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ON EXISTENCE OF SCATTERING SOLUTIONS FOR DISSIPATIVE SYSTEMS

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In this report we shall give two frameworks (Theorem 1 and 3) for the existence of scattering solutions of dissipative systems and apply these to some dissipative wave equations.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot , \cdot \rangle_{\mathcal{H}}$. This norm is denoted by $\| \cdot \|_{\mathcal{H}}$. Let $\{V(t)\}_{t \geq 0}$ and $\{U_0(t)\}_{t \in \mathbb{R}}$ be a contraction semi-group and a unitary group in $\mathcal{H}$, respectively. We denote these generators by $A$ and $A_0$ ($V(t) = e^{-itA}$ and $U_0(t) = e^{-itA_0}$). We make the following assumptions on $A$ and $A_0$.

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R}$ or $[0, \infty)$.

(A2) $(A-i)^{-1} - (A_0-i)^{-1}$ defined as a form is extended to a compact operator $K$ in $\mathcal{H}$.

(A3) There exist non-zero projection operators $P_+$ and $P_-$ such that $P_+ + P_- = I_d$ and

(A3.1) $\| KU_0(t)\psi(A_0)P_+ \| \in L^1(\mathbb{R}_+),$

(A3.2) $\| K^*U_0(t)\psi(A_0)P_+ \| \in L^1(\mathbb{R}_+),$

(A3.3) $\| K^*U_0(-t)\psi(A_0)P_- \| \in L^1(\mathbb{R}_+),$

(A3.4) $\lim_{t \to +\infty} U_0(-t)\psi(A_0)P_- f_t = 0,$

for each $\psi \in C_0^\infty(\mathbb{R}\setminus 0)$ and $\{f_t\}_{t \in \mathbb{R}}$ satisfying $\sup_{t \in \mathbb{R}} \| f_t \|_{\mathcal{H}} < \infty$, where $\| \cdot \|$ is the operator norm of bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

Let $\mathcal{H}_b$ be the space generated by the eigenvectors of $A$ with real eigenvalues.

**Theorem 1.** Assume that (A1) $\sim$ (A3). For any $f \in \mathcal{H}_b^\perp$, the wave operator

$$Wf = \lim_{t \to \infty} U_0(-t)V(t)f$$

exists. Moreover $W$ is not zero as an operator in $\mathcal{H}$.

To prove Theorem 1 we shall use the following facts (see [17] and [14]):

(F1) $\{(A - i)^{-2}Af \in \mathcal{H} : f \in D(A) \cap \mathcal{H}_b^\perp\}$ is dense in $\mathcal{H}_b^\perp$.

(F2) There exists a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} t_n = \infty$$

and
Theorem 1 implies that there exists scattering states of $\frac{dV(t)f}{dt} = -iAV(t)f, f \in D(A)$ as follows:

**Corollary 2.** Assume that (A1) $\sim$ (A3). Then there exist non-trivial initial data $f \in \mathcal{H}$ and $f_+ \in \mathcal{H}$ such that for any $k = 0, 1, 2, \cdots$, and $\zeta_0 \in \mathbb{C}$ satisfying $\Re \zeta_0 > 0$

$$\lim_{t \to \infty} \|V(t)(A - \zeta_0)^{-k}f - U_0(t)(A_0 - \zeta_0)^{-k}f_+\|_{\mathcal{H}} = 0.$$

Theorem 1 is proven by using Enss's approach [3] and [17]. Examples of Theorem 1 contain scattering problem for elastic wave equation with dissipative boundary condition in a half space of $\mathbb{R}^3$ (cf. [2]). To show (A3) we use the Mellin transformation (cf.[13]). Theorem 1 is not applied to acoustic wave equations with dissipative terms in stratified media(cf. [19]). Since generalized eigenfunctions of acoustic wave propagation in stratified media are not smooth at thresholds, the key estimates (A3.1)$\sim$(A3.3) have not been obtained in the neighborhood of each threshold. So we consider the following assumptions to deal with such equations.

Let $B_0$ be non-negative operator.

(A4) $B_0$ is $A_0$-compact.

(A5) Let $\zeta$ belong to $\mathbb{C}\backslash\mathbb{R}$. $\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}$ can be extended to a bounded operator $Q(\zeta)$ which satisfies that for any $\beta > \alpha > 0$, there exist positive constants $C_{\alpha,\beta}$ and $\eta$ such that

$$\sup_{\alpha \leq |Re\zeta| \leq \beta, 0 < |Im\zeta| < \eta} \|Q(\zeta)\| \leq C_{\alpha,\beta}.$$

We reset $A = A_0 - iB_0$, $D(A) = D(A_0)$. Then [15] (see Theorem X-50) implies that $A$ generates a contraction semi-group $\{V(t)\}_{t \geq 0}, (V(t) = e^{-itA})$.

We have the following theorem.

**Theorem 3.** Assume that (A1), (A4) and (A5). Then

1. $A$ has no real eigenvalues.
2. The wave operator

$$W = s - \lim_{t \to \infty} U_0(-t)V(t)$$

exists. Moreover $W$ is not zero as an operator in $\mathcal{H}$.

**Corollary 4.** Assume that (A1), (A4) and (A5). Then we have the same conclusion of Corollary 2.

To prove Theorem 3 we shall used Mochizuki's idea [12] due to Kato's smooth perturbation theory[8].

In §4 we shall apply our frameworks to elastic wave equation with dissipative boundary condition in a half space of $\mathbb{R}^3$ and acoustic wave equation with dissipative term in stratified media. It seems that there is little literature concerning such dissipative systems (cf. [7]).
2. Proof of Theorem 1 and Corollary 2.

In this section we deal the case $\sigma(A_{0}) = \sigma_{ac}(A_{0}) = \mathbb{R}$ only. The another case can be dealt in the same way. We set $F(\lambda) = (\lambda - i)^{-2}\lambda$ and $W(t) = U_{0}(-t)V(t)$. In this section $C$ is used as positive constants.

Below we shall give the proof of Theorem 1. First we prove the existence of $W$ by refering to [3], [17], [10], [13], [4], [18] and [14]. But we sometimes omit to note the above references.

**proof of the existence of $W$.** For any $f \in \mathcal{H}_{b}^{1} \cap D(A)$ and $t, s > t_{n}$, note (F1) and

$$
\|(W(t) - W(s))F(A)^{2}f\|_{\mathcal{H}} \leq \|(W(t) - W(t_{n}))F(A)^{2}f\|_{\mathcal{H}} + \|(W(s) - W(t_{n}))F(A)^{2}f\|_{\mathcal{H}}.
$$

Thus in order to prove the existence of $W$, it is sufficient to show

(2.1) $$
\lim_{n \to \infty} \lim_{t \to \infty} \|(W(t) - W(t_{n}))F(A)^{2}f\|_{\mathcal{H}} = 0
$$

(cf. [4])

We estimate $\|(W(t) - W(t_{n}))F(A)^{2}f\|_{\mathcal{H}}$ as follows (cf. [17]):

$$
\|(W(t) - W(t_{n}))F(A)^{2}f\|_{\mathcal{H}} = \|U_{0}(-t)(V(t - t_{n}) - U_{0}(t - t_{n}))(F(A)^{2} - F(A_{0})^{2})V(t_{n})f\|_{\mathcal{H}}
$$

$$
\leq \sum_{j=1}^{5} \|T_{j}\|_{\mathcal{H}},
$$

where

$$
T_{1} = (V(t - t_{n}) - U_{0}(t - t_{n}))(F(A)^{2} - F(A_{0})^{2})V(t_{n})f,
$$

$$
T_{2} = (V(t - t_{n}) - U_{0}(t - t_{n}))(I_{d} - \psi_{M}(A_{0}))F(A_{0})^{2}V(t_{n})f,
$$

$$
T_{3} = (V(t - t_{n}) - U_{0}(t - t_{n}))(\psi_{M}F)(A_{0})P_{+}F(A_{0})V(t_{n})f,
$$

$$
T_{4} = (V(t - t_{n}) - U_{0}(t - t_{n}))(\psi_{M}F)(A_{0})P_{-}F(A_{0})(I_{d} - \psi_{M}(A_{0}))V(t_{n})f,
$$

$$
T_{5} = (V(t - t_{n}) - U_{0}(t - t_{n}))(\psi_{M}F)(A_{0})P_{-}F(A_{0})(I_{d} - \psi_{M}(A_{0}))V(t_{n})f
$$

and $\psi_{M}(\lambda) \in C_{0}^{\infty}(\mathbb{R})$ satisfies $0 \leq \psi_{M}(\lambda) \leq 1$, $\psi_{M}(\lambda) = 0(|\lambda| < 1/2M, |\lambda| > 2M)$ and $\psi_{M}(\lambda) = 1(1/M < |\lambda| < M)$.

First, we note that for any $\varepsilon$, there exists $M > 0$ such that

$$
\|T_{j}\|_{\mathcal{H}} \leq C \|(1 - \psi_{M})F\|_{L^{\infty}(\mathbb{R})} < \varepsilon \quad (j = 2, 4)
$$

Therefore once the limits

(2.2) $$
\lim_{n \to \infty} \lim_{t \to \infty} \|T_{j}\|_{\mathcal{H}} = 0, \quad (j = 1, 3, 5)
$$

are proved, we obtain (2.1). Below we shall show (2.2). For $j = 1$ (A2) implies that $F(A)^{2} - F(A_{0})^{2}$ is a compact operator in $\mathcal{H}$. Using (F2) we have

$$
\|T_{1}\|_{\mathcal{H}} \leq C \|(F(A)^{2} - F(A_{0})^{2})V(t_{n})f\|_{\mathcal{H}} \to 0 \quad (n \to \infty)
$$
For $j = 3$, we decompose $T_3$ as follows

$$T_3 = T_{31} + T_{32} + T_{33},$$

where

$$T_{31} = V(t-t_n)(F(A_0) - F(A))(\psi_M F)(A_0)P_+ F(A_0)V(t_n)f$$

$$T_{32} = (F(A) - F(A_0))U_0(t-t_n)(\psi_M F)(A_0)P_+ F(A_0)V(t_n)f$$

$$T_{33} = F(A)(V(t-t_n) - U_0(t-t_n))\psi_M(A_0)P_+ F(A_0)V(t_n)f$$

Same argument as in the proof of $T_1$ implies

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_{31}\|_\mathcal{H} = 0.$$

We have by (A1)

$$w - \lim_{t \to \infty} U_0(t-t_n)f = 0.$$

Thus (A2) implies

$$\lim_{t \to \infty} \|T_{32}\|_\mathcal{H} = 0.$$

To estimate $T_{33}$, we use Cook-Kuroda method. We have by (A2)

$$\left\langle T_{33}, g \right\rangle_\mathcal{H} = -i \int_0^{t-t_n} \langle V(t-t_n-s)A(A-i)^{-1}KU_0(s)\tilde{\psi}_M(A_0)P_+ F(A_0)f_n, g \rangle_{\mathcal{H}} ds.$$

where $g \in \mathcal{H}$, $f_n = V(t_n)f$ and $\tilde{\psi}_M(\lambda) = (\lambda - i)\psi_M(\lambda)$.

Therefore we obtain

$$\|T_{33}\|_\mathcal{H} \leq C \int_0^{\infty} \|KU_0(s)\tilde{\psi}_M(A_0)P_+ F(A_0)f_n\| ds.$$

For each $s \geq 0$ we have by (F2) and (A2),

$$\lim_{n \to \infty} \|KU_0(s)\tilde{\psi}_M(A_0)P_+ F(A_0)f_n\|_\mathcal{H} = 0.$$

Therefore (A3.1) and Lebesgue's theorem imply

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_{33}\|_\mathcal{H} = 0.$$

Now we obtain

$$\lim_{n \to \infty} \lim_{t \to \infty} \|T_3\|_\mathcal{H} = 0.$$

We estimate $T_5$ as follows:

$$\|T_5\|_\mathcal{H}^2 \leq C \|P_-(F\psi_M)(A_0)V(t_n)f\|_\mathcal{H}^2$$

$$= C \sum_{j=1}^3 T_{5j},$$
where
\[ T_{51} = (\psi_M(A_0)P_- h_n, (F(A_0) - F(A))V(t_n)f)_\mathcal{H} \]
\[ T_{52} = (\psi_M(A_0)P_- h_n, (V(t_n) - U_0(t_n))F(A)f)_\mathcal{H} \]
\[ T_{53} = (U_0(-t_n)\psi_M(A_0)P_- h_n, F(A)f)_\mathcal{H} \]
and \( h_n = (F\psi_M)(A_0)V(t_n)f \).

(A2) and (F2) imply
\[ \lim_{n \to \infty} T_{51} = 0. \]

(A3.4) implies
\[ \lim_{n \to \infty} T_{53} = 0. \]

To estimate \( T_{52} \), again we use Cook-Kuroda method. Note that
\[ |T_{52}| \leq C\|f\|_\mathcal{H} \int_0^{\infty} \|K^*U_0(-s)\overline{\tilde{\psi}_M(A_0)P_- l_{l_n}}\|_\mathcal{H} ds. \]

Using (A2), (F2) and (A3.2) we have by Lebesgue's theorem
\[ \lim_{n \to \infty} T_{52} = 0. \]

Now we obtain
\[ \lim \lim_{n \to \infty} \|T_5\|_\mathcal{H} = 0. \]

Therefore the proof of the existence of \( W \) is completed. \( \square \)

To show \( W \neq 0 \), we introduce a subspace of \( \mathcal{H} \), \( D \), as follows:
\[ D = \{ f \in \mathcal{H} : \lim_{t \to \infty} V(t)f = 0 \}. \]

Since
\[ Af = \lambda f, \lambda \in \mathbb{R}, f \in \mathcal{H} \implies A^*f = \lambda f \]
(see Lemma 1.1.5 of [14]), we can easily show
\[ D \subset \mathcal{H}_b^\perp. \]

We prepare the following proposition without the proof.

**Proposition 2.1.** Assume that
\[ \mathcal{H}_b^\perp \cap D = \{0\}. \]

Then one has
\[ w - \lim_{t \to \infty} U_0(-t)V(t)f = 0 \]
for any \( f \in \mathcal{H} \).

Below we shall show \( W \not\equiv 0 \) (cf. [12] §3).

**Proof of \( W \not\equiv 0 \).** For any \( f \in \mathcal{H} \) and \( g \in \mathcal{H} \), note that

\[
\langle U_0(-t)V(t)(A-i)^{-1}f, (A_0+i)^{-1}g \rangle_{\mathcal{H}} = \langle (A-i)^{-1}f, (A_0+i)^{-1}g \rangle_{\mathcal{H}} + i \int_0^t \langle V(\tau)f, K^*U_0(\tau)g \rangle_{\mathcal{H}} d\tau.
\]

We assume that \( W \equiv 0 \), i.e., for any \( f \in \mathcal{H} \),

\[
\|Wf\|_{\mathcal{H}} = \lim_{t \to \infty} \|V(t)f\|_{\mathcal{H}} = 0.
\]

(2.7) means

\[
\mathcal{H}_b^\perp \ominus D = \{0\}.
\]

Hence Proposition 2.1 and (2.6) imply

\[
\langle (A-i)^{-1}f, (A_0+i)^{-1}g \rangle_{\mathcal{H}} = -i \int_0^\infty \langle V(\tau)f, K^*U_0(\tau)g \rangle_{\mathcal{H}} d\tau.
\]

Putting

\[
f = (A_0 - i)U_0(s)\psi_M(A_0)P_+h \quad \text{and} \quad g = (A_0 + i)U_0(s)\psi_M(A_0)P_+h
\]

for any \( h \in \mathcal{H} \), we have

\[
\|\psi_M(A_0)P_+h\|^2_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}\|((A-i)^{-1}-(A_0-i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}
\]

\[
+ C_M \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} d\tau.
\]

(A1) and (A2) imply

\[
\lim_{s \to \infty} \|((A-i)^{-1}-(A_0-i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} = 0
\]

and (A3.2) implies

\[
\lim_{s \to \infty} \int_0^\infty \|K^*U_0(\tau+s)\tilde{\psi}_M(A_0)P_+\| d\tau = 0.
\]

Therefore we have

\[
\|\psi_M(A_0)P_+h\|_{\mathcal{H}} = 0,
\]

for any \( h \in \mathcal{H}_0 \) and any \( M > 0 \).

(2.8) means \( P_+ \equiv 0 \). This is a contradiction with (A3). Now we complete the proof of \( W \not\equiv 0 \). \( \square \)

We give a brief sketch of the proof of Corollary 2.

**Proof of Corollary 2.** Noting that \( U_0(t) \) is unitary in \( \mathcal{H} \) we have the case \( k = 0 \) by Theorem 1. It follows from the case \( k = 0 \) and (A1) that the case \( k = 1 \).

We can show the cases \( k = 2, 3, 4, \cdots \) by the induction. \( \square \)

For the sake of simplicity, we shall also restrict ourselves to the case $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R}$ only.

Let $E(\lambda)$ be the spectral family of $A_0$. Then we have

$$A_0 = \int_{-\infty}^{\infty} \lambda dE(\lambda).$$

For $\beta > \alpha > 0$, we denote $E((-\beta, -\alpha) \cup (\alpha, \beta))$ by $E_{\alpha, \beta}(A_0)$.

(A3) means that $\sqrt{B_0}E_{\alpha, \beta}(A_0)$ is $A_0$-smooth, i.e. for any $g \in \mathcal{H}$

$$\int_{-\infty}^{\infty} \|\sqrt{B_0}U_0(t)E_{\alpha, \beta}(A_0)g\|_{\mathcal{H}}^2 dt \leq \tilde{C}_{\alpha, \beta} \|g\|_{\mathcal{H}}^2$$

(cf. [8] or [16]), where $\tilde{C}_{\alpha, \beta}$ is a positive constant which depends on $\alpha$ and $\beta$ only. Moreover we note the following identity of $V(t)f, f \in D(A)$ :

$$\int_{-\infty}^{\infty} \|\sqrt{B_0}U_0(t)E_{\alpha, \beta}(A_0)g\|_{\mathcal{H}}^2 dt \leq \tilde{C}_{\alpha, \beta} \|g\|_{\mathcal{H}}^2$$

Using (3.1) and (3.2) we prove the following lemma.

**Lemma 3.1.** Let $\beta > \alpha > 0$. Then for any $f \in D(A)$ one has

$$\lim_{t, s \to \infty} \|E_{\alpha, \beta}(A_0)(U_0(-t)V(t) - U_0(-s)V(s))f\|_{\mathcal{H}} = 0.$$

**proof.** See [12] §3.

By Lemma 3.1 and (A1) we have the following lemma.

**Lemma 3.2.** One has

$$\lim_{t \to \infty} V(t) = 0.$$

Using Lemma 3.2 we prove Theorem 3(1) as follows.

**proof of Theorem 3(1).** Assume that there exists $f \in D(A), \lambda \in \mathbb{R}$ such that $Af = \lambda f$. Then we have

$$\langle V(t)f, f \rangle_{\mathcal{H}} = e^{-it\lambda} \|f\|_{\mathcal{H}}^2$$

This yields a contradiction with Lemma 3.2. \qed

Theorem 3(1) and (F1) imply that

$$\{(A - i)^{-2}Af \in \mathcal{H} : f \in D(A)\}$$

is dense in $\mathcal{H}$.

Below we prove Theorem 3(2).
proof of Theorem 3(2). First we show the existence of $W$. Set $F(\lambda) = (\lambda - i)$.

By (2.6) it is sufficient to show that \(\{U_0(-t)V(t)F(A)f\}_{t \geq 0}\) is Cauchy in $t \to \infty$, where $f \in D(A)$. We estimate as follows (cf. [17]):

\[
||(U_0(-t)V(t) - U_0(-s)V(s))F(A)f||_{\mathcal{H}} \leq \sum_{j=1}^{4} ||T_j||_{\mathcal{H}},
\]

where

\[
T_1 = U_0(-t)(F(A) - F(A_0))V(t)f, \\
T_2 = U_0(-s)(F(A) - F(A_0))V(s)f, \\
T_3 = F(A_0)(I_d - E_{1/M,M}(A_0))(U_0(-t)V(t) - U_0(-s)V(s))f \\
\text{and} \\
T_4 = F(A_0)E_{1/M,M}(A_0)(U_0(-t)V(t) - U_0(-s)V(s))f.
\]

We note that for any $\varepsilon$, there exists $M > 1$ such that

\[
||(1 - \chi_{(-M,-1/M)\cup(1/M,M)})F||_{L^\infty(\mathbb{R})} < \varepsilon.
\]

Thus we have

(3.5) \[ ||T_3||_{\mathcal{H}} < \varepsilon ||f||_{\mathcal{H}}. \]

By (A4), $F'(A) - F'(A_0)$ is a compact operator. Hence Lemma 3.2 implies

(3.6) \[ \lim_{t \to \infty} ||T_1||_{\mathcal{H}} = \lim_{s \to \infty} ||T_2||_{\mathcal{H}} = 0. \]

Lemma 3.1 implies

(3.7) \[ \lim_{t,s \to \infty} ||T_4||_{\mathcal{H}} = 0. \]

(3.5), (3.6) and (3.7) imply the existence of $W$.

Next we prove $W \not\equiv 0$ (cf. [12]§3). Assume that $W \equiv 0$ i.e. for any $f \in \mathcal{H}$

(3.8) \[ \lim_{t \to \infty} ||V(t)f||_{\mathcal{H}} = 0. \]

We set $G(\lambda) = (\lambda - i)^{-1}$. Then noting

\[
\langle U_0(-t)V(t)G(A)f, G(A_0)f \rangle_{\mathcal{H}} \\
= \langle G(A)f, G(A_0)f \rangle_{\mathcal{H}} - \int_0^{t} \langle U_0(-\tau)BV(\tau)G(A)f, G(A_0)f \rangle_{\mathcal{H}} d\tau,
\]

we have by (3.8) and Schwartz inequality

(3.9) \[ |\langle G(A)f, G(A_0)f \rangle_{\mathcal{H}}| \leq \left( \int_0^{\infty} ||\sqrt{B}V(\tau)G(A)f||_{\mathcal{H}}^2 d\tau \right)^{\frac{1}{2}} \times \left( \int_0^{\infty} ||\sqrt{B}U_0(\tau)G(A_0)f||_{\mathcal{H}}^2 d\tau \right)^{\frac{1}{2}}. \]
(3.2) and (3.8) imply

\begin{equation}
2 \int_0^\infty ||\sqrt{B}V(\tau)G(A)f||_{\mathcal{H}}^2 d\tau = ||G(A)f||_{\mathcal{H}}^2.
\end{equation}

Hence we have by (3.9) and (3.10)

\begin{equation}
||G(A_0)f||_{\mathcal{H}}^2 \leq ||f\|_{\mathcal{H}} \{||G(A)-G(A_0)||f\|_{\mathcal{H}} + \frac{1}{2} \int_0^\infty ||\sqrt{B}U_0(\tau)G(A_0)f||_{\mathcal{H}}^2 d\tau \}.
\end{equation}

Let fix $M > 1$. Put $f = U_0(s)g, g$ satisfying $E_{1/M,M}(A_0)g = g$. Then we have

\begin{equation}
||G(A_0)g||_{\mathcal{H}}^2 \leq ||g\|_{\mathcal{H}} \{||G(A)-G(A_0)||U_0(s)g\|_{\mathcal{H}} + \frac{1}{2} \int_s^\infty ||\sqrt{B}E_{1/M,M}(A_0)U_0(\tau)G(A_0)g||_{\mathcal{H}}^2 d\tau \}.
\end{equation}

(A1) and (A4) imply

\begin{equation}
\lim_{s \to \infty} ||(G(A)-G(A_0))U_0(s)c||_{\mathcal{H}} = 0.
\end{equation}

(3.1) implies

\begin{equation}
\lim_{s \to \infty} \int_s^\infty ||\sqrt{B}E_{1/M,M}(A_0)U_0(\tau)G(A_0)g||_{\mathcal{H}}^2 d\tau = 0.
\end{equation}

Therefore it follows from (3.11), (3.12) and (3.13) that $g \equiv 0$. This is a contradiction. Therefore we have $W \neq 0$. \Box

To prove Corollary 4 we should repeat the same way as in the proof of Corollary 2. Here we omit to do it.

4. Applications.

Application 1 (Elastic wave equation with dissipative boundary condition in a half space of $\mathbf{R}^3$).

We shall apply Theorem 1. In this section we also use $C$ as positive constants.

Let $x = (x_1, x_2, x_3) = (y, x_3) \in \mathbf{R}^2 \times \mathbf{R}_+$ and $\mu_0 > 0, \rho_0 > 0, \lambda_0 \in \mathbf{R}$ satisfying $3\lambda_0 + 2\mu_0 > 0$. We use $O_{3 \times 3}$ and $I_{3 \times 3}$ as zero and unit matrix of $3 \times 3$ type, respectively.

We set

$$\epsilon_{hj}(u(x)) = \frac{1}{2}(\frac{\partial u_h}{\partial x_j} + \frac{\partial u_j}{\partial x_h})$$

and

$$\sigma_{hj}(u(x)) = \lambda_0(\nabla_x \cdot u)\delta_{hj} + 2\mu_0 \epsilon_{hj}(u)$$

where $h, j = 1, 2, 3, u(x) = (u_1(x), u_2(x), u_3(x)) \in C^3$ and $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3).$
We define operators $\tilde{L}_0$ as
\[
(\tilde{L}_0 u)_h = -\sum_{j=1}^{3} \frac{1}{\rho_0} \frac{\partial \sigma_{hj}(u(x))}{\partial x_j} \quad (h = 1, 2, 3).
\]

We consider two elastic wave equations as follows:
\[
\begin{aligned}
\{ & \partial_t^2 u(x, t) + \tilde{L}_0 u(x, t) = 0, (x, t) \in \mathbb{R}_+^3 \times [0, \infty), \\
& (\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u)) |_{x_3=0} = B(y) \partial_t u |_{x_3=0}
\end{aligned}
\tag{4.1}
\]
and
\[
\begin{aligned}
\{ & \partial_t^2 u(x, t) + \tilde{L}_0 u(x, t) = 0, (x, t) \in \mathbb{R}_+^3 \times \mathbb{R}, \\
& \sigma_{ii}(u) |_{x_3=0} = 0(i = 1, 2, 3)
\end{aligned}
\tag{4.2}
\]

To set assumptions for $B(y)$ we introduce a function space $B_k^k(\Omega)$ as follows:
\[
B_k^k(\Omega) = \{ \tau \in C^k(\Omega); \sum_{|\alpha| = k} \| \partial^\alpha \tau \|_{L^\infty(\Omega)} < \infty \}
\]
where $\Omega \subset \mathbb{R}^n$.

Assume that
\[
B(y) \text{ belongs to } B^1(\mathbb{R}^2, \mathbb{M}_{3 \times 3}) \text{ and satisfies}
\]
\[
O_{3 \times 3} \leq B(y) \leq \varphi(|y|) I_{3 \times 3},
\]
where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbb{R}_+)$.

The following operator $L_0$ in $\mathcal{G} = L^2(\mathbb{R}_+^3, \mathbb{C}^3; \rho_0 dx)$:
\[
L_0 u = \tilde{L}_0 u
\]
and
\[
D(L_0) = \{ u \in H^1(\mathbb{R}_+^3, \mathbb{C}^3); \tilde{L}_0 u \in \mathcal{G}, \sigma_{h3}(u) |_{x_3=0} = 0(h = 1, 2, 3) \}
\]
is a non-negative self-adjoint operator.

Let $\mathcal{H}$ be Hilbert space with inner product:
\[
\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+^3} \left( \sum_{h,j,k,l=1}^{3} a_{hijkl} \xi_{kl}(f_1 \overline{\xi}_{hj}(g_1) + f_2 \overline{g}_2 \rho_0) dx, \right.
\]
where $a_{hijkl} = \lambda_0 \delta_{hk} \delta_{kl} + \mu_0 (\delta_{hk} \delta_{jl} + \delta_{hl} \delta_{jk})$ and $f = (f_1, f_2), g = (g_1, g_2)$. By Korn's inequality (cf. [5]) we note that $\mathcal{H}$ is equivalent to $H^1(\mathbb{R}_+^3, \mathbb{C}^3) \times L^2(\mathbb{R}_+^3, \mathbb{C}^3)$ as Banach space.

We set $f = \xi(u(x,t), u_t(x,t))$, where $u(x,t)$ is the solution to (4.1) (resp. (4.2)) with a initial data $f_0 = \xi(u(x,0), u_t(x,0)) \in \mathcal{H}$. Then (4.1) (resp. (4.2)) can be written as
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\[ \partial_t f = -iAf \quad (\text{resp. } \partial_t f = -iA_0f), \]

where

\[ A = i \begin{pmatrix} 0 & I_{3 \times 3} \\ -\bar{L}_0 & 0 \end{pmatrix}, \quad A_0 = i \begin{pmatrix} 0 & I_{3 \times 3} \\ -\bar{L}_0 & 0 \end{pmatrix}, \]

\[ D(A) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \bar{L}_0 f_1 \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}_+^3, \mathbb{C}^3) \}, \]

\[ D(A_0) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \bar{L}_0 f_1 \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}_+^3, \mathbb{C}^3) \}, \]

\[ \sigma_{h3}(f_1) \mid_{x_3=0} = B(y)f_2 \mid_{x_3=0} \}

According to P210-P211 of [11] or Corollary 1.1.4 of [14] we can show that \( A \) generates a contraction semi-group \( \{V(t)\}_{t \geq 0} \) (resp. a unitary group \( \{U_0(t)\}_{t \in \mathbb{R}} \)) in \( \mathcal{H} \). Using \( \{V(t)\}_{t \geq 0} \) (resp. \( \{U_0(t)\}_{t \in \mathbb{R}} \)) we solve \( \partial_t f = -iAf \) (resp. \( \partial_t f = -iA_0f \)) as follows

\[ f = V(t)f_0 \quad (\text{resp. } f = U_0(t)f_0). \]

Below we make a check on Assumptions (A1),(A2) and (A3) [2] implies \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \). Therefore we have (A1).

Next we show (A2). For \( f, g \in \mathcal{H}, \) we have by easy calculation

\[ \langle ((A - i)^{-1} - (A_0 - i)^{-1})f, g \rangle_{\mathcal{H}} = i \int_{\mathbb{R}^2} B(y)\Gamma_0((A_0 - i)^{-1}f_2)\overline{\Gamma_0((A^* + i)^{-1}g_2)}dy, \]

where \( \Gamma_0 \) is a target operator which is defined by

\[ (\Gamma_0u)(y) = u(y, 0). \]

Note that \( \Gamma_0((A_0 - i)^{-1}f_2) \) and \( \Gamma_0((A^* + i)^{-1}f_2) \) belong to \( H^{1-s}(\mathbb{R}_+^3, \mathbb{C}^3) \) by Korn's inequality for any \( s \in (1/2, 1). \) Since \( B(y)\Gamma_0\Pi_2(A_0 - i)^{-1} \) is a compact operator from \( \mathcal{H} \) to \( L^2(\mathbb{R}^2, \mathbb{C}^3) \) by Rellich's theorem, where \( \Pi_j \iota(f_1, f_2) = f_j(j = 1, 2), \) the form \( (A - i)^{-1} - (A_0 - i)^{-1} \) can be extended to a compact operator, \( (\Gamma_0\Pi_2(A^* + i)^{-1}B(y)\Gamma_0\Pi_2(A_0 - i)^{-1}) \), in \( \mathcal{H}. \)

To show (A3) we state a result from [2]. There exist \( F_{Pu}, F_{Su}, F_{Shu} \) and \( F_R \) which are partially isometric operators from \( G = L^2(\mathbb{R}_+^3, \mathbb{C}^3; \rho_0dx) \) onto \( L^2(\mathbb{R}_+^3, \mathbb{C}^3) \) and \( L^2(\mathbb{R}^2, \mathbb{C}^3) \), respectively. Defining the operator \( F \) as follows:

\[ Fu = (F_{Pu}, F_{Su}, F_{Shu}, F_{Ru}) \quad \text{for} \quad u \in G, \]

we have by Theorem 3.6 of [2]
Lemma A. $F$ is unitary operator from $\mathcal{G}$ to
\[ \mathcal{H} = \bigoplus_{j=1}^{3} L^2(\mathbb{R}^3_+, \mathbb{C}^3) \bigoplus L^2(\mathbb{R}^2, \mathbb{C}^3) \]
and for every $u \in D(L_0)$
\[ F L_0 u = (c_P^2|k|^2 F_P u, c_S^2|k|^2 F_S u, c_S^2|k|^2 F_{SH} u, c_R^2|p|^2 F_R u), \]
where $k = (p, p_3) \in \mathbb{R}^2 \times \mathbb{R}_+.$

Using $F_j (j = P, S, SH, R)$ as above, we construct $P_\pm$ as follows:
\[ P_{\pm} = T^{-1} \left\{ \sum_{j = P, S, SH} \begin{pmatrix} F_j^* P_{\pm}^{(3)} I_{3 \times 3} F_j & O_{3 \times 3} \\ O_{3 \times 3} & F_j^* P_{\mp}^{(3)} I_{3 \times 3} F_j \end{pmatrix} \\ + \begin{pmatrix} F_{R_+}^* P_{\pm}^{(2)} I_{3 \times 3} F_R & O_{3 \times 3} \\ O_{3 \times 3} & F_{R_-}^* P_{\mp}^{(2)} I_{3 \times 3} F_R \end{pmatrix} \right\} T \]
where
\[ T = \frac{1}{\sqrt{2}} \begin{pmatrix} L_0^{1/2} & iI_{3 \times 3} \\ L_0^{1/2} & -iI_{3 \times 3} \end{pmatrix} \]
and $P_{\pm}^{(3)}$ (resp. $P_{\mp}^{(3)}$) and $P_{\pm}^{(2)}$ (resp. $P_{\mp}^{(2)}$) are negative (resp. positive) spectral projections of
\[ D^{(3)} = \frac{1}{2i}(k \cdot \nabla_k + \nabla_k \cdot k) \quad \text{and} \quad D^{(2)} = \frac{1}{2i}(p \cdot \nabla_p + \nabla_p \cdot p), \quad \text{respectively.} \]

Using the representation of the generalized eigenfunction of $L_0$ (see [2]) and the Mellin transformation we show (A3.1)~(A3.4) (cf. [13] and [6]). The Mellin transformations for $D^{(3)}, D^{(2)}$ are given as
\[ (M^{(3)} u)(\lambda, \omega) = (2\pi)^{-1/2} \int_0^{+\infty} r^{1/2-i\lambda} u(r \omega) dr \]
and
\[ (M^{(2)} v)(\lambda, \nu) = (2\pi)^{-1/2} \int_0^{+\infty} r^{-i\lambda} v(r \nu) dr, \]
where $u(k) \in C^\infty_0(\mathbb{R}^3_+ \setminus \{0\}), v(p) \in C^\infty_0(\mathbb{R}^2 \setminus \{0\}), \omega \in S^2_+ = \{(\omega_1, \omega_2, \omega_3) = (\overline{\omega}, \omega_3) \in S^2 : \omega_3 > 0\}$ and $\nu \in S^1.$

Then $M^{(3)}$ (resp. $M^{(2)}$) is extended to a unitary operator from $L^2(\mathbb{R}^3_+)$ (resp. $L^2(\mathbb{R}^2)$) to $L^2(\mathbb{R} \times S^2_+)$ (resp. $L^2(\mathbb{R} \times S^1)$) (cf. [13] Lemma 2).

**Proposition 4.1.** $P_{\pm}$ as in (4.5) satisfy (A3).

To show Proposition 4.1 we prepare
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Lemma 4.2. Let \( \psi(\lambda) \) be same as in (A3) and \( 0 < \delta < c_R \) (for \( c_R \), see Appendix). Then for any positive integer \( N \) and \( t \in \mathbb{R}_\pm \), there exists a positive constant \( C_{N,\psi} \) which is independent of \( t \) such that

\[
\| \nabla_x (e^{-itA_0}\psi(A_0)P_{\pm}f)_{1} \|_{L^2(\mathbb{R}^3_+,\mathbb{C}^3)} \leq C_{N,\psi}(1+|t|)^{-N}\|f\|_{\mathcal{H}},
\]

(4.6)

\[
\| (e^{-itA_0}\psi(A_0)P_{\pm}f)_{2} \|_{L^2(\mathbb{R}^3_+,\mathbb{C}^3)} \leq C_{N,\psi}(1+|t|)^{-N}\|f\|_{\mathcal{H}},
\]

(4.7)

and

\[
\| \Gamma_0 (e^{-itA_0}\psi(A_0)P_{\pm}f)_{2} \|_{L^\infty(\mathbb{R}^3_+,\mathbb{C}^3)} \leq C_{N,\psi}(1+|t|)^{-N}\|f\|_{\mathcal{H}},
\]

(4.8)

for any \( f \in \mathcal{H}_0 \), where

\[
\|u\|_{L^2(\mathbb{R}^3_+,\mathbb{C}^3)}^B = \left( \int_B |u|^2 \, dx \right)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{L^2(\mathbb{R}^3_+,\mathbb{C}^3)}^{B} = \left( \int_B |v|^2 \, dy \right)^{\frac{1}{2}}.
\]

This lemma is the key lemma to show (A3). The proof is done by using \( M^{(3)}, M^{(2)} \) and Lemma A. But we omit to prove (cf. [13] or [6]).

Proof of Proposition 4.1. Lemma A of Appendix implies that \( P_+ \) and \( P_- \) are projection operators and satisfy \( P_+ + P_- = Id \) in \( \mathcal{H} \). Below we show (A3.1)\textasciitilde(A3.4).

For any \( f, g \in \mathcal{H} \) we have by (4.4)

\[
\|K e^{-itA_0}\psi(A_0)P_{+}f,g\|_{\mathcal{H}} \leq C I(t) \times \left( \|A^*(A^*+i)^{-1}g\|_{\mathcal{H}} + \|(A^*+i)^{-1}g\|_{\mathcal{H}} \right),
\]

where

\[
I(t) = \left( \int_{\mathbb{R}^2} |B(y)\Gamma_0(\psi(A_0)P_{+}f)_{2}|^2 \, dy \right)^{\frac{1}{2}} \times \left( \|A^*(A^*+i)^{-1}g\|_{\mathcal{H}} + \|(A^*+i)^{-1}g\|_{\mathcal{H}} \right).
\]

Decomposing \( I(t) \) as follows:

\[
I(t) \leq C \left\{ \left( \int_{\mathbb{R}^2 \cap \{|y| \leq \delta t\}} |\Gamma_0(e^{-itA_0}(A_0-i)^{-1}\psi(A_0)f)_{2}|^2 \, dy \right)^{\frac{1}{2}} \times \right. \\
+ \left. \left( \int_{\mathbb{R}^2 \cap \{|y| \geq \delta t\}} |B(y)\Gamma_0(e^{-itA_0}(A_0-i)^{-1}\psi(A_0)f)_{2}|^2 \, dy \right)^{\frac{1}{2}} \right\},
\]

we have by (4.8) of Lemma 4.2 and (4.3)

\[
I(t) \leq C_{N,\psi}\{(1+t)^{-N} + \varphi(\delta t)\}\|f\|_{\mathcal{H}}.
\]

Therefore (A3.1) is proven.
To prove (A3.2) and (A3.3) we note
\[ (f, K^* g)_\mathcal{H} = \langle((A - i)^{-1} - (A_0 - i)^{-1})f, g\rangle_{\mathcal{H}} \]
for any \( f, g \in \mathcal{H} \).

By easy calculation we have
\[ (4.9) \]
\[ \langle((A - i)^{-1} - (A_0 - i)^{-1})f, g\rangle_{\mathcal{H}} = i \int_{\mathbb{R}^2} \Gamma_0((A - i)^{-1}f, y, t) \overline{B(y) \Gamma_0((A_0 + i)^{-1}g, y, t)} dy. \]

Then using (4.9) and the same way as in the proof of (A3.1), we obtain (A3.2) and (A3.3). Here we omit the detail.

We show (A3.4). Lemma 4.2 implies
\[ |\langle e^{itA_0} \psi(A_0) P_{-} f_{t}, g\rangle_{\mathcal{H}}| \leq C_{N, \psi} \left\{ (1 + t)^{-N} \||g\||_{\mathcal{H}} + \||\nabla_x g\||_{L^2(\mathbb{R}^n; \mathbb{C}^3)} + \||g_2\||_{L^2(\mathbb{R}^n; \mathbb{C}^3)} \right\} \||f_{t}\||_{\mathcal{H}}, \]
for any \( g \in \mathcal{H} \) and any positive integer \( N \). Thus, noting \( \sup_{t \in \mathbb{R}} \||f_{t}\||_{\mathcal{H}} < \infty \), we have (A3.4). \( \square \)

Application 2 (Acoustic wave equations with dissipative terms in stratified media).

We shall apply Theorem 3. First we explain acoustic operator.

Let \( n \geq 1 \) and \( (x, y) \in \mathbb{R}^n \times \mathbb{R} \). We set
\[ c_0(y) = \begin{cases} \bar{c}_+ & (y \geq h) \\ c_{-} & (0 < y < h) \\ c_- & (y \leq 0), \end{cases} \]

for some positive constants \( h \) and \( c_+, c_{-}, c_\). Acoustic operators are
\[ L_0 = -c_0(y)^2 \triangle, \]
where
\[ \triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2}. \]

Considering the case \( c_0 < \min(c_+, c_{-}) \) we find the guided waves (cf. [18] or [19]). But we do not restrict ourselves to such cases.

\( L_0 \) is a non-negative self-adjoint operator in \( \mathcal{G} = L^2(\mathbb{R}^{n+1}; c_0(y)^{-2} dx dy) \). \( D(L_0) \) is given by \( H^2(\mathbb{R}^{n+1}); H^s(\mathbb{R}^{n+1}) \) being Sobolev space of order \( s \) over \( \mathbb{R}^{n+1} \).

We deal with the following dissipative wave equations:
\[ (4.10) \]
\[ \partial^2_t u(x, y, t) + b(x, y) \partial_t u(x, y, t) + L_0 u(x, y, t) = 0 \]
and
\[ (4.11) \]
\[ \partial^2_t u(x, y, t) + (\partial_t u, \varphi)_G \varphi(x, y) + L_0 u(x, y, t) = 0, \]
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where \((x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times [0, \infty)\) and \((\cdot, \cdot)_{\mathcal{G}}\) is the inner-product of \(\mathcal{G}\).

We assume that \(b(x, y)\) and \(\varphi(x, y)\) are measurable functions which satisfy

\[0 \leq b(x, y) \leq C(1 + |x|^2 + y^2)^{-\frac{\theta}{2}}\]

and

\[\varphi(x, y) \in L^2(\mathbb{R}^{n+1}; (1 + |x|^2 + y^2)^{\frac{\theta}{2}}) dx dy\]

for some \(\theta > 1\) and \(C > 0\).

We shall show the existence of the scattering states for (4.10) and (4.11) which are considered as the perturbed systems of

\[(4.12) \quad \partial_t^2 u(x, y, t) + L_0 u(x, y, t) = 0, \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}\]

In [19], [1], and [21], we can find local resolvent estimates as follows: for any \(\beta > \alpha > 0\), there exists positive constants \(C_{\alpha, \beta}\) and \(\eta\) such that

\[(4.13) \quad \sup_{\alpha \leq |\text{Re} \zeta| \leq \beta, 0 < |\text{Im} \zeta| < \eta} \|X_{\frac{\alpha}{2}}(L_0 - \zeta^2)^{-1} X_{\frac{\alpha}{2}}\|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})} \leq C_{\alpha, \beta}.

where \(\zeta \in \mathbb{C}\), \(X_\gamma = (1 + |x|^2 + y^2)^{-\frac{\gamma}{2}}\) and \(\| \cdot \|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})}\) is the norm of the bounded operator in \(L^2(\mathbb{R}^{n+1})\).

[12] has already dealt with the case \(c_+ = c_- = 1\) and \(n \geq 2\) of (4.10). His proof has been based on Kato's smooth perturbation theory [10] and global resolvent estimates for \(L_0\) (see also [10] Theorem 4.4.1).

We apply Theorem 3 (Corollary 4) to (4.10). We set \(f(t) = (u(t, x, y), \partial_t u(t, x, y))\).

Then (4.12) and (4.10) can be written as \(\partial_t f = -iA_0 f\) and \(\partial_t f = -iA f\) respectively, where

\[A_0 = i \begin{pmatrix} 0 & 1 \\ -L_0 & 0 \end{pmatrix}, \quad A = i \begin{pmatrix} 0 & 1 \\ -L_0 & -b(x, y) \end{pmatrix}.

Let \(\mathcal{H}\) be Hilbert spaces with inner product

\[\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{n+1}} (\nabla f_1(x, y) \nabla g_1(x, y) + f_2(x, y)g_2(x, y)c_0^{-2}(y)) dx dy,

and \(\| \cdot \|_{\mathcal{H}}\) is the corresponding norm, where \(f = (f_1, f_2), g = (g_1, g_2)\).

The domains of \(A_0\) is

\[D(A_0) = \{ f \in \mathcal{H}; \Delta f_1 \in L^2(\mathbb{R}^{n+1}), f_2 \in H^1(\mathbb{R}^{n+1}) \}.

Then \(A_0\) is a self-adjoint operator in \(\mathcal{H}\) and generates a unitary group \(\{U_0(t)\}_{t \in \mathbb{R}}\) in \(\mathcal{H}\). Below we make a check on (A1), (A4) and (A5).

Note that

\[T_0 A_0 T_0^{-1} = \begin{pmatrix} \sqrt{L_0} & 0 \\ 0 & -\sqrt{L_0} \end{pmatrix},

where

\[T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{L_0} & i \\ \sqrt{L_0} & -i \end{pmatrix} \]
and $T_0$ is a unitary operator from $\mathcal{H}$ onto $\mathcal{G} \times \mathcal{G}$. It follows from (4.13) that for any $u \in \mathcal{G}$
\[
\sup_{\alpha \leq |Re\zeta| \leq \beta, 0 < |Im\zeta| < \eta} |\text{Im}(\pm \sqrt{L_0 - \zeta})^{-1} X_{\frac{\theta}{2}} u, X_{\frac{\theta}{2}} u)_\mathcal{G}| < \infty.
\]
Therefore we have (A1) by [16] Theorem XIII-20.

Since
\[
B_0 = \begin{pmatrix} 0 & 0 \\ 0 & b(x, y) \end{pmatrix}
\]
is $A_0$-compact by Rellich's theorem, we have (A2). Therefore $A$ generates a contraction semi-group \(\{V(t)\}_{t \geq 0}\) in $\mathcal{H}$.

In the same argument as in [12]\S 3 we can show (A5) as follow. Let $g = (g_1, g_2) \in \mathcal{H}$. We set
\[
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (A_0 - \zeta)^{-1} \sqrt{B_0} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.
\]
Then we have
\[
(L_0 - \zeta^2) u_2 = \zeta \sqrt{b(x, y)} g_2
\]
and
\[
\sqrt{B_0} (A_0 - \zeta)^{-1} \sqrt{B_0} g = \sqrt{B_0} u = ^t (0, \sqrt{b(x, y)} u_2).
\]
Therefore we can calculate as follows:

(4.14) \[\|\sqrt{B_0} (A_0 - \zeta)^{-1} \sqrt{B_0} g\|_\mathcal{H} = |\zeta|\|\sqrt{b(x, y)} (L_0 - \zeta^2)^{-1} \sqrt{b(x, y)} g_2\|_{\mathcal{G}_0}.\]

(4.13) and (4.14) imply (A5). Thus we have the conclusion of Theorem 3(Corollary 4) for (4.10) and (4.12).

Next we apply Theorem 3(Corollary 4) to (4.11). we set
\[
B_0 = \begin{pmatrix} 0 & 0 \\ 0 & \langle \cdot, \varphi \rangle_{\mathcal{G}} \varphi \end{pmatrix}
\]
Then $B$ is a compact operator in $\mathcal{H}$. We shall show (A5). Note that

(4.15) \[|\text{Im}\zeta|\|\sqrt{B} (A_0 - \zeta)^{-1} f\|_\mathcal{H}^2 \leq |\text{Im}\zeta|\|X_{\frac{\theta}{2}} ((A_0 - \zeta)^{-1} f)_2\|_\mathcal{G}^2 \times \|X_{-\frac{\theta}{2}} \varphi\|_\mathcal{G}^2\]

for any $f \in \mathcal{H}$. We set
\[
B_1 = \begin{pmatrix} 0 & 0 \\ 0 & X_{\theta} \end{pmatrix}.
\]
Then we have
\[
|\text{Im}\zeta|\|X_{\frac{\theta}{2}} ((A_0 - \zeta)^{-1} f)_2\|_\mathcal{G}^2 = |\text{Im}\zeta|\|\sqrt{B_1} (A_0 - \zeta)^{-1} f\|_\mathcal{H}^2 \leq \|\sqrt{B_1} ((A_0 - \zeta)^{-1} - (A_0 - \bar{\zeta})^{-1}) \sqrt{B_1} f\|_\mathcal{H}^2.\]

Noting (4.14) which is changed $B_0$ and $b(x, y)$ to $B_1$ and $X_{\theta}$, respectively we get (A5). Therefore we have the conclusion of Theorem 3(Corollary 4) for (4.11) and
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