| Title | Binding through coupling to a radiation field (Spectral and Scattering Theory and Related Topics) |
| Author(s) | Hiroshima, Fumio |
| Citation | 数理解析研究所講究録 (2001), 1208: 69-79 |
| Issue Date | 2001-05 |
| URL | http://hdl.handle.net/2433/41059 |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

Kyoto University
Binding through coupling to a radiation field

Fumio Hiroshima

April 12, 2001

1 Introduction

1.1 Definition

This is a joint work with H. Spohn\(^1\). We consider a system of one electron interacting with a quantized radiation field. In particular we investigate the so called Pauli-Fierz [13] model\(^2\). Although the Pauli-Fierz model is a nonrelativistic model, it correctly describes the interaction between low energy electrons and photons in a sense. Actually the Lamb shift and gyromagnetic ratio shift were described by using the Pauli-Fierz model. See [2, 14, 12].

In this paper we take the dipole approximation for simplicity. Moreover we suppose that the electron is spinless, moves in the \(d\)-dimensional space, and has the \(d - 1\) transverse degrees of freedom. Throughout this paper we assume

\[ d \geq 3. \]

The Hamiltonian of the system is of the form

\[ H(\alpha) = \frac{1}{2m} (p \otimes I - \alpha I \otimes A)^2 + V \otimes I + I \otimes H_f \]  (1.1)

acting on the Hilbert space

\[ \mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\mathrm{EM}}. \]
Here $\mathcal{F}_{\mathrm{EM}}$ denotes the Boson Fock space over $W := \bigoplus^{d-1} L^2(\mathbb{R}^d)$

$$\mathcal{F}_{\mathrm{EM}} := \bigoplus_{n=0}^\infty [\otimes_{s}^{n}W],$$

where $\otimes_{s}^{n}W$ denotes the $n$-fold symmetric tensor product of $W$ with $\otimes_{s}^{0}W := c$. $m$ is the bare mass of the electron and $\alpha$ a coupling constant. We adopt the unit $\hbar = 1 = c$. Then $\alpha \approx \sqrt{137}$. $p = -i\vec{\nabla}$ is the momentum operator canonically conjugate to the position operator $x$ in $L^2(\mathbb{R}^d)$, and $V = V(x)$ an external potential for which precise conditions will be specified below. The smeared radiation field is defined by

$$A_{\mu} := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \int e_{\mu}^{\ell}(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d \omega(k)}} a^{\uparrow \ell}(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d \omega(k)}} a^{\ell}(k) \right\} d^dk,$$

and the free Hamiltonian by

$$H_{\mathrm{f}} := \sum_{r=1}^{d-1} \int \omega(k) a^{\uparrow \ell}(k) a^{\ell}(k) dk,$$

where the dispersion relation is given by

$$\omega(k) := |k|.$$
(1.2) and (1.3) ensure that $H(\alpha)$ is a well defined symmetric operator in $\mathcal{H}$. It is known that

$$\text{Spec}(H_f) = [0, \infty)$$

and

$$\text{Spec}_p(H_f) = \{0\}.$$ 

The multiplicity of $\{0\}$ is one, and $H_f \Omega = 0$, where $\Omega := 1 \oplus 0 \oplus 0 \oplus \cdots$ is the Fock vacuum in $\mathcal{F}_{\text{EM}}$.

1.2 Problems

Suppose that $V$ is relatively bounded with respect to $-\Delta$ with a sufficiently small relative bound. Then it is proven [8] that $H(\alpha)$ is self-adjoint on $D(\Delta \otimes I) \cap D(I \otimes H_f)$ and bounded from below for arbitrary couplings. Moreover by investigating the integral kernel of $e^{-tH(\alpha)}$, $t \geq 0$, the uniqueness of the ground state, if it exists, is established in [6]3.

In the case when $-\frac{1}{2m}\Delta + V$ has the positive spectral gap,

$$\inf \text{Spec}_{\text{ess}}\left(-\frac{1}{2m}\Delta + V\right) - \inf \text{Spec}\left(-\frac{1}{2m}\Delta + V\right) > 0,$$

the existence of the ground state of the full Pauli-Fierz Hamiltonian is established in [3, 5, 9, 4]. In particular, Bach, Fröhlich and Sigal [3] proved it under no assumption of infrared cutoff condition4 but sufficiently weak couplings. For arbitrary couplings, it is established in [4] due to Griesemer, Lieb and Loss.

The main purpose of this paper is to prove the existence of the ground state of $H(\alpha)$ under no assumption of the positive spectral gap. In the

3 For the full Pauli-Fierz Hamiltonian, self-adjointness and the uniqueness of the ground state are established in [8] and [6], respectively.

4 The condition $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 / \omega(k)^3 dk < \infty$ is called the infrared cutoff condition. In the case of $d = 3$ this condition implies $0 = \varphi(0) = (2\pi)^{-3/2} \int \varphi(x) dx$, i.e., physically the electron charge turns out to be negative.
zero spectral gap case, $-\frac{1}{2m} \Delta + V$ may have no ground state. That is, we show that strong couplings produce the ground state. The physical reasoning behind such a result is as follows. As the electron binds photons it acquires the effective mass

$$m \to m + \delta m(\alpha^2)$$

which is increasing in $|\alpha|$. Roughly speaking $H(\alpha)$ may be replaced by

$$H(\alpha) \sim -\frac{1}{2(m + \delta m(\alpha^2))} \Delta + V,$$

(1.4)

and, for the sufficiently large $|\alpha|$, the right hand side of (1.4) may have ground states. Needless to say (1.4) has no sharp mathematical meaning, we show, however, the associated phenomena in this paper.

---

**Figure 1:** $H(0)$

**Figure 2:** $H(\alpha)$
This paper is organized as follows. In Section 2 we prove the binding. In Section 3 we give some examples of the external potentials. Finally in Section 4 we give some remarks.

2 Binding

We suppose the following assumptions on \( V \).

(1) \( \|Vf\| \leq a\|\Delta f\| + b\|f\| \) for \( f \in D(\Delta) \) with sufficiently small \( a \geq 0 \), and positive \( b \geq 0 \).

(2) \( V \in C^1(\mathbb{R}^d) \) and \( \partial_\mu V \in L^\infty(\mathbb{R}^d), \mu = 1, ..., d \).

(3) There exist \( \mu_0 \geq 1 \) and \( r_0 > 0 \) such that for all \( \eta > \mu_0 \)

\[
\inf \text{Spec}
\left(-\frac{1}{2m}\Delta + \eta V\right) \leq -r_0,
\]

and

\[
\text{Spec}_{\text{ess}}
\left(-\frac{1}{2m}\Delta + \eta V\right) = [0, \infty).
\]

It is of interest to investigate sufficiently shallow external potentials. Since \( d \geq 3 \), for such a shallow \( V \), \( -\frac{1}{2m}\Delta + V \) may have no ground state. If \( -\frac{1}{2m}\Delta + V \) has no ground state, then the decoupled Hamiltonian

\[
H(\alpha = 0) = \left(-\frac{1}{2m}\Delta + V\right) \otimes I + I \otimes H_f
\]

also has no ground state.

For later use we define the dilatation unitary of \( L^2(\mathbb{R}^d) \) by

\[
D(\kappa)f(k) := \kappa^{d/2}f(k/\kappa),
\]

where \( \kappa > 0 \) denotes the scaling parameter. The scaled Hamiltonian is defined by

\[
H(\alpha, \kappa) := \kappa^2D(\kappa)^{-1}\left\{ \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + I \otimes H_f + \frac{1}{\kappa^2}V(x/\kappa) \otimes I \right\} D(\kappa)
\]
\[ \frac{1}{2m} (p \otimes I - \kappa \alpha I \otimes A)^2 + V \otimes I + \kappa^2 I \otimes H_f. \]

We suppose the following technical assumptions on \( \hat{\varphi} \).

1. \( \hat{\varphi}(k) = \hat{\varphi}(|k|) \).
2. \( \omega^{n/2} \hat{\varphi} \in L^2(\mathbb{R}^d) \) for \( n = -5, -4, -3, -2, -1, 0, 1, 2 \).
3. \( |\hat{\varphi}(\sqrt{s})| s^{(d-1)/2} \in L^\epsilon([0, \infty), ds), 0 < \epsilon < 1 \), and is Lipschitz continuous of order strictly less than one.
4. \( \|\hat{\varphi} \omega^{(d-2)/2}\|_\infty < \infty \) and \( \|\hat{\varphi} \omega^{(d-1)/2}\|_\infty < \infty \).
5. \( \hat{\varphi}(k) \neq 0 \) for all \( k \neq 0 \).

Thus (1)–(5) ensure the following lemmas\(^5\).

**Lemma 2.1** There exist the unitary operator \( U(\kappa) \) such that

\[ U(\kappa)^{-1} H(\alpha, \kappa) U(\kappa) = H_{\text{eff}} + \kappa^2 H_f + \kappa^2 \alpha^2 g + \delta V, \]

where

\[ H_{\text{eff}} := -\frac{1}{2m_{\text{eff}}} \Delta + V, \]

\[ m_{\text{eff}} = m_{\text{eff}}(\alpha^2) := m + \alpha^2 \left( \frac{d-1}{d} \right) ||\hat{\varphi}/\omega||^2, \]

and

\[ g := \frac{d-1}{2\pi} \int_{-\infty}^{\infty} \frac{t^2 ||\hat{\varphi}/(t^2 + \omega^2)||^2}{m + \alpha^2 \left( \frac{d-1}{d} \right) ||\hat{\varphi}/\sqrt{t^2 + \omega^2}||^2} dt. \]

Moreover

\[ \delta V = \delta V(\alpha, \kappa) := U(\kappa)^{-1} (V \otimes I) U(\kappa) - V \otimes I. \]

**Lemma 2.2** We have

\[ -\frac{D(\alpha)}{\kappa} (H_f + I) \leq \delta V \leq \frac{D(\alpha)}{\kappa} (H_f + I) \]

in the sense of form, where \( D(\alpha) \) is a real number satisfying

\[ \lim_{|\alpha| \to \infty} D(\alpha) = 0. \]

\(^5\) See [1, 11] for details.
Let
\[\alpha_{\text{critical}} := \sqrt{m(\mu_0 - 1)} \sqrt{\frac{d}{d - 1}} \|\phi/\omega\|^{-1}.\]
We see
\[H_{\text{eff}} = \frac{m}{m_{\text{eff}}} \left(-\frac{1}{2m} \Delta + \frac{m_{\text{eff}}}{m} V\right).\]
Then, in the case of \(\alpha > \alpha_{\text{critical}}\), it follows that
\[\inf_x V(x) \leq \inf \text{Spec}(H_{\text{eff}}) < -r_0 \frac{m}{m_{\text{eff}}}.\]
In particular the ground states of \(H_{\text{eff}}\) exist.

**Theorem 2.3** Let \(k = 1\). There exists \(\alpha_* > \alpha_{\text{critical}}\) such that for all \(|\alpha| \geq \alpha_*\) the ground state of \(H(\alpha)\) exists and it is unique.

**Proof:** Let \(N\) be the number operator in \(\mathcal{F}_{\text{EM}}\) and \(0 < \nu\). By a momentum lattice approximation we see that \(H(\alpha) + \nu N\) has the normalized ground state \(\Phi_\nu\). Let \(E_I\) denote the spectral projection of \(H_{\text{eff}}\) to \(I \subset \mathbb{R}\) and \(P_\Omega\) the projection to \(\Omega\). Let \(P = E(-\infty,-r_0 m/m_{\text{eff}}) \otimes P_\Omega\) and \(\Sigma := \inf \text{Spec}(H_{\text{eff}})\). Then we can see that
\[
(\Phi_\nu, P\Phi_\nu) \geq 1 - \left(\frac{|\alpha| \epsilon}{m_{\text{eff}}}\right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}
\]
with some constant \(\epsilon\). Note that
\[
\lim_{|\alpha| \to 0} \frac{|\alpha|}{m_{\text{eff}}(\alpha^2)} = 0
\]
and
\[
\lim_{|\alpha| \to 0} \Sigma = \inf_x V(x).
\]
Thus for sufficiently large \(|\alpha|\) the right hand side of (2.1) is strictly positive. Take a subsequence \(\nu'\) such that \(\Phi_{\nu'} \to \Phi\) as \(\nu \to 0\) weakly. Since \(P\) is a finite rank operator, \(P\Phi_\nu\) strongly converges to \(P\Phi\) and
\[
(\Phi, P\Phi) \geq 1 - \left(\frac{|\alpha| \epsilon}{m_{\text{eff}}}\right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}
\]
holds. In particular $\Phi \neq 0$. Hence $\Phi$ is the ground state. \qed

By the assumptions $H_{\text{eff}}$ has ground states for $|\alpha| > \alpha_{\text{critical}}$. We have to make sure that $H(\alpha)$ has the same properties.

**Theorem 2.4** We suppose that $\kappa$ is sufficiently large. Set $V_\kappa(x) := \kappa^{-2}V(x/\kappa)$. Then the ground state of

$$H_\kappa(\alpha) = \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + V_\kappa \otimes I + I \otimes H_f$$

exists for all $|\alpha| > \alpha_{\text{critical}}$ and it is unique.

**Proof:** We have

$$H_\kappa(\alpha) = \frac{1}{\kappa^2}D(\kappa)H(\alpha, \kappa)D(\kappa)^{-1}.$$  

Thus it is enough to prove the existence of the ground states of $H(\alpha, \kappa)$. From the momentum lattice approximation we see that $H(\alpha, \kappa) + \nu N$ has the ground state $\Phi_\nu$. Moreover we have the inequality

$$(\Phi_\nu, P\Phi_\nu) \geq 1 - \frac{1}{\kappa^6} \left(\frac{|\alpha|\epsilon}{m_{\text{eff}}}\right)^2 - \frac{1}{\kappa^2} \left(\frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}\right).$$

Then the theorem follows in the same way as in Theorem 2.3. \qed

**3 Example**

Suppose that

$$V(x) \leq 0.$$  

Let

$$N(V) := a_d \int_{\mathbb{R}^d} |mV(x)|^{d/2} dx,$$

where $a_d$ is a universal constant. The following is known as the Lieb-Thirring equality

$$N(V) = \#\{\text{the nonnegative eigenvalues of } -\frac{1}{2m}\Delta + V\}.$$
Suppose that
\[ N(V) < 1. \]
Then \( H(0) \) has no ground state, \( H(\alpha) \) for sufficiently large \(|\alpha|\), however, has the ground state and it is unique by Theorem 2.3.

**Remark 3.1** If \(-\frac{1}{2m}\Delta + V\) has the ground state with a positive spectral gap, then \( H(\alpha) \) has the ground state for arbitrary \( \alpha \in \mathbb{R} \).

### 4 Concluding remarks

(1) The full Pauli-Fierz Hamiltonian is defined by
\[
H(\alpha) = \frac{1}{2m}(p \otimes I - \alpha A)^2 + V \otimes I + I \otimes H_f.
\]
Here under the identification \( \mathcal{H} \cong \mathcal{F}_{\mathrm{EM}dx} \)
\[
A_\mu := \int_{\mathbb{R}^d} A_\mu(x)dx,
\]
and
\[
A_\mu(x) := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \int e_\mu^r(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d\omega(k)}} e^{-ikx_\mu^r} a^r(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d\omega(k)}} e^{ikx_\mu^r} a^r(k) \right\} d^d k.
\]
For the full Pauli-Fierz Hamiltonian, it seems to be unknown the binding.

(2) For \( \alpha \) such that \( 0 < |\alpha| < \alpha_{\text{critical}} \), no existence of the ground state is not known.

(3) The Nelson Hamiltonian with two charged particles is defined by
\[
H_{\text{Nelson}}(\alpha) := \left(-\frac{1}{2m}\Delta + V\right) \otimes I + I \otimes H_N + \alpha \phi
\]
acting on
\[
\mathcal{H} := L^2(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathcal{F},
\]
where $\mathcal{F}$ denotes the Boson Fock space over $L^2(\mathbb{R}^d)$. The free Hamiltonian is defined by

$$H_N := \int \omega(k) a^\dagger(k) a(k) dk$$

and the scalar field by

$$\phi := \int_{\mathbb{R}^{d_\mathbb{R} \times d}} \phi(x) dx,$$

$$\phi(x) := \sum_{j=1}^{2} \frac{1}{\sqrt{2}} \int \lambda(-k) e^{-ikx^j} a^\dagger(k) + \hat{\lambda}(k) e^{ikx^j} a(k) dk.$$

Roughly speaking $H_{\text{Nelson}}(\alpha)$ may be replaced by

$$H_{\text{Nelson}}(\alpha) \sim -\frac{1}{2m} \Delta + V + V_{\text{eff}},$$

$$V_{\text{eff}}(x^1, x^2) = -\frac{\alpha^2}{2} \int_{\mathbb{R}^d} \frac{\hat{\lambda}(k)^2}{\omega(k)} e^{-ik(x^1 - x^2)} dk.$$
Then we can also prove the binding of the Nelson Hamiltonian under certain conditions. We omit details.

References


