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ON EXISTENCE OF SCATTERING SOLUTIONS FOR DISSIPATIVE SYSTEMS

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In this report we shall give two frameworks (Theorem 1 and 3) for the existence of scattering solutions of dissipative systems and apply these to some dissipative wave equations.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. This norm is denoted by $\| \cdot \|_{\mathcal{H}}$. Let $\{V(t)\}_{t \geq 0}$ and $\{U_{0}(t)\}_{t \in \mathbb{R}}$ be a contraction semi-group and a unitary group in $\mathcal{H}$, respectively. We denote these generators by $A$ and $A_{0}$ ($V(t) = e^{-itA}$ and $U_{0}(t) = e^{-itA_{0}}$). We make the following assumptions on $A$ and $A_{0}$.

(A1) $\sigma(A_{0}) = \sigma_{ac}(A_{0}) = \mathbb{R}$ or $[0, \infty)$.

(A2) $(A - i)^{-1} - (A_{0} - i)^{-1}$ defined as a form is extended to a compact operator $K$ in $\mathcal{H}$.

(A3) There exist non-zero projection operators $P_{+}$ and $P_{-}$ such that $P_{+} + P_{-} = I_{d}$ and

\[(A3.1) \quad \| KU_{0}(t)\psi(A_{0})P_{+} \| \in L^{1}(\mathbb{R}_{+}), \]
\[(A3.2) \quad \| K^{*}U_{0}(t)\psi(A_{0})P_{+} \| \in L^{1}(\mathbb{R}_{+}), \]
\[(A3.3) \quad \| K^{*}U_{0}(-t)\psi(A_{0})P_{-} \| \in L^{1}(\mathbb{R}_{+}), \]
\[(A3.4) \quad \text{w-}\lim_{t \to +\infty} U_{0}(-t)\psi(A_{0})P_{-} f_{t} = 0, \]

for each $\psi \in C_{0}^{\infty}(\mathbb{R}\backslash 0)$ and $\{f_{t}\}_{t \in \mathbb{R}}$ satisfying $\sup_{t \in \mathbb{R}} \| f_{t} \|_{\mathcal{H}} < \infty$, where $\| \cdot \|$ is the operator norm of bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

Let $\mathcal{H}_{b}$ be the space generated by the eigenvectors of $A$ with real eigenvalues.

Theorem 1. Assume that (A1) ~ (A3). For any $f \in \mathcal{H}_{b}^{\perp}$, the wave operator

$Wf = \lim_{t \to \infty} U_{0}(-t)V(t)f$

exists. Moreover $W$ is not zero as an operator in $\mathcal{H}$.

To prove Theorem 1 we shall use the following facts (see [17] and [14]):

(F1) $\{(A - i)^{-2}Af \in \mathcal{H} : f \in D(A) \cap \mathcal{H}_{b}^{\perp}\}$ is dense in $\mathcal{H}_{b}^{\perp}$.

(F2) There exists a sequence $\{t_{n}\}$ such that

$\lim_{n \to \infty} t_{n} = \infty$

and

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$$w - \lim_{n \to \infty} V(t_n) f = 0, \text{ for any } f \in \mathcal{H}_d^+.$$ 

Theorem 1 implies that there exists scattering states of $$\frac{dV(t)f}{dt} = -iAV(t)f, f \in D(A)$$ as follows:

**Corollary 2.** Assume that (A1) ~ (A3). Then there exist non-trivial initial data $$f \in \mathcal{H}$$ and $$f_+ \in \mathcal{H}$$ such that for any $$k = 0, 1, 2, \cdots$$ and $$\zeta \in \mathbb{C}$$ satisfying $$\Re \zeta > 0$$

$$\lim_{t \to \infty} \| V(t)(A - \zeta)^{-k}f - U_0(t)(A_0 - \zeta)^{-k}f_+ \|_{\mathcal{H}} = 0.$$ 

Theorem 1 is proven by using Enss's approach [3] and [17]. Examples of Theorem 1 contain scattering problem for elastic wave equation with dissipative boundary condition in a half space of $$\mathbb{R}^3$$ (cf. [2]). To show (A3) we use the Mellin transformation (cf.[13]). Theorem 1 is not applied to acoustic wave equations with dissipative terms in stratified media(cf. [19]). Since generalized eigenfunctions of acoustic wave propagation in stratified media are not smooth at thresholds, the key estimates (A3.1)~(A3.3) have not been obtained in the neighborhood of each threshold. So we consider the following assumptions to deal with such equations.

Let $$B_0$$ be non-negative operator.

1. $$B_0$$ is $$A_0$$-compact. 
2. Let $$\zeta$$ belong to $$\mathbb{C} \setminus \mathbb{R}$$. $$\sqrt{B}_0(A_0 - \zeta)^{-1}\sqrt{B}_0$$ can be extended to a bounded operator $$Q(\zeta)$$ which satisfies that for any $$\beta > \alpha > 0$$, there exist positive constants $$C_{\alpha, \beta}$$ and $$\eta$$ such that

$$\sup_{\alpha \leq |\Re \zeta| \leq \beta, 0 < |\Im \zeta| < \eta} \| Q(\zeta) \| \leq C_{\alpha, \beta}.$$ 

We reset $$A = A_0 - iB_0$$, $$D(A) = D(A_0)$$. Then [15] (see Theorem X-50) implies that $$A$$ generates a contraction semi-group $$\{ V(t) \}_{t \geq 0} (V(t) = e^{-itA})$$.

We have the following theorem.

**Theorem 3.** Assume that (A1), (A4) and (A5). Then

1. $$A$$ has no real eigenvalues.
2. The wave operator

$$W = s - \lim_{t \to \infty} U_0(-t)V(t)$$

exists. Moreover $$W$$ is not zero as an operator in $$\mathcal{H}$$.

**Corollary 4.** Assume that (A1), (A4) and (A5). Then we have the same conclusion of Corollary 2.

To prove Theorem 3 we shall used Mochizuki's idea [12] due to Kato's smooth perturbation theory[8].

In §4 we shall apply our frameworks to elastic wave equation with dissipative boundary condition in a half space of $$\mathbb{R}^3$$ and acoustic wave equation with dissipative term in stratified media. It seems that there is little literature concerning such dissipative systems (cf. [7]).
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2. Proof of Theorem 1 and Corollary 2.
In this section we deal the case \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \) only. The another case can be dealt in the same way. We set \( F(\lambda) = (\lambda - i)^{-2} \lambda \) and \( W(t) = U_0(-t)V(t) \).

In this section \( C \) is used as a positive constant.

Below we shall give the proof of Theorem 1. First we prove the existence of \( W \) by referring to [3], [17], [10], [13], [4], [18] and [14]. But we sometimes omit to note the above references.

**proof of the existence of \( W \).** For any \( f \in \mathcal{H}^1_b \cap D(A) \) and \( t, s > t_n \), note (F1) and

\[
\|(W(t) - W(s))F(A)^2 f\|_{\mathcal{H}} \leq \|(W(t) - W(t_n))F(A)^2 f\|_{\mathcal{H}} + \|(W(s) - W(t_n))F(A)^2 f\|_{\mathcal{H}}.
\]

Thus in order to prove the existence of \( W \), it is sufficient to show

\[
\lim_{n \to \infty} \lim_{t \to \infty} \|(W(t) - W(t_n))F(A)^2 f\|_{\mathcal{H}} = 0
\]

(cf. [4])

We estimate \( \|(W(t) - W(t_n))F(A)^2 f\|_{\mathcal{H}} \) as follows (cf. [17]):

\[
\|(W(t) - W(t_n))F(A)^2 f\|_{\mathcal{H}} = \|U_0(-t)(V(t-t_n) - U_0(t-t_n))F(A)^2V(t_n)f\|_{\mathcal{H}} \leq \sum_{j=1}^{5} \|T_j\|_{\mathcal{H}},
\]

where

\[
T_1 = (V(t-t_n) - U_0(t-t_n))(F(A)^2 - F(A_0)^2)V(t_n)f,
T_2 = (V(t-t_n) - U_0(t-t_n))(I_d - \psi_M(A_0))F(A_0)^2V(t_n)f,
T_3 = (V(t-t_n) - U_0(t-t_n))(\psi_M F)(A_0)P_+F(A_0)V(t_n)f,
T_4 = (V(t-t_n) - U_0(t-t_n))(\psi_M F)(A_0)P_-F(A_0)(I_d - \psi_M(A_0))V(t_n)f,
T_5 = (V(t-t_n) - U_0(t-t_n))(\psi_M F)(A_0)P_-(\psi_M F)(A_0)V(t_n)f.
\]

and \( \psi_M(\lambda) \in C_0^\infty(\mathbb{R}) \) satisfies \( 0 \leq \psi_M(\lambda) \leq 1, \psi_M(\lambda) = 0(|\lambda| < 1/2M, |\lambda| > 2M) \) and \( \psi_M(\lambda) = 1(1/M < |\lambda| < M) \).

First, we note that for any \( \epsilon \), there exists \( M > 0 \) such that

\[
\|T_j\|_{\mathcal{H}} \leq C\|(1 - \psi_M)F\|_{L^\infty(\mathbb{R})} < \epsilon \quad (j = 2, 4)
\]

Therefore once the limits

\[
\lim_{n \to \infty} \lim_{t \to \infty} \|T_j\|_{\mathcal{H}} = 0, \quad (j = 1, 3, 5)
\]

are proved, we obtain (2.1). Below we shall show (2.2). For \( j = 1 \) (A2) implies that \( F(A)^2 - F(A_0)^2 \) is a compact operator in \( \mathcal{H} \). Using (F2) we have

\[
\|T_1\|_{\mathcal{H}} \leq C\|(F(A)^2 - F(A_0)^2)V(t_n)f\|_{\mathcal{H}} \to 0 \quad (n \to \infty)
\]
For $j = 3$, we decompose $T_3$ as follows

$$T_3 = T_{31} + T_{32} + T_{33},$$

where

$$T_{31} = V(t - t_n)(F(A_0) - F(A))(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$

$$T_{32} = (F(A) - F(A_0))U_0(t - t_n)(\psi_M F)(A_0)P_+F(A_0)V(t_n)f$$

$$T_{33} = F(A)(V(t - t_n) - U_0(t - t_n))\psi_M(A_0)P_+F(A_0)V(t_n)f$$

Same argument as in the proof of $T_1$ implies

$$\lim_{n \to \infty} \lim_{t \to \infty} ||T_{31}||_\mathcal{H} = 0.$$

We have by (A1)

$$w - \lim_{t \to \infty} U_0(t - t_n)f = 0.$$

Thus (A2) implies

$$\lim_{t \to \infty} ||T_{32}||_\mathcal{H} = 0.$$

To estimate $T_{33}$, we use Cook-Kuroda method. We have by (A2)

$$\langle T_{33}, g \rangle_\mathcal{H} = -i \int_0^{t-t_n} \langle V(t-t_n-s)A(A-i)^{-1}KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n, g \rangle_{\ell} ds.$$  

where $g \in \mathcal{H}$, $f_n = V(t_n)f$ and $\tilde{\psi}_M(\lambda) = (\lambda - i)\psi_M(\lambda)$.

Therefore we obtain

$$||T_{33}||_\mathcal{H} \leq C \int_0^{\infty} ||KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n||_\mathcal{H} ds.$$

For each $s \geq 0$ we have by (F2) and (A2),

$$\lim_{n \to \infty} ||KU_0(s)\tilde{\psi}_M(A_0)P_+F(A_0)f_n||_\mathcal{H} = 0.$$

Therefore (A3.1) and Lebesgue’s theorem imply

$$\lim_{n \to \infty} \lim_{t \to \infty} ||T_{33}||_\mathcal{H} = 0.$$

Now we obtain

$$\lim_{n \to \infty} \lim_{t \to \infty} ||T_3||_\mathcal{H} = 0.$$

We estimate $T_5$ as follows:

$$||T_5||_\mathcal{H}^2 \leq C ||P_-(F\psi_M)(A_0)V(t_n)f||_\mathcal{H}^2$$

$$= C \sum_{j=1}^3 T_{5j},$$
where

\[ T_{51} = (\psi_{M}(A_{0})P_{-}h_{n}, (F(A_{0}) - F(A))V(t_{n})f)_{\mathcal{H}} \]
\[ T_{52} = (\psi_{M}(A_{0})P_{-}h_{n}, (V(t_{n}) - U_{0}(t_{n}))F(A)f)_{\mathcal{H}} \]
\[ T_{53} = (U_{0}(-t_{n})\psi_{M}(A_{0})P_{-}h_{n}, F(A)f)_{\mathcal{H}} \]

and \( h_{n} = (F\psi_{M})(A_{0})V(t_{n})f \).

(A2) and (F2) imply

\[ \lim_{n \to \infty} T_{51} = 0. \]

(A3.4) implies

\[ \lim_{n \to \infty} T_{53} = 0. \]

To estimate \( T_{52} \), again we use Cook-Kuroda method. Note that

\[ |T_{52}| \leq C||f||_{\mathcal{H}} \int_{0}^{\infty} ||K^{*}U_{0}(-s)\tilde{\psi}_{M}(A_{0})P_{-}l_{l_{n}}||_{\mathcal{H}} ds. \]

Using (A2), (F2) and (A3.2) we have by Lebesgue’s theorem

\[ \lim_{n \to \infty} T_{52} = 0. \]

Now we obtain

\[ \lim_{n \to \infty} \lim_{t \to \infty} ||T_{5}||_{\mathcal{H}} = 0. \]

Therefore the proof of the existence of \( W \) is completed. \( \square \)

To show \( W \not\equiv 0 \), we introduce a subspace of \( \mathcal{H} \), \( D \), as follows:

\[ D = \{ f \in \mathcal{H} : \lim_{t \to \infty} V(t)f = 0 \}. \]

Since

\[ Af = \lambda f, \lambda \in \mathbb{R}, f \in \mathcal{H} \Rightarrow A^{*}f = \lambda f \]

(see Lemma 1.1.5 of [14]), we can easily show

\[ D \subset \mathcal{H}_{b}^{\perp}. \]

We prepare the following proposition without the proof.

**Proposition 2.1.** Assume that

\[ \mathcal{H}_{b}^{\perp} \cap D = \{0\}. \]

Then one has

\[ w - \lim_{t \to \infty} U_{0}(-t)V(t)f = 0 \]

(2.3)
for any \( f \in \mathcal{H} \).

Below we shall show \( W \not\equiv 0 \) (cf. [12]§3).

**proof of \( W \not\equiv 0 \).** For any \( f \in \mathcal{H} \) and \( g \in \mathcal{H} \), note that

\[
\langle U_0(-t)V(t)(A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}} = \langle (A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}} + i \int_0^t \langle V(\tau)f, K^*U_0(\tau)g, \rangle_{\mathcal{H}}d\tau.
\]

We assume that \( W \equiv 0 \), i.e., for any \( f \in \mathcal{H} \),

\[
\|Wf\|_{\mathcal{H}} = \lim_{t \to \infty} \|V(t)f\|_{\mathcal{H}} = 0.
\]

(2.7) means

\[
\mathcal{H}_b^+ \ominus D = \{0\}.
\]

Hence Proposition 2.1 and (2.6) imply

\[
\langle (A - i)^{-1}f, (A_0 + i)^{-1}g \rangle_{\mathcal{H}} = -i \int_0^\infty \langle V(\tau)f, K^*U_0(\tau)g, \rangle_{\mathcal{H}}d\tau.
\]

Putting

\[
f = (A_0 - i)U_0(s)\psi_M(A_0)P_+h \quad \text{and} \quad g = (A_0 + i)U_0(s)\psi_M(A_0)P_+h
\]

for any \( h \in \mathcal{H} \), we have

\[
\|\psi_M(A_0)P_+h\|^2_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}\|((A - i)^{-1} - (A_0 - i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}
\]

\[
+ C_M \int_0^\infty \|K^*U_0(\tau + s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}}d\tau.
\]

(A1) and (A2) imply

\[
\lim_{s \to \infty} \|((A - i)^{-1} - (A_0 - i)^{-1})U_0(s)\tilde{\psi}_M(A_0)P_+h\|_{\mathcal{H}} = 0
\]

and (A3.2) implies

\[
\lim_{s \to \infty} \int_0^\infty \|K^*U_0(\tau + s)\tilde{\psi}_M(A_0)P_+\|_{\mathcal{H}}d\tau = 0.
\]

Therefore we have

\[
\|\psi_M(A_0)P_+h\|_{\mathcal{H}} = 0,
\]

for any \( h \in \mathcal{H}_0 \) and any \( M > 0 \).

(2.8) means \( P_+ \equiv 0 \). This is a contradiction with (A3). Now we complete the proof of \( W \not\equiv 0 \). \( \square \)

We give a brief sketch of the proof of Corollary 2.

**proof of Corollary 2.** Noting that \( U_0(t) \) is unitary in \( \mathcal{H} \) we have the case \( k = 0 \) by Theorem 1. It follows from the case \( k = 0 \) and (A1) that the case \( k = 1 \).

We can show the cases \( k = 2, 3, 4, \cdots \) by the induction. \( \square \)

For the sake of simplicity, we shall also restrict ourselves to the case \( \sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R} \) only.

Let \( E(\lambda) \) be the spectral family of \( A_0 \). Then we have

\[
A_0 = \int_{-\infty}^{\infty} \lambda dE(\lambda).
\]

For \( \beta > \alpha > 0 \), we denote \( E((-\beta, -\alpha) \cup (\alpha, \beta)) \) by \( E_{\alpha, \beta}(A_0) \).

(A3) means that \( \sqrt{B_0} E_{\alpha, \beta}(A_0) \) is \( A_0 \)-smooth, i.e. for any \( g \in \mathcal{H} \)

\[(3.1) \quad \int_{-\infty}^{\infty} \| \sqrt{B_0} U_0(t) E_{\alpha, \beta}(A_0) g \|^2 dt \leq \tilde{C}_{\alpha, \beta} \| g \|^2_{\mathcal{H}}
\]

(cf. [8] or [16]), where \( \tilde{C}_{\alpha, \beta} \) is a positive constant which depends on \( \alpha \) and \( \beta \) only.

Moreover we note the following identity of \( V(t) f, f \in D(A) \) :

\[(3.2) \quad \| V(t) f \|^2_{\mathcal{H}} + 2 \int_0^t \| \sqrt{B_0} V(\tau) f \|^2_{\mathcal{H}} d\tau = \| f \|^2_{\mathcal{H}},
\]

Using (3.1) and (3.2) we prove the following lemma.

**Lemma 3.1.** Let \( \beta > \alpha > 0 \). Then for any \( f \in D(A) \) one has

\[
\lim_{t, s \to \infty} \| E_{\alpha, \beta}(A_0)(U_0(-t)V(t) - U_0(-s)V(s)) f \|_{\mathcal{H}} = 0.
\]

**proof.** See [12] §3.

By Lemma 3.1 and (A1) we have the following lemma.

**Lemma 3.2.** One has

\[
w - \lim_{t \to \infty} V(t) = 0.
\]

Using Lemma 3.2 we prove Theorem 3(1) as follows.

**proof of Theorem 3(1).** Assume that there exists \( f \in D(A), \lambda \in \mathbb{R} \) such that \( Af = \lambda f \). Then we have

\[
(V(t)f, f)_{\mathcal{H}} = e^{-it\lambda} \| f \|^2_{\mathcal{H}}
\]

This yields a contradiction with Lemma 3.2. \( \square \)

Theorem 3(1) and (F1) imply that

\[(3.4) \quad \{(A - i)^{-2}Af \in \mathcal{H} : f \in D(A)\} \quad \text{is dense in} \quad \mathcal{H}.
\]

Below we prove Theorem 3(2).
proof of Theorem 3(2). First we show the existence of $W$. Set $F(\lambda) = (\lambda - i)^{-1}$ By (2.6) it is sufficient to show that $\{U_0(-t)V(t)F(A)f\}_{t \geq 0}$ is Cauchy in $t \to \infty$, where $f \in D(A)$. We estimate as follows (cf. [17]):

$$
\|(U_0(-t)V(t) - U_0(-s)V(s))F(A)f\|_{\mathcal{H}} \leq \sum_{j=1}^{4} \|T_j\|_{\mathcal{H}},
$$

where

$$
T_1 = U_0(-t)(F(A) - F(A_0))V(t)f
$$

$$
T_2 = U_0(-s)(F(A) - F(A_0))V(s)f
$$

$$
T_3 = F(A_0)(I - E_{1/M,M}(A_0))(U_0(-t)V(t) - U_0(-s)V(s))f
$$

and

$$
T_4 = F(A_0)E_{1/M,M}(A_0)(U_0(-t)V(t) - U_0(-s)V(s))f.
$$

We note that for any $\varepsilon$, there exists $M > 1$ such that

$$
\|(1 - \chi_{(-M,-1/M) \cup (1/M,M)}(\cdot))F\|_{L^\infty(\mathbb{R})} < \varepsilon.
$$

Thus we have

$$
(3.5) \quad \|T_3\|_{\mathcal{H}} < \varepsilon \|f\|_{\mathcal{H}}.
$$

By (A4), $F'(A) - F(A_0)$ is a compact operator. Hence Lemma 3.2 implies

$$
(3.6) \quad \lim_{t \to \infty} \|T_1\|_{\mathcal{H}} = \lim_{s \to \infty} \|T_2\|_{\mathcal{H}} = 0.
$$

Lemma 3.1 implies

$$
(3.7) \quad \lim_{t,s \to \infty} \|T_4\|_{\mathcal{H}} = 0.
$$

(3.5), (3.6) and (3.7) imply the existence of $W$.

Next we prove $W \not\equiv 0$ (cf. [12]§3). Assume that $W \equiv 0$ i.e. for any $f \in \mathcal{H}$

$$
(3.8) \quad \lim_{t \to \infty} \|V(t)f\|_{\mathcal{H}} = 0.
$$

We set $G(\lambda) = (\lambda - i)^{-1}$. Then noting

$$
(U_0(-t)V(t)G(A)f,G(A_0)f)_{\mathcal{H}}
$$

$$
= (G(A)f,G(A_0)f)_{\mathcal{H}} - \int_{0}^{t} (U_0(-\tau)BV(\tau)G(A)f,G(A_0)f)_{\mathcal{H}} d\tau,
$$

we have by (3.8) and Schwartz inequality

$$
(3.9) \quad |\langle G(A)f,G(A_0)f\rangle_{\mathcal{H}}|
$$

$$
\leq \left( \int_{0}^{\infty} \|\sqrt{B}V(\tau)G(A)f\|_{\mathcal{H}}^2 d\tau \right)^{\frac{1}{2}} \times \left( \int_{0}^{\infty} \|\sqrt{BU_0(\tau)G(A_0)f}\|_{\mathcal{H}}^2 d\tau \right)^{\frac{1}{2}}.
$$
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(3.2) and (3.8) imply

\begin{equation}
2 \int_{0}^{\infty} \|\sqrt{B}V(\tau)G(A)f\|_{H}^{2}d\tau = \|G(A)f\|_{H}^{2}.
\end{equation}

Hence we have by (3.9) and (3.10)

\[ \|G(A_{0})f\|_{H}^{2} \leq \|f\|_{H}\{\|(G(A) - G(A_{0}))f\|_{H} + (\frac{1}{2} \int_{0}^{\infty} \|\sqrt{B}U_{0}(\tau)G(A_{0})f\|_{H}^{2}d\tau)^{\frac{1}{2}}\} \].

Let fix \( M > 1 \). Put \( f = U_{0}(s)g, g \) satisfying \( E_{1/M,M}(A_{0})g = g \). Then we have

\begin{equation}
\|G(A_{0})g\|_{H}^{2} \leq \|g\|_{H}\{\|(G(A) - G(A_{0}))U_{0}(s)g\|_{H} + \frac{1}{2} \int_{s}^{\infty} \|E_{1/M,M}(A_{0})U_{0}(\tau)G(A_{0})g\|_{H}^{2}d\tau)^{\frac{1}{2}}\}.
\end{equation}

(A1) and (A4) imply

\begin{equation}
\lim_{s \to \infty} \|(G(A) - G(A_{0}))U_{0}(s)g\|_{H} = 0.
\end{equation}

(3.1) implies

\begin{equation}
\lim_{s \to \infty} \int_{s}^{\infty} \|E_{1/M,M}(A_{0})U_{0}(\tau)G(A_{0})g\|_{H}^{2}d\tau = 0.
\end{equation}

Therefore it follows from (3.11), (3.12) and (3.13) that \( g \equiv 0 \). This is a contradiction. Therefore we have \( W \neq 0 \).

To prove Corollary 4 we should repeat the same way as in the proof of Corollary 2. Here we omit to do it.

4. Applications.

Application 1 (Elastic wave equation with dissipative boundary condition in a half space of \( \mathbb{R}^{3} \)).

We shall apply Theorem 1. In this section we also use \( C \) as positive constants.

Let \( x = (x_{1}, x_{2}, x_{3}) = (y, x_{3}) \in \mathbb{R}^{2} \times \mathbb{R}_{+} \) and \( \mu_{0} > 0, \rho_{0} > 0, \lambda_{0} \in \mathbb{R} \) satisfying \( 3\lambda_{0} + 2\mu_{0} > 0 \). We use \( O_{3 \times 3} \) and \( I_{3 \times 3} \) as zero and unit matrix of \( 3 \times 3 \) type, respectively.

We set

\[ \epsilon_{hj}(u(x)) = \frac{1}{2} \frac{\partial u_{h}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{h}} \]

and

\[ \sigma_{hj}(u(x)) = \lambda_{0}(\nabla_{x} \cdot u)\delta_{hj} + 2\mu_{0}\epsilon_{hj}(u) \]

where \( h, j = 1, 2, 3, u(x) = (u_{1}(x), u_{2}(x), u_{3}(x)) \in C^{3} \) and \( \nabla_{x} = (\partial/\partial_{1}, \partial/\partial_{2}, \partial/\partial_{3}) \).
We define operators $\tilde{L}_0$ as

$$(\tilde{L}_0 u)_h = -\sum_{j=1}^{3} \frac{1}{\rho_0} \frac{\partial \sigma_{hj}(u(x))}{\partial x_j} \quad (h = 1, 2, 3).$$

We consider two elastic wave equations as follows:

\begin{equation}
\begin{aligned}
\{ \partial_t^2 u(x, t) + \tilde{L}_0 u(x, t) &= 0, (x, t) \in \mathbb{R}_+^3 \times [0, \infty), \\
\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u) |_{x_3=0} &= B(y) \partial_t u |_{x_3=0}
\end{aligned}
\end{equation}

(4.1)

and

\begin{equation}
\begin{aligned}
\{ \partial_t^2 u(x, t) + \tilde{L}_0 u(x, t) &= 0, (x, t) \in \mathbb{R}_+^3 \times \mathbb{R}, \\
\sigma_{i3}(u) |_{x_3=0} &= 0(i = 1, 2, 3)
\end{aligned}
\end{equation}

(4.2)

To set assumptions for $B(y)$ we introduce a function space $B^k(\Omega)$ as follows:

$B^k(\Omega) = \{ u \in C^k(\Omega); \sum_{|\alpha| \leq k} ||\partial^\alpha u||_{L^\infty(\Omega)} < \infty\}$,

where $\Omega \subset \mathbb{R}^n$.

Assume that

\begin{equation}
B(y) \text{ belongs to } B^1(\mathbb{R}_+^3, \mathbb{M}_{3\times3}) \text{ and satisfies}
\end{equation}

$O_{3\times3} \leq B(y) \leq \varphi(|y|) I_{3\times3},$

where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbb{R}_+^3)$. $\mathbb{M}_{3\times3}$ is the class of $3 \times 3$ matrix.

The following operator $L_0$ in $\mathcal{G} = L^2(\mathbb{R}_+^3, \mathbb{C}^3; \rho_0 dx)$:

$L_0 u = \tilde{L}_0 u$

and

$$D(L_0) = \{ u \in H^1(\mathbb{R}_+^3, \mathbb{C}^3); \tilde{L}_0 u \in \mathcal{G}, \sigma_{h3}(u) |_{x_3=0} = 0(h = 1, 2, 3) \}$$

is a non-negative self-adjoint operator.

Let $\mathcal{H}$ be Hilbert space with inner product:

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+^3} \left( \sum_{h,j,k,l=1}^{3} a_{hijkl} \varepsilon_{kl} f_1 \varepsilon_{hj} g_1 + f_2 g_2 \rho_0 \right) dx,$$

where $a_{hijkl} = \lambda_0 \delta_{hj} \delta_{kl} + \mu_0 (\delta_{hk} \delta_{jl} + \delta_{hl} \delta_{jk})$ and $f = (f_1, f_2), g = (g_1, g_2)$. By Korn’s inequality (cf. [5]) we note that $\mathcal{H}$ is equivalent to $\dot{H}^1(\mathbb{R}_+^3, \mathbb{C}^3) \times L^2(\mathbb{R}_+^3, \mathbb{C}^3)$ as Banach space.

We set $f = \tau(u(x, t), u_t(x, t))$, where $u(x, t)$ is the solution to (4.1) (resp. (4.2)) with a initial data $f_0 = \tau(u(x, 0), u_t(x, 0)) \in \mathcal{H}$. Then (4.1) (resp. (4.2)) can be written as
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$$\partial_t f = -iAf \quad (\text{resp. } \partial_t f = -iA_0 f),$$
where

$$A = i\begin{pmatrix} 0 & I_{3 \times 3} \\ -\tilde{L}_0 & 0 \end{pmatrix}, \quad A_0 = i\begin{pmatrix} 0 & I_{3 \times 3} \\ -L_0 & 0 \end{pmatrix},$$

$$D(A) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \tilde{L}_0 f_1 \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}_+^3, \mathbb{C}^3), \iota(\sigma_{13}(f_1), \sigma_{23}(f_1), \sigma_{33}(f_1))\, |_{x_3=0}= B(y)f_2\, |_{x_3=0}\},$$
and

$$D(A_0) = \{ f = \iota(f_1, f_2) \in \mathcal{H}; \tilde{L}_0 f_1 \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), f_2 \in H^1(\mathbb{R}_+^3, \mathbb{C}^2), \sigma_{h3}(f_1)|_{x_3=0} = 0(h = 1, 2, 3)\}$$

According to P210-P211 of [11] or Corollary 1.1.4 of [14] we can show that $A$ generates a contraction semi-group $\{V(t)\}_{t \geq 0}$ (resp. a unitary group $\{U_0(t)\}_{t \in \mathbb{R}}$) in $\mathcal{H}$. Using $\{V(t)\}_{t \geq 0}$ (resp. $\{U_0(t)\}_{t \in \mathbb{R}}$) we solve $\partial_t f = -iAf$ (resp. $\partial_t f = -iA_0 f$) as follows

$$f = V(t)f_0 \quad (\text{resp. } f = U_0(t)f_0).$$

Below we make a check on Assumptions (A1),(A2) and (A3) [2] implies $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbb{R}$. Therefore we have (A1).

Next we show (A2). For $f, g \in \mathcal{H}$, we have by easy calculation

$$(4.4) \quad \langle ((A - i)^{-1} - (A_0 - i)^{-1})f, g \rangle_{\mathcal{H}}$$

$$= i \int_{\mathbb{R}^2} B(y)\Gamma_0((A_0 - i)^{-1}f)\overline{\Gamma_0((A^* + i)^{-1}g)}(x)dy,$$
where $\Gamma_0$ is a target operator which is defined by

$$(\Gamma_0 u)(y) = u(y, 0).$$

Note that $\Gamma_0((A_0 - i)^{-1}f)\overline{\Gamma_0((A^* + i)^{-1}g)}$ belong to $H^{1-s}(\mathbb{R}_+^3, \mathbb{C}^3)$ by Korn's inequality for any $s \in (1/2, 1)$. Since $B(y)\Gamma_0 \Pi_2(A_0 - i)^{-1}$ is a compact operator from $\mathcal{H}$ to $L^2(\mathbb{R}_+^3, \mathbb{C}^3)$ by Rellich's theorem, where $\Pi_j \iota(f_1, f_2) = f_j(j = 1, 2)$, the form $(A - i)^{-1} - (A_0 - i)^{-1}$ can be extended to a compact operator, $(\Gamma_0 \Pi_2(A^* + i)^{-1})B(y)\Gamma_0 \Pi_2(A_0 - i)^{-1}$, in $\mathcal{H}$.

To show (A3) we state a result from [2]. There exist $F_{Pu}, F_{Su}, F_{Shu}$ and $F_R$ which are partially isometric operators from $\mathcal{G} = L^2(\mathbb{R}_+^3, \mathbb{C}^3; \rho_0\rho dx)$ onto $L^2(\mathbb{R}_+^3, \mathbb{C}^3)$ and $L^2(\mathbb{R}_+^3, \mathbb{C}^3)$, respectively. Defining the operator $F$ as follows:

$$F = (F_{Pu}, F_{Su}, F_{Shu}, F_R u) \quad \text{for } u \in \mathcal{G},$$
we have by Theorem 3.6 of [2]
Lemma A. $F$ is unitary operator from $G$ to
\[
\mathcal{H} = \bigoplus_{j=1}^{3} L^2(\mathbb{R}^3_+, \mathbb{C}^3) \bigoplus L^2(\mathbb{R}^2, \mathbb{C}^3)
\]
and for every $u \in D(L_0)$
\[
FL_0 u = (c_P^2 |k|^2 F_P u, c_S^2 |k|^2 F_S u, c_{SH}^2 |k|^2 F_{SH} u, c_R^2 |p|^2 F_R u),
\]
where $k = (p, p_3) \in \mathbb{R}^2 \times \mathbb{R}_+$. Using $F_j (j = P, S, SH, R)$ as above, we construct $P_{\pm}$ as follows:

\[
P_{\pm} = T^{-1} \left\{ \sum_{j=P, S, SH} \left( F_j^* P_{\mp}^{(3)} I_{3 \times 3} F_j \begin{pmatrix} O_{3 \times 3} & F_j^* P_{\mp}^{(3)} I_{3 \times 3} F_j \\ O_{3 \times 3} & F_j^* P_{\mp}^{(3)} I_{3 \times 3} F_j \end{pmatrix} + \begin{pmatrix} F_R^* P_{\pm}^{(2)} I_{3 \times 3} F_R \\ O_{3 \times 3} & F_R^* P_{\pm}^{(2)} I_{3 \times 3} F_R \end{pmatrix} \right) \right\} T
\]

where
\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix} L_0^{1/2} & iI_{3 \times 3} \\ L_0^{1/2} & -iI_{3 \times 3} \end{pmatrix}
\]

and $P_{\mp}^{(3)}$ (resp. $P_{\pm}^{(3)}$) and $P_{\mp}^{(2)}$ (resp. $P_{\pm}^{(2)}$) are negative (resp. positive) spectral projections of

\[
D^{(3)} = \frac{1}{2i} (k \cdot \nabla_k + \nabla_k \cdot k) \quad \text{and} \quad D^{(2)} = \frac{1}{2i} (p \cdot \nabla_p + \nabla_p \cdot p),
\]

respectively. Using the representation of the generalized eigenfunction of $L_0$ (see [2]) and the Mellin transformation we show (A3.1)~-(A3.4) (cf. [13] and [6]). The Mellin transformations for $D^{(3)}, D^{(2)}$ are given as

\[
(M^{(3)} u)(\lambda, \omega) = (2\pi)^{-1/2} \int_{0}^{+\infty} r^{1/2 - i\lambda} u(r\omega) dr
\]

and

\[
(M^{(2)} v)(\lambda, \nu) = (2\pi)^{-1/2} \int_{0}^{+\infty} r^{-i\lambda} v(r\nu) dr,
\]

where $u(k) \in C_0^\infty(\mathbb{R}^3_+ \backslash \{0\}), v(p) \in C_0^\infty(\mathbb{R}^2 \backslash \{0\}), \omega \in S_+^2 = \{ (\omega_1, \omega_2, \omega_3) = (\overline{\omega}, \omega_3) \in S^2 : \omega_3 > 0 \}$ and $\nu \in S^1$. Then $M^{(3)}$ (resp. $M^{(2)}$) is extended to a unitary operator from $L^2(\mathbb{R}^3_+)$ (resp. $L^2(\mathbb{R}^2)$) to $L^2(\mathbb{R} \times S_+^2)$ (resp. $L^2(\mathbb{R} \times S^1)$) (cf. [13] Lemma 2).

Proposition 4.1. $P_{\pm}$ as in (4.5) satisfy (A3).

To show Proposition 4.1 we prepare
**Lemma 4.2.** Let $\psi(\lambda)$ be same as in (A3) and $0 < \delta < c_R$ (for $c_R$, see Appendix). Then for any positive integer $N$ and $t \in \mathbb{R}_\pm$, there exists a positive constant $C_{N,\psi}$ which is independent of $t$ such that

\begin{align*}
(4.6) \quad & \| \nabla_x (e^{-itA_0} \psi(A_0) P_{\pm} f) \|_{L^2(\mathbb{R}^3_+, \mathbb{C}^3)} \leq C_{N,\psi} (1 + |t|)^{-N} \| f \|_{\mathcal{H}}, \\
(4.7) \quad & \| (e^{-itA_0} \psi(A_0) P_{\pm} f) \|_{L^2(\mathbb{R}^3_+, \mathbb{C}^3)} \leq C_{N,\psi} (1 + |t|)^{-N} \| f \|_{\mathcal{H}}, \\
(4.8) \quad & \| \Gamma_0 (e^{-itA_0} \psi(A_0) P_{\pm} f) \|_{L^2(\mathbb{R}^3_+, \mathbb{C}^3)} \leq C_{N,\psi} (1 + |t|)^{-N} \| f \|_{\mathcal{H}},
\end{align*}

for any $f \in \mathcal{H}_0$, where

\begin{align*}
&\| u \|_{L^2(\mathbb{R}^3_+, \mathbb{C}^3)} = (\int_{\mathbb{R}^3_+} |u|^2 dx)^{\frac{1}{2}} \quad \text{and} \quad \| v \|_{L^2(\mathbb{R}^3_+, \mathbb{C}^3)} = (\int_{\mathbb{R}^3_+} |v|^2 dy)^{\frac{1}{2}}.
\end{align*}

This lemma is the key lemma to show (A3). The proof is done by using $M^{(3)}, M^{(2)}$ and Lemma A. But we omit to prove (cf. [13] or [6]).

**proof of Proposition 4.1.** Lemma A of Appendix implies that $P_+$ and $P_-$ are projection operators and satisfy $P_+ + P_- = \text{Id}$ in $\mathcal{H}$. Below we show (A3.1)$\sim$(A3.4).

For any $f, g \in \mathcal{H}$ we have by (4.4)

\begin{align*}
|\langle Ke^{-itA_0} \psi(A_0) P_+ f, g \rangle_{\mathcal{H}}| & \leq CI(t) \times (\| A^* (A^* + i)^{-1} g \|_{\mathcal{H}} + \| (A^* + i)^{-1} g \|_{\mathcal{H}}),
\end{align*}

where

\begin{align*}
I(t) = (\int_{\mathbb{R}^3} |B(y) \Gamma_0 (e^{-itA_0} (A_0 - i)^{-1} \psi(A_0) f) \|_{\mathcal{H}}^2 dy)^{\frac{1}{2}} \times \\
\times (\| A^* (A^* + i)^{-1} g \|_{\mathcal{H}} + \| (A^* + i)^{-1} g \|_{\mathcal{H}}).
\end{align*}

Decomposing $I(t)$ as follows:

\begin{align*}
I(t) \leq & C \{ (\int_{\mathbb{R}^2 \cap \{|y| \leq \delta t\}} \| \Gamma_0 (e^{-itA_0} (A_0 - i)^{-1} \psi(A_0) f) \|_{\mathcal{H}}^2 dy)^{\frac{1}{2}} \\
& + (\int_{\mathbb{R}^2 \cap \{|y| \geq \delta t\}} |B(y) \Gamma_0 (e^{-itA_0} (A_0 - i)^{-1} \psi(A_0) f) \|_{\mathcal{H}}^2 dy)^{\frac{1}{2}} \},
\end{align*}

we have by (4.8) of Lemma 4.2 and (4.3)

\begin{align*}
I(t) \leq C_{N,\psi} \{(1 + t)^{-N} + \varphi(\delta t)\} \| f \|_{\mathcal{H}}.
\end{align*}

Therefore (A3.1) is proven.
To prove (A3.2) and (A3.3) we note

$$\langle f, K^* g \rangle_{\mathcal{H}} = \langle ((A - i)^{-1} - (A_0 - i)^{-1}) f, g \rangle_{\mathcal{H}}$$

for any $f, g \in \mathcal{H}$.

By easy calculation we have

$$(4.9) \quad \langle ((A - i)^{-1} - (A_0 - i)^{-1}) f, g \rangle_{\mathcal{H}} = i \int_{\mathbb{R}^2} \Gamma_0((A - i)^{-1} f) \overline{B(y) \Gamma_0((A_0 + i)^{-1} g)} dy.$$

Then using (4.9) and the same way as in the proof of (A3.1), we obtain (A3.2) and (A3.3). Here we omit the detail.

We show (A3.4). Lemma 4.2 implies

$$|\langle e^{itA_0} \psi(A_0) P_- f_t, g \rangle_{\mathcal{H}}| \leq C_N \psi \left( (1 + t)^{-N} ||g||_{\mathcal{H}} + ||\nabla_x f||_{L^2(\mathbb{R}_t^1, \mathbb{C}^3)} \right),$$

for any $g \in \mathcal{H}$ and any positive integer $N$. Thus, noting $\sup_{t \in \mathbb{R}} ||f_t||_{\mathcal{H}} < \infty$, we have (A3.4). \(\square\)

Application 2 (Acoustic wave equations with dissipative terms in stratified media).

We shall apply Theorem 3. First we explain acoustic operator.

Let $n \geq 1$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. We set

$$c_0(y) = \begin{cases} 
  c_+ & (y \geq h) \\
  c_h & (0 < y < h) \\
  c_- & (y \leq 0),
\end{cases}$$

for some positive constants $h$ and $c_+, c_-, c_h$.

Acoustic operators are

$$L_0 = -c_0(y)^2 \Delta,$$

where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2}.$$

Considering the case $c_h < \min(c_+, c_-)$ we find the guided waves (cf. [18] or [19]).

But we do not restrict ourselves to such cases.

$L_0$ is a non-negative self-adjoint operator in $\mathcal{G} = L^2(\mathbb{R}_t^{n+1}; c_0(y)^{-2} dx dy)$, $D(L_0)$ is given by $H^2(\mathbb{R}_t^{n+1})$, $H^s(\mathbb{R}_t^{n+1})$ being Sobolev space of order $s$ over $\mathbb{R}_t^{n+1}$.

We deal with the following dissipative wave equations:

$$(4.10) \quad \partial_t^2 u(x, y, t) + b(x, y) \partial_t u(x, y, t) + L_0 u(x, y, t) = 0$$

and

$$(4.11) \quad \partial_t^2 u(x, y, t) + \langle \partial_t u, \varphi \rangle_\mathcal{G} \varphi(x, y) + L_0 u(x, y, t) = 0,$$
where \((x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times [0, \infty)\) and \(\langle \cdot, \cdot \rangle_{\mathcal{G}}\) is the inner-product of \(\mathcal{G}\).

We assume that \(b(x, y)\) and \(\varphi(x, y)\) are measurable functions which satisfy

\[0 \leq b(x, y) \leq C(1 + |x|^2 + y^2)^{-\frac{\theta}{2}}\]

and

\[\varphi(x, y) \in L^2(\mathbb{R}^{n+1}; (1 + |x|^2 + y^2)^{\frac{\theta}{2}}) dxdy\]

for some \(\theta > 1\) and \(C > 0\).

We shall show the existence of the scattering states for (4.10) and (4.11) which are considered as the perturbed systems of

\[(4.12) \quad \partial_t^2 u(x, y, t) + L_0 u(x, y, t) = 0, \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}
\]

In [19], [1], and [21], we can find local resolvent estimates as follows: for any \(\beta > \alpha > 0\), there exists positive constants \(C_{\alpha, \beta}\) and \(\eta\) such that

\[(4.13) \quad \sup_{\alpha \leq |\text{Re} \zeta| \leq \beta, 0 < |\text{Im} \zeta| < \eta} \| X_\frac{\zeta}{\gamma} (L_0 - \zeta^2)^{-1} X_{\frac{\zeta}{\gamma}} \|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})} \leq C_{\alpha, \beta}.
\]

where \(\zeta \in \mathbb{C}, X_\gamma = (1 + |x|^2 + y^2)^{-\frac{\gamma}{2}}\) and \(\| \cdot \|_{L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})}\) is the norm of the bounded operator in \(L^2(\mathbb{R}^{n+1})\).

[12] has already dealt with the case \(c_h = c_+ = c_- = 1\) and \(n \geq 2\) of (4.10). His proof has been based on Kato's smooth perturbation theory [10] and global resolvent estimates for \(L_0\) (see also [10] Theorem 4.4.1).

We apply Theorem 3 (Corollary 4) to (4.10). We set \(f(t) = (u(t, x, y), \partial_t u(t, x, y))\).

Then (4.12) and (4.10) can be written as \(\partial_t f = -iA_0 f\) and \(\partial_t f = -iAf\) respectively, where

\[A_0 = i \begin{pmatrix} 0 & 1 \\ -L_0 & 0 \end{pmatrix}, \quad A = i \begin{pmatrix} 0 & 1 \\ -L_0 & -b(x, y) \end{pmatrix}.
\]

Let \(\mathcal{H}\) be Hilbert spaces with inner product

\[\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{n+1}} (\nabla f_1(x, y) \overline{\nabla g_1(x, y)} + f_2(x, y) \overline{g_2(x, y)} c_0^{-2}(y)) dxdy,
\]

and \(\| \cdot \|_{\mathcal{H}}\) is the corresponding norm, where \(f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\).

The domains of \(A_0\) is

\[D(A_0) = \{ f \in \mathcal{H}; \Delta f_1 \in L^2(\mathbb{R}^{n+1}), f_2 \in H^1(\mathbb{R}^{n+1}) \}.
\]

Then \(A_0\) is a self-adjoint operator in \(\mathcal{H}\) and generates a unitary group \(\{ U_0(t) \}_{t \in \mathbb{R}}\) in \(\mathcal{H}\). Below we make a check on (A1), (A4) and (A5).

Note that

\[T_0 A_0 T_0^{-1} = \begin{pmatrix} \sqrt{L_0} & 0 \\ 0 & -\sqrt{L_0} \end{pmatrix},
\]

where

\[T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{L_0} & i \\ \sqrt{L_0} & -i \end{pmatrix}.
\]
and $T_0$ is a unitary operator from $\mathcal{H}$ onto $\mathcal{G} \times \mathcal{G}$. It follows from (4.13) that for any $u \in \mathcal{G}$

$$\sup_{\alpha \leq |Re \zeta| \leq \beta, 0 < |Im \zeta| < \eta} |\text{Im}((\pm \sqrt{L_0} - \zeta)^{-1}X_{\frac{\theta}{2}}u, X_{\frac{\theta}{2}}u)|_\mathcal{G} < \infty.$$ 

Therefore we have (A1) by [16] Theorem XIII-20.

Since

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & b(x, y) \end{pmatrix}$$

is $A_0$-compact by Rellich's theorem, we have (A2). Therefore $A$ generates a contraction semi-group $\{V(t)\}_{t \geq 0}$ in $\mathcal{H}$.

In the same argument as in [12]§3 we can show (A5) as follow. Let $g = (g_1, g_2) \in \mathcal{H}$. We set

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (A_0 - \zeta)^{-1}\sqrt{B_0} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$ 

Then we have

$$(L_0 - \zeta^2)u_2 = \zeta \sqrt{b(x, y)}g_2$$

and

$$\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}g = \sqrt{B_0}u =^t (0, \sqrt{b(x, y)}u_2).$$

Therefore we can calculate as follows:

(4.14) $$||\sqrt{B_0}(A_0 - \zeta)^{-1}\sqrt{B_0}g||_\mathcal{H} = ||\zeta||\sqrt{b(x, y)}(L_0 - \zeta^2)^{-1}\sqrt{b(x, y)}g_2||_\mathcal{G}.$$ 

(4.13) and (4.14) imply (A5). Thus we have the conclusion of Theorem 3(Corollary 4) for (4.10) and (4.12).

Next we apply Theorem 3(Corollary 4) to (4.11). we set

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & \langle \cdot, \varphi \rangle_\mathcal{G} \varphi \end{pmatrix}$$

Then $B$ is a compact operator in $\mathcal{H}$. We shall show (A5). Note that

(4.15) $$||\text{Im}\zeta||\sqrt{B}(A_0 - \zeta)^{-1}f||_\mathcal{H}^2 \leq ||\text{Im}\zeta||X_{\frac{\theta}{2}}((A_0 - \zeta)^{-1}f)_2||_\mathcal{G}^2 \times ||X_{-\frac{\theta}{2}}\varphi||_\mathcal{G}^2$$

for any $f \in \mathcal{H}$. We set

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & X_\theta \end{pmatrix}.$$ 

Then we have

$$||\text{Im}\zeta||X_{\frac{\theta}{2}}((A_0 - \zeta)^{-1}f)_2||_\mathcal{G}^2 \leq ||\text{Im}\zeta||\sqrt{B_1}(A_0 - \zeta)^{-1}f||_\mathcal{H}^2 \leq ||\sqrt{B_1}((A_0 - \zeta)^{-1} - (A_0 - \overline{\zeta})^{-1})\sqrt{B_1}||_\mathcal{H}||f||_\mathcal{H}^2.$$ 

Noting (4.14) which is changed $B_0$ and $b(x, y)$ to $B_1$ and $X_\theta$, respectively we get (A5). Therefore we have the conclusion of Theorem 3(Corollary 4) for (4.11) and
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