (\int_{0}^{\infty} \|\sqrt{B}V(\tau)G(A)f\|_{\mathcal{H}}^{2} d\tau)^{\frac{1}{2}} \times (\int_{0}^{\infty} \|\sqrt{B}U_0(\tau)G(A_0)f\|_{\mathcal{H}}^{2} d\tau)^{\frac{1}{2}}.
\]
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(3.2) and (3.8) imply

\[ 2 \int_{0}^{\infty} \| \sqrt{B}V(\tau)G(A)f \|_{\mathcal{H}}^2 d\tau = \| G(A)f \|_{\mathcal{H}}^2. \]

Hence we have by (3.9) and (3.10)

\[ \| G(A_0)f \|_{\mathcal{H}}^2 \leq \| f \|_{\mathcal{H}} \{ \| (G(A) - G(A_0))f \|_{\mathcal{H}} + \left( \frac{1}{2} \int_{0}^{\infty} \| \sqrt{B}U_0(\tau)G(A_0)f \|_{\mathcal{H}}^2 d\tau \right)^{1/2} \}. \]

Let fix \( M > 1 \). Put \( f = U_0(s)g, g \) satisfying \( E_{1/M,M}(A_0)g = g \). Then we have

\[ \| G(A_0)g \|_{\mathcal{H}}^2 \leq \| g \|_{\mathcal{H}} \{ \| (G(A) - G(A_0))U_0(s)g \|_{\mathcal{H}} + \left( \frac{1}{2} \int_{s}^{\infty} \| \sqrt{B}E_{1/M,M}(A_0)U_0(\tau)G(A_0)g \|_{\mathcal{H}}^2 d\tau \right)^{1/2} \}. \]

(A1) and (A4) imply

\[ \lim_{s \to \infty} \| (G(A) - G(A_0))U_0(s)g \|_{\mathcal{H}} = 0. \]

(3.1) implies

\[ \lim_{s \to \infty} \int_{s}^{\infty} \| \sqrt{B}E_{1/M,M}(A_0)U_0(\tau)G(A_0)g \|_{\mathcal{H}}^2 d\tau = 0. \]

Therefore it follows from (3.11), (3.12) and (3.13) that \( g \equiv 0 \). This is a contradiction. Therefore we have \( W \not\equiv 0 \). \( \square \)

To prove Corollary 4 we should repeat the same way as in the proof of Corollary 2. Here we omit to do it.

4. Applications.

Application 1 (Elastic wave equation with dissipative boundary condition in a half space of \( \mathbb{R}^3 \)).

We shall apply Theorem 1. In this section we also use \( C \) as positive constants.

Let \( x = (x_1, x_2, x_3) = (y, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+ \) and \( \mu_0 > 0, \rho_0 > 0, \lambda_0 \in \mathbb{R} \) satisfying

\[ \lambda_0 + 2\mu_0 > 0. \]

\( \rho_0 \), \( \lambda_0 \), \( \mu_0 \) as zero and unit matrix of \( 3 \times 3 \) type, respectively.

We use \( O_{3 \times 3} \) and \( I_{3 \times 3} \) as zero and unit matrix of \( 3 \times 3 \) type, respectively.

We set

\[ \epsilon_{hj}(u(x)) = \frac{1}{2} \left( \frac{\partial u_h}{\partial x_j} + \frac{\partial u_j}{\partial x_h} \right) \]

and

\[ \sigma_{hj}(u(x)) = \lambda_0 (\nabla_x \cdot u) \delta_{hj} + 2\mu_0 \epsilon_{hj}(u) \]

where \( h, j = 1, 2, 3, u(x) = (u_1(x), u_2(x), u_3(x)) \in C^3 \) and \( \nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) \).
We define operators $\tilde{L}_0$ as

$$(\tilde{L}_0u)_h = -\sum_{j=1}^{3} \frac{1}{\rho_0} \frac{\partial \sigma_{hj}(u(x))}{\partial x_j} \quad (h = 1, 2, 3).$$

We consider two elastic wave equations as follows:

(4.1) \quad \left\{ \begin{array}{l}
\partial_t^2 u(x, t) + \tilde{L}_0u(x, t) = 0, \quad (x, t) \in \mathbb{R}_+^3 \times [0, \infty), \\
t(\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u))|_{x_3=0} = B(y)\partial_t u|_{x_3=0} \end{array} \right.

and

(4.2) \quad \left\{ \begin{array}{l}
\partial_t^2 u(x, t) + \tilde{L}_0u(x, t) = 0, \quad (x, t) \in \mathbb{R}_+^3 \times \mathbb{R}, \\
\sigma_{i3}(u)|_{x_3=0} = 0(i = 1, 2, 3). \end{array} \right.

To set assumptions for $B(y)$ we introduce a function space $B^k(\Omega)$ as follows:

$$B^k(\Omega) = \{ u \in C^k(\Omega); \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)} < \infty \},$$

where $\Omega \subset \mathbb{R}^n$.

Assume that

(4.3) $B(y)$ belongs to $B^1(\mathbb{R}^2, M_{3 \times 3})$ and satisfies

$$O_{3 \times 3} \leq B(y) \leq \varphi(|y|)I_{3 \times 3},$$

where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbb{R}_+ \times \mathbb{R})$. $M_{3 \times 3}$ is the class of $3 \times 3$ matrix.

The following operator $L_0$ in $\mathcal{G} = L^2(\mathbb{R}_+^3, \mathbb{C}^3; \rho_0 dx)$:

$$L_0u = \hat{\tilde{L}}_0u$$

and

$$D(L_0) = \{ u \in H^1(\mathbb{R}_+^3, \mathbb{C}^3); \hat{\tilde{L}}_0u \in \mathcal{G}, \sigma_{h3}(u)|_{x_3=0} = 0(h = 1, 2, 3) \}$$

is a non-negative self-adjoint operator.

Let $\mathcal{H}$ be Hilbert space with inner product :

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \left( \sum_{h,j,k,l=1}^{3} a_{hjkl} \epsilon_{kl}(f_1) \epsilon_{hj}(g_1) + f_2 \overline{g_2} \rho_0 \right) dx,$$

where $a_{hjkl} = \lambda_0 \delta_{hj} \delta_{kl} + \mu_0 (\delta_{hk} \delta_{jl} + \delta_{hl} \delta_{jk})$ and $f = (f_1, f_2), g = (g_1, g_2)$. By Korn's inequality (cf. [5]) we note that $\mathcal{H}$ is equivalent to $\dot{H}^1(\mathbb{R}_+^3, \mathbb{C}^3) \times L^2(\mathbb{R}_+^3, \mathbb{C}^3)$ as Banach space.

We set $f = ^t(u(x, t), u_t(x, t))$, where $u(x, t)$ is the solution to (4.1) (resp. (4.2)) with a initial data $f_0 = ^t(u(x, 0), u_t(x, 0)) \in \mathcal{H}$. Then (4.1) (resp. (4.2)) can be written as
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\[ \partial_t f = -iAf \quad (\text{resp. } \partial_t f = -iA_0f), \]
where

\[ A = i \begin{pmatrix} 0 & I_{3\times 3} \\ -\tilde{L}_0 & 0 \end{pmatrix}, \quad A_0 = i \begin{pmatrix} 0 & I_{3\times 3} \\ -\tilde{L}_0 & 0 \end{pmatrix}, \]

\[ D(A) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \tilde{L}_0f_1 \in L^2(\mathbb{R}^3_+, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}^3_+, \mathbb{C}^3), \]
\[ \iota(\sigma_{13}(f_1), \sigma_{23}(f_1), \sigma_{33}(f_1)) |_{x_3=0} = B(y)f_2 |_{x_3=0} \}
and

\[ D(A_0) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \tilde{L}_0f_1 \in L^2(\mathbb{R}^3_+, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}^3_+, \mathbb{C}^2), \]
\[ \sigma_{h3}(f_1) |_{x_3=0} = 0(h = 1, 2, 3) \}

According to P210-P211 of [11] or Corollary 1.1.4 of [14] we can show that \( A \) generates a contraction semi-group \( \{V(t)\}_{t \geq 0} \) (resp. a unitary group \( \{U_0(t)\}_{t \in \mathbb{R}} \)) in \( \mathcal{H} \). Using \( \{V(t)\}_{t \geq 0} \) (resp. \( \{U_0(t)\}_{t \in \mathbb{R}} \)) we solve \( \partial_t f = -iAf \) (resp. \( \partial_t f = -iA_0f \)) as follows

\[ f = V(t)f_0 \quad (\text{resp. } f = U_0(t)f_0). \]

Below we make a check on Assumptions (A1),(A2) and (A3) [2] implies \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \). Therefore we have (A1).

Next we show (A2). For \( f, g \in \mathcal{H} \), we have by easy calculation

\[ (4.4) \quad \langle((A - i)^{-1} - (A_0 - i)^{-1})f, g\rangle_{\mathcal{H}} \]
\[ = i \int_{\mathbb{R}^2} B(y)\Gamma_0((A_0 - i)^{-1}f)(A^* + i)^{-1})d_{2}d_2 \]
\[ \Gamma_0((A - i)^{-1} - (A_0 - i)^{-1})f, g\rangle_{\mathcal{H}} \]

where \( \Gamma_0 \) is a target operator which is defined by

\[ (\Gamma_0u)(y) = u(y, 0). \]

Note that \( \Gamma_0((A_0 - i)^{-1}f) \) and \( \Gamma_0((A^* + i)^{-1}f) \) belong to \( H^{1-s}(\mathbb{R}^3_+, \mathbb{C}^3) \) by Korn's inequality for any \( s \in (1/2, 1) \). Since \( B(y)\Gamma_0\Pi_2(A_0 - i)^{-1} \) is a compact operator from \( \mathcal{H} \) to \( L^2(\mathbb{R}^2, \mathbb{C}^3) \) by Rellich's theorem, where \( \Pi_j \iota(f_1, f_2) = f_j(j = 1, 2) \), the form \( (A - i)^{-1} - (A_0 - i)^{-1} \) can be extended to a compact operator, \( (\Gamma_0\Pi_2(A^* + i)^{-1})B(y)\Gamma_0\Pi_2(A_0 - i)^{-1} \), in \( \mathcal{H} \).

To show (A3) we state a result from [2]. There exist \( F_{pu}, F_{Su}, F_{SHu} \) and \( F_R \) which are partially isometric operators from \( \mathcal{G} = L^2(\mathbb{R}^3_+, \mathbb{C}^3; \rho_0dx) \) onto \( L^2(\mathbb{R}^3_+, \mathbb{C}^3) \) and \( L^2(\mathbb{R}^2, \mathbb{C}^3) \), respectively. Defining the operator \( F \) as follows

\[ Fu = (F_{pu}, F_{Su}, F_{SHu}, F_{Ru}) \quad \text{for } u \in \mathcal{G}, \]
we have by Theorem 3.6 of [2]
Lemma A. \(F\) is unitary operator from \(\mathcal{G}\) to

\[
\mathcal{H} = \bigoplus_{j=1}^{3} L^2(\mathbb{R}^3_+, \mathbb{C}^3) \bigoplus L^2(\mathbb{R}^2, \mathbb{C}^3)
\]

and for every \(u \in D(L_0)\)

\[
FL_0u = (c_P^2 |k|^2 F_P u, c_S^2 |k|^2 F_S u, c_{SH}^2 |k|^2 F_{SH} u, c_R^2 |p|^2 F_R u),
\]

where \(k = (p, p_3) \in \mathbb{R}^2 \times \mathbb{R}_+.\)

Using \(F_j (j = P, S, SH, R)\) as above, we construct \(P_{\pm}\) as follows:

\[
P_{\pm} = T^{-1} \{ \sum_{j=P,S,SH} (F_j^* P_{\pm}^{(3)} I_{3\times3} F_j O_{3\times3}) + (F_R^* P_{\pm}^{(2)} I_{3\times3} F_R O_{3\times3}) \} T
\]

where

\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix}
L_0^{1/2} & iI_{3\times3} \\
L_0^{-1/2} & -iI_{3\times3}
\end{pmatrix}
\]

and \(P_{\pm}^{(3)}\) (resp. \(P_{\pm}^{(2)}\)) are negative (resp. positive) spectral projections of

\[
D^{(3)} = \frac{1}{2i} (k \cdot \nabla_k + \nabla_k \cdot k) \quad \text{and} \quad D^{(2)} = \frac{1}{2i} (p \cdot \nabla_p + \nabla_p \cdot p), \quad \text{respectively.}
\]

Using the representation of the generalized eigenfunction of \(L_0\) (see [2]) and the Mellin transformation we show (A3.1)~(A3.4) (cf. [13] and [6]). The Mellin transformations for \(D^{(3)}, D^{(2)}\) are given as

\[
(M^{(3)}u)(\lambda, \omega) = (2\pi)^{-1/2} \int_0^{+\infty} r^{1/2-i\lambda} u(r\omega) dr
\]

and

\[
(M^{(2)}v)(\lambda, \nu) = (2\pi)^{-1/2} \int_0^{+\infty} r^{-i\lambda} v(r\nu) dr,
\]

where \(u(k) \in C_0^\infty(\mathbb{R}_+^3 \setminus \{0\}), v(p) \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \omega \in S^2_+ = \{ (\omega_1, \omega_2, \omega_3) = (\overline{\omega}, \omega_3) \in S^2 : \omega_3 > 0 \} \) and \(\nu \in S^1.\)

Then \(M^{(3)}\) (resp. \(M^{(2)}\)) is extended to a unitary operator from \(L^2(\mathbb{R}_+^3)\) (resp. \(L^2(\mathbb{R}^2)\)) to \(L^2(\mathbb{R} \times S^2_+)\) (resp. \(L^2(\mathbb{R} \times S^1)\)) (cf.[13] Lemma 2).

**Proposition 4.1.** \(P_{\pm}\) as in (4.5) satisfy \((A3)\).

To show Proposition 4.1 we prepare
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Lemma 4.2. Let $\psi(\lambda)$ be same as in (A3) and $0 < \delta < c_R$ (for $c_R$, see Appendix). Then for any positive integer $N$ and $t \in \mathbb{R}_\pm$, there exists a positive constant $C_{N,\psi}$ which is independent of $t$ such that

\begin{equation}
\|\nabla_x (e^{-itA_0}\psi(A_0)P_\pm f)_1\|_{L^2(\mathbb{R}_+^3, \mathbb{C}^3)} \leq C_{N,\psi}(1 + |t|)^{-N} \|f\|_\mathcal{H},
\end{equation}

\begin{equation}
\|e^{-itA_0}\psi(A_0)P_\pm f\|_{L^2(\mathbb{R}_+^3, \mathbb{C}^3)} \leq C_{N,\psi}(1 + |t|)^{-N} \|f\|_\mathcal{H}
\end{equation}

and

\begin{equation}
\|\Gamma_0 e^{-itA_0}\psi(A_0)P_\pm f\|_{L^2(\mathbb{R}_+^3, \mathbb{C}^3)} \leq C_{N,\psi}(1 + |t|)^{-N} \|f\|_\mathcal{H}
\end{equation}

for any $f \in \mathcal{H}_0$, where

$$\|u\|_{L^2(\mathbb{R}_+^3, \mathbb{C}^3)}^B = (\int_B |u|^2 dx)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{L^2(\mathbb{R}^3, \mathbb{C}^3)}^B = (\int_B |v|^2 dy)^{\frac{1}{2}}.$$

This lemma is the key lemma to show (A3). The proof is done by using $M^{(3)}, M^{(2)}$ and Lemma A. But we omit to prove (cf. [13] or [6]).

Proof of Proposition 4.1. Lemma A of Appendix implies that $P_+$ and $P_-$ are projection operators and satisfy $P_+ + P_- = Id$ in $\mathcal{H}$. Below we show (A3.1)~(A3.4).

For any $f, g \in \mathcal{H}$ we have by (4.4)

$$|\langle Ke^{-itA_0}\psi(A_0)P_+ f, g\rangle_\mathcal{H}| \leq CI(t) \times (\|A^*(A^* + i)^{-1}g\|_\mathcal{H} + \|(A^* + i)^{-1}g\|_\mathcal{H}),$$

where

$$I(t) = (\int_{\mathbb{R}^2} |B(y)\Gamma_0 (e^{-itA_0}(A_0 - i)^{-1}\psi(A_0)f)_2|^2 dy)^{\frac{1}{2}} \times (\|A^*(A^* + i)^{-1}g\|_\mathcal{H} + \|(A^* + i)^{-1}g\|_\mathcal{H}).$$

Decomposing $I(t)$ as follows:

\begin{align*}
I(t) & \leq C\{(\int_{\mathbb{R}^2 \cap \{|y| \leq \delta t\}} |\Gamma_0 (e^{-itA_0}(A_0 - i)^{-1}\psi(A_0)f)_2|^2 dy)^{\frac{1}{2}} \times \|A^*(A^* + i)^{-1}g\|_\mathcal{H} + \|(A^* + i)^{-1}g\|_\mathcal{H}) \}
+ (\int_{\mathbb{R}^2 \cap \{|y| \geq \delta t\}} |B(y)\Gamma_0 (e^{-itA_0}(A_0 - i)^{-1}\psi(A_0)f)_2|^2 dy)^{\frac{1}{2}} \}.
\end{align*}

we have by (4.8) of Lemma 4.2 and (4.3)

$$I(t) \leq C_{N,\psi}\{(1 + t)^{-N} + \varphi(\delta t)\} \|f\|_\mathcal{H}.$$

Therefore (A3.1) is proven.
To prove (A3.2) and (A3.3) we note
\[
\langle f, K^* g \rangle_{\mathcal{H}} = \langle ((A - i)^{-1} - (A_0 - i)^{-1})f, g \rangle_{\mathcal{H}}
\]
for any \( f, g \in \mathcal{H} \).

By easy calculation we have
\[
(4.9) \quad \langle ((A - i)^{-1} - (A_0 - i)^{-1})f, g \rangle_{\mathcal{H}_0} = i \int_{\mathbb{R}^2} \Gamma_0((A - i)^{-1}f)(y_2)\overline{B(y)\Gamma_0((A_0 + i)^{-1}g)(y_2)}dy.
\]

Then using (4.9) and the same way as in the proof of (A3.1), we obtain (A3.2) and (A3.3). Here we omit the detail.

We show (A3.4). Lemma 4.2 implies
\[
|\langle e^{itA_0} \psi'(A_0)P_{-}f, g \rangle_{\mathcal{H}}| \leq C_N \psi \{(1 + t)^{-N}||g||_{\mathcal{H}} + ||\nabla_{x^{(/1}}.||_{L^{r}(\mathbb{R}^3, C^{3})} + ||g_2||_{L^{r}(\mathbb{R}^3, C^{3})}||f_1.||_{\mathcal{H}}
\]
for any \( g \in \mathcal{H} \) and any positive integer \( N \). Thus, noting \( \sup_{t \in \mathbb{R}}||f_1.||_{\mathcal{H}} < \infty \), we have (A3.4). \( \square \)

Application 2 (Acoustic wave equations with dissipative terms in stratified media).

We shall apply Theorem 3. First we explain acoustic operator.

Let \( n \geq 1 \) and \((x, y) \in \mathbb{R}^n \times \mathbb{R} \). We set
\[
c_0(y) = \begin{cases} 
c_+(y \geq h) \
c_h \quad (0 < y < h) \
c_- \quad (y \leq 0),
\end{cases}
\]

for some positive constants \( h \) and \( c_+, c_-, c_h \).

Acoustic operators are
\[
L_0 = -c_0(y)^2 \Delta,
\]
where
\[
\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2}.
\]

Considering the case \( c_h < \min(c_+, c_-) \) we find the guided waves (cf. [18] or [19]). But we do not restrict ourselves to such cases.

\( L_0 \) is a non-negative self-adjoint operator in \( \mathcal{G} = L^2(\mathbb{R}^{n+1}; c_0(y)^{-2}dx dy) \). \( D(L_0) \) is given by \( H^2(\mathbb{R}^{n+1}), H^s(\mathbb{R}^{n+1}) \) being Sobolev space of order \( s \) over \( \mathbb{R}^{n+1} \).

We deal with the following dissipative wave equations :
\[
(4.10) \quad \partial_t^2 u(x, y, t) + b(x, y)\partial_t u(x, y, t) + L_0 u(x, y, t) = 0
\]
and
\[
(4.11) \quad \partial_t^2 u(x, y, t) + \langle \partial_t u, \varphi \rangle_{\mathcal{G}} \varphi(x, y) + L_0 u(x, y, t) = 0,
\]
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where \((x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times [0, \infty)\) and \((\cdot, \cdot)_G\) is the inner-product of \(G\).

We assume that \(b(x, y)\) and \(\varphi(x, y)\) are measurable functions which satisfy

\[
0 \leq b(x, y) \leq C(1 + |x|^2 + y^2)^{-\frac{\beta}{2}}
\]

and

\[
\varphi(x, y) \in L^2(\mathbb{R}^{n+1}; (1 + |x|^2 + y^2)^{\frac{\beta}{2}} dx dy)
\]

for some \(\beta > 1\) and \(C > 0\).

We shall show the existence of the scattering states for (4.10) and (4.11) which are considered as the perturbed systems of

\[
\partial_t^2 u(x, y, t) + L_0 u(x, y, t) = 0, \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}
\]

In [19], [1], and [21], we can find local resolvent estimates as follows: for any \(\beta > \alpha > 0\), there exists positive constants \(C_{\alpha, \beta}\) and \(\eta\) such that

\[
\sup_{\alpha \leq |\text{Re}\zeta| \leq \beta, 0 < |\text{Im}\zeta| < \eta} \|X_{\zeta}^{-1}(L_0 - \zeta^2)^{-1}X_{\zeta}\|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})} \leq C_{\alpha, \beta}.
\]

where \(\zeta \in \mathbb{C}, X_\gamma = (1 + |x|^2 + y^2)^{-\frac{\gamma}{2}}\) and \(\| \cdot \|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})}\) is the norm of the bounded operator in \(L^2(\mathbb{R}^{n+1})\).

[12] has already dealt with the case \(c_h = c_+ = c_- = 1\) and \(n \geq 2\) of (4.10). His proof has been based on Kato’s smooth perturbation theory [10] and global resolvent estimates for \(L_0\) (see also [10] Theorem 4.4.1).

We apply Theorem 3 (Corollary 4) to (4.10). We set \(f(t) = (u(t, x, y), \partial_t u(t, x, y))\). Then (4.12) and (4.10) can be written as \(\partial_t f = -iA_0 f\) and \(\partial_t f = -iAf\) respectively, where

\[
A_0 = i \begin{pmatrix} 0 & 1 \\ L_0 & 0 \end{pmatrix}, \quad A = i \begin{pmatrix} 0 & 1 \\ -L_0 & -b(x, y) \end{pmatrix}.
\]

Let \(\mathcal{H}\) be Hilbert spaces with inner product

\[
\langle f, g \rangle_\mathcal{H} = \int_{\mathbb{R}^{n+1}} (\nabla f_1(x, y) \overline{\nabla g_1(x, y)} + f_2(x, y) \overline{g_2(x, y)} c_0^{-2}(y)) dx dy,
\]

and \(\| \cdot \|_\mathcal{H}\) is the corresponding norm, where \(f = (f_1, f_2), g = (g_1, g_2)\).

The domains of \(A_0\) is

\[
D(A_0) = \{ f \in \mathcal{H}; \triangle f_1 \in L^2(\mathbb{R}^{n+1}), f_2 \in H^1(\mathbb{R}^{n+1}) \}.
\]

Then \(A_0\) is a self-adjoint operator in \(\mathcal{H}\) and generates a unitary group \(\{U_0(t)\}_{t \in \mathbb{R}}\) in \(\mathcal{H}\). Below we make a check on (A1), (A4) and (A5).

Note that

\[
T_0 A_0 T_0^{-1} = \begin{pmatrix} \sqrt{L_0} & 0 \\ 0 & -\sqrt{L_0} \end{pmatrix},
\]

where

\[
T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{L_0} & i \\ \sqrt{L_0} & -i \end{pmatrix}.
\]
and $T_0$ is a unitary operator from $\mathcal{H}$ onto $\mathcal{G} \times \mathcal{G}$. It follows from (4.13) that for any $u \in \mathcal{G}$

$$\sup_{\alpha \leq |\text{Re} \zeta| \leq \beta, 0 < |\text{Im} \zeta| < \eta} |\text{Im}((\pm \sqrt{L_0} - \zeta)^{-1}X_{\overline{\frac{\theta}{2}}}u, X_{\frac{\theta}{2}}u)_{\mathcal{G}}| < \infty.$$ 

Therefore we have (A1) by [16] Theorem XIII-20.

Since

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & b(x, y) \end{pmatrix}$$

is $A_0$-compact by Rellich's theorem, we have (A2). Therefore $A$ generates a contraction semi-group $\{V(t)\}_{t \geq 0}$ in $\mathcal{H}$.

In the same argument as in [12]§3 we can show (A5) as follow. Let $g = (g_1, g_2) \in \mathcal{H}$. We set

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (A_0 - \zeta)^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$ 

Then we have

$$(L_0 - \zeta^2)u_2 = \zeta \sqrt{b(x, y)}g_2$$

and

$$\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}g = \sqrt{B_0}u = t(0, \sqrt{b(x, y)}u_2).$$ 

Therefore we can calculate as follows:

(4.14) \[ \|\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}g\|_\mathcal{H} = |\zeta|\|\sqrt{b(x, y)}(L_0 - \zeta^2)^{-1}\sqrt{b(x, y)}g_2\|_{\mathcal{G}_0}. \]

(4.13) and (4.14) imply (A5). Thus we have the conclusion of Theorem 3(Corollary 4) for (4.10) and (4.12).

Next we apply Theorem 3(Corollary 4) to (4.11). we set

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & (\cdot, \varphi)_{\mathcal{G}} \varphi \end{pmatrix}$$

Then $B$ is a compact operator in $\mathcal{H}$. We shall show (A5). Note that

(4.15) \[ |\text{Im} \zeta|\sqrt{B}(A_0 - \zeta)^{-1}f\|_{\mathcal{H}}^2 \leq |\text{Im} \zeta|\|X_{\frac{\theta}{2}}((A_0 - \zeta)^{-1}f)_{\mathcal{G}}\|_{\mathcal{G}_0}^2 \times \|X_{-\frac{\theta}{2}}\varphi\|_{\mathcal{G}}^2 \]

for any $f \in \mathcal{H}$. We set

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & X_{\theta} \end{pmatrix}.$$ 

Then we have

$$|\text{Im} \zeta|\|X_{\frac{\theta}{2}}((A_0 - \zeta)^{-1}f)_{\mathcal{G}}\|_{\mathcal{G}_0}^2 = |\text{Im} \zeta|\sqrt{B_1}(A_0 - \zeta)^{-1}f\|_{\mathcal{H}}^2 \leq \sqrt{B_1}((A_0 - \zeta)^{-1}(A_0 - \overline{\zeta})^{-1})\sqrt{B_1}\|f\|_{\mathcal{H}}^2.$$ 

Noting (4.14) which is changed $B_0$ and $b(x, y)$ to $B_1$ and $X_{\theta}$, respectively we get (A5). Therefore we have the conclusion of Theorem 3(Corollary 4) for (4.11) and
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REFERENCES

1. ______, Analyticity properties and estimates of resolvent kernels near thresholds, to appear in Comm.PDE.
5. H.Ito, Extended Korn's inequalities and the associated best possible constants, J. Elasticity 24 (1990), 43-78.

Tokyo Metropolitan College of Aeronautical Engineering