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ON AN ABSTRACT RADIATION CONDITION

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INTRODUCTION

We shall present an abstract radiation condition in terms of the Mourre theory of conjugate operator method.

Let $\mathcal{H}$ be a Hilbert space and $A$ be a self-adjoint operator in $\mathcal{H}$. For $s \geq 0$ consider the Hilbert space $A^s = D((A^*A)^{1/2})$ with the graph norm, and if $s < 0$, $A^s = (A^{-s})^*$. Then, if $s \geq 0$, $A^s \subseteq \mathcal{H} \subseteq A^{-s}$ continuously and densely, and the scalar product of $\mathcal{H}$ extends to a natural duality $(\cdot, \cdot)_{A^{s}} : A^s \times A^{-s} \to \mathbb{C}$ for all $s \in \mathbb{R}$. We denote by $P_{\pm}$ the spectral projectors of $A$ associated to the half-lines $[0, +\infty)$ and $(-\infty, 0]$, respectively.

We recall now some (Besov) spaces of operators (see [ABG]). Let $S$ be a bounded operator on $\mathcal{H}$. We say that $S \in C^{k}(A)$, $k$ positive integer, if the application $\mathbb{R} \ni \tau \mapsto W(\tau)[S] = e^{i\tau A}S e^{-i\tau A} \in \mathcal{B}(\mathcal{H})$ is strongly $C^{k}$; in this case $ad_{A}^s S$ can be extended as a bounded operator on $\mathcal{H}$. Consider $\theta \in (0, 1], p \in [1, \infty]$; we say that $S \in C^{\theta,p}(A)$ if $(\tau \to (W(\tau) - I)^m[S])/|\tau|^{|\theta|+1/p} \in L^{p}((0, \infty))$, where $m = 1$ if $\theta < 1$, and $m = 2$ if $\theta = 1$. (If $p = \infty$, this condition should be read as $\sup_{\tau > 0}||(W(\tau) - I)^m[S])/|\tau|^{|\theta| < \infty$.) For general $\theta > 0$, we say that $S \in C^{\theta,p}(A)$ if $S \in C^{1}(A)$ and $ad_{A}^s S \in C^{\theta-1,p}(A)$, where $l$ is the largest integer $l < \theta$.

Let $L$ be a self-adjoint operator in $\mathcal{H}$. Then $L \in C^{\theta,p}(A)$ (or $C^{k}(A)$) if $(L - z)^{-1} \in C^{\theta,p}(A)$ (or $C^{k}(A)$) for some (and hence all) $z \in \mathbb{C} \setminus \sigma(L)$.

If $L$ is a self-adjoint operator of class $C^{1}(A)$, then the commutator $i[L, A]$ is defined as a continuous form on the domain of $L$. Then one can define the strict Mourre set $\mu^{A}(L)$ of $L$ with respect to $A$ as the set of $\lambda \in \mathbb{R}$ with the property that there exists $J = (\lambda - \delta, \lambda + \delta) \neq \emptyset$ and $d > 0$ such that

$$E_{L}(J) \langle [L, A] E_{L}(J) \rangle \geq d E_{L}(J).$$

We recall that if $L$ has a spectral gap and $L \in C^{1,1}(A)$, then there exist $R_{L}(\lambda \pm i0) = \lim_{\epsilon \to 0} (L - \lambda \mp i\epsilon)^{-1}$ uniformly in $\mathcal{B}(A^{s}, A^{-s})$, whenever $s > 1/2$.

The following theorem was given in [BGS1] (for the proof see [BGS2]; see also [J] for some earlier results).

THEOREM 1. Let $s > 1/2$ be a real number and $L$ be a self-adjoint operator with a spectral gap and of class $C^{s+1,2}(A)$. Then we have $P_{\mp} R_{L}(\lambda \pm i0) A^{s} \subseteq A^{s-1}$ for each $\lambda \in \mu^{A}(L)$.

It turns out that in some stronger hypotheses this condition characterizes $R_{L}(\lambda \pm i0)$. Namely, we prove the following theorem, extending some results of [B2], [M].

THEOREM 2. Let $1 \geq \theta > 1/2$ be a real number, $L \geq -M$ be a bounded from below self-adjoint operator of class $C^{1+\theta,\infty}(A)$ such that $i[L, A] \in \mathcal{B}(\mathcal{G}, \mathcal{G}^{*})$, where $\mathcal{G}$ is the form domain of $L$, and $\lambda \in \mu^{A}(L)$. Suppose $u \in A^{-s}$, $s \in (1/2, \theta)$ satisfies:

a) $u, (L - \lambda) \varphi_{-s, s} = 0$ for all $\varphi \in (L + M)^{-1}A^{s}$,

b) there exists $\alpha < \theta/2$ such that $\langle A \rangle^{-\alpha} P_{-}(A)u \in \mathcal{H}$ (or $\langle A \rangle^{-\alpha} P_{+}(A)u \in \mathcal{H}$).

Then $u = 0$. 
The proof follows Isozaki's proof of some type of radiation conditions which are strongly related to those presented here. (See [I1], [I2], [I3].) We only remark here that Theorem 2 provides some useful results in the study of the layered media.

One of the tools needed here is the functional calculus using almost analytic extensions of symbols. Let \( m \in \mathbb{R} \). We denote by \( S^m \) the set of symbols \( f \in C^\infty(\mathbb{R}) \) that satisfy

\[
  p_k(f) = \sup_{x \in \mathbb{R}} (x)^{m-k} |f^{(k)}(x)| < \infty.
\]

Then \( S^m \), endowed with the seminorms \( p_k \) is a Fréchet space. The following result can be found in [B2], [M] (see also [DG] for the main idea).

**PROPOSITION 3.** Consider a bounded family of symbols \( \{f_\epsilon\} \subset S^m \). Then there exists a family of functions (the almost analytic extensions) \( \{\tilde{f}_\epsilon\} \subset C^\infty(\mathbb{C}) \) such that:

1. \( |\text{Im} z| \leq \langle \text{Re} z \rangle \) on \( \text{supp} \tilde{f}_\epsilon \),
2. \( |\overline{\partial} \tilde{f}_\epsilon(z)| \leq C_N(z)^{m-N-1} |\text{Im} z|^{N-1} \) for all \( N \geq 0 \) and all \( z \in \mathbb{C} \), where the constants \( C_N \) do not depend on \( z \) and \( \epsilon \).

This construction provides an useful representation for the functional calculus of a self-adjoint operator, due to Helffer-Sjöstrand ([HS]): Let \( A \) be a self-adjoint operator on \( \mathcal{H} \) and \( f \in S^{-\delta}, \delta > 0 \). Then

\[
  f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{f}(z)(A-z)^{-1} \, dx \, dy,
\]

where \( z = x + iy \) and \( \tilde{f} \) is an almost analytic extension of \( f \). If \( B \) is a bounded operator with \( ad_A^n \) is a bounded form on the domain of \( A \), and \( ad_A k f^{(k)}(A) \) (respectively \( f^{(k)}(A)ad_A k \)) \( k = 1, \ldots, n-1 \), are bounded operators, then

\[
  [B, f(A)] = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} ad_A^k(B)f^{(k)}(A) + R_n(A,B)
\]

where

\[
  R_n(A,B) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{f}(z)(A-z)^{-1} ad_A^n(B)(A-z)^{-n} \, dx \, dy,
\]

\[
  R'_n(A,B) = \frac{(-1)^n}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{f}(z)(A-z)^{-n} ad_A^n(B)(A-z)^{-1} \, dx \, dy.
\]

For a proof, see for instance [M].

1. **COMMUTATORS**

**LEMMA 1.1.** Let \( B \in C^{\theta,\infty}(A), 0 < \theta < 1, \) be a bounded operator and \( \alpha_1, \alpha_2 \) positive numbers such that \( 0 < \alpha_1 + \alpha_2 < \theta \). Then

\[
  \| (A)^{\alpha_1} [B, (A-z)^{-1}] (A)^{\alpha_2} \| \leq C (|\text{Im} z|^{-\delta-1} + |\text{Im} z|^{-1} + \langle z \rangle |\text{Im} z|^{-2} + \langle z \rangle^2 |\text{Im} z|^{-3})
\]
whenever $|\text{Im}z| \neq 0$.

**Proof.** Consider $0 < \alpha < \theta$. We consider first the operator $(A)^\alpha [B, (A - z)^{-1}]$. Suppose $\text{Im}z > 0$; the case $\text{Im}z < 0$ is similar.

(i) We have (weakly)

$$[B, (A - z)^{-1}] = \int_{-\infty}^{0} e^{\mu t} e^{i\lambda t} [B, e^{itA}] \, dt,$$

where $z = \lambda + i\mu$. Using that $B \in \mathcal{C}^{\theta, \infty}(A)$, we get

$$||[B, (A - z)^{-1}]|| \leq \int_{-\infty}^{0} e^{\mu t} t^\theta \, dt \leq C\mu^{-\theta-1} \int_{-\infty}^{0} e^{t} |t|^\theta \, dt,$$

hence

$$||[B, (A - z)^{-1}]|| \leq C\mu^{-\theta-1}. \tag{1.2}$$

(ii) Denote $\nu(\lambda) = (\lambda)^\alpha$. Helffer-Sjöstrand formula gives (first as bounded operators between $A^\alpha$ and $A^{-\alpha}$)

$$[B, (A)^\alpha] = \frac{1}{\pi} \int_{C} \overline{\nu}(z) [B, (A - z)^{-1}] \, dz \, dy. \tag{1.3}$$

The norm of the integrand in (3) can be bounded by

$$||\overline{\nu}(z)[B, (A - z)^{-1}]|| \leq C\langle z \rangle^{\alpha-1-N} |\text{Im}z|^{N-\theta-1}.$$

If one takes $N = \theta + 1$ to avoid the singularities, we get

$$||\overline{\nu}(z)[B, (A - z)^{-1}]|| \leq C\langle z \rangle^{\alpha-2-\theta},$$

which is integrable if $\alpha < \theta$. Hence

$$[B, (A)^\alpha] \in \mathcal{B}(\mathcal{H}). \tag{1.4}$$

(iii) We can write then

$$[A]^\alpha (B, (A - z)^{-1}] = [B, (A)^\alpha (A - z)^{-1}] - [(A)^\alpha, B](A - z)^{-1}. \tag{1.5}$$

The norm of the second hand in the rhs of (4) is bounded by $C|\text{Im}z|^{-1}$.

(iv) We estimate now the first term in the rhs of (4). Let $g$ be a smooth function on $\mathbb{R}$, $g(t) = 1$ if $|t| \geq 1$ and $g(t) = 0$ if $|t| < 1/2$. Then

$$[B, (A)^\alpha (A - z)^{-1}] = [B, g(A)(A)^\alpha (A - z)^{-1}] + [B, (1 - g(A))(A)^\alpha (A - z)^{-1}].$$

The second term of the rhs of (1.5) equals

$$[B, (A)^\alpha][A - z]^{-1} + [B, (A - z)^{-1}](1 - g(A))(A)^\alpha.$$
and has the norm less then (using (2))

\[(1.6)\quad C(|\text{Im}z|^{-1} + |\text{Im}z|^{\theta+1}).\]

We denote \(g_z(\lambda) = g(\lambda)(\lambda)^{\alpha}(\lambda - z)^{-1}\). We shall use the following form of the Helffer–Sjöstrand form (see [BGS2], section 4):

\[(1.7)\quad [B, g_z(A)] = \frac{1}{\pi} \int_{\mathbb{R}} ((g_z(\lambda) - \lambda g_z'(\lambda))[B, \text{Im}R_{A}(\lambda + i\lambda)] - \partial_{\lambda}(\lambda g_z(\lambda))[B, \text{Im}R_{A}(\lambda + i\lambda)]) \, d\lambda
- \frac{1}{\pi} \int_{0}^{\lambda} g_z^{(2)}(\lambda))[B, \text{Im}R_{A}(\lambda + i\mu)]\mu \, d\mu \, d\lambda.

The norm of the integrand in the first term of (1.7) can be estimated by (using (2) and on \text{supp}g)

\[C \left( \frac{(\lambda)^{\alpha}}{|\lambda - z|} + \frac{(\lambda)^{\alpha+1}}{|\lambda - z|^{2}} \right) (\lambda)^{-\theta-1} \leq C(\langle \lambda \rangle^{-1} + \langle \text{Im}z \rangle |\text{Im}z|^{-2}).\]

Hence the first integral in (7) can be bounded as follows

\[(1.8)\quad \| \int_{\mathbb{R}} (g_z(\lambda) - \lambda g_z'(\lambda) + 2(i + 1)^{-1}\partial_{\lambda}(\lambda g_z(\lambda))[B, R_{A}(\lambda + i\lambda)] \, d\lambda \| \leq C(|\text{Im}z|^{-1} + \langle z \rangle |\text{Im}z|^{-2}).\]

To estimate the second integral we note first that

\[(1.9)\quad \| \int_{0}^{\lambda} g_z^{(2)}(\lambda))[B, R_{A}(\lambda + i\mu)]\mu \, d\mu \, d\lambda \| \leq C(\langle \lambda \rangle^{1-\theta})\]
on \text{supp}g. Then

\[\| g_z^{(2)}(\lambda))[B, R_{A}(\lambda + i\mu)]\mu \, d\mu \, d\lambda \| \leq C(\langle \lambda \rangle)^{1-\theta} \left( \frac{(\lambda)^{\alpha}}{|\lambda - z|^{3}} + \frac{(\lambda)^{\alpha-1}}{|\lambda - z|^{2}} + \frac{(\lambda)^{\alpha-2}}{|\lambda - z|} \right) \]

\[\leq C(\lambda)^{-1+\alpha-\theta}(|\text{Im}z|^{-1} + \langle z \rangle |\text{Im}z|^{-2} + \langle z \rangle^{3}|\text{Im}z|^{-3})\]

Summing up:

\[\|[B, g_z(A)]\| \leq C((\langle z \rangle^{2}|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda)^{-1+\alpha-\theta} |\text{Im}z|^{-1}).\]

Then one gets

\[(1.11)\quad \|\langle A \rangle^{\alpha}[B, (A - z)^{-1}]\| \leq C((\langle z \rangle^{2}|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda)^{-1+\alpha-\theta} |\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).\]

In the same way

\[(1.12)\quad \|[B, (A - z)^{-1}\langle A \rangle^{\alpha}]\| \leq C((\langle z \rangle^{2}|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda)^{-1+\alpha-\theta} |\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).\]

The general result follows by interpolation.
LEMMA 1.2. Let \( \{\chi_t\} \in S^\alpha, \alpha < 1 \) be a bounded family of symbols, and \( B \in C^{1+\theta,\infty}(A) \) a bounded operator. Then
\[
i[B,\chi_t(A)] = i[B,A]\chi_t'(A) + R_{1,t},
i[B,\chi_t(A)] = \chi_t'(A)i[B,A] + R_{2,t},
\]
where
\[
(A)^{\alpha_1}R_{1,t}(A)^{\alpha_2} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \|(A)^{\alpha_1}R_{1,t}(A)^{\alpha_2}\| \leq C
\]
\[
(A)^{\alpha_2}R_{2,t}(A)^{\alpha_1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \|(A)^{\alpha_2}R_{2,t}(A)^{\alpha_1}\| \leq C,
\]
whenever \( \alpha_1 + \alpha_2 + \alpha < 1 + \theta, \alpha_1 + \alpha_2 < 1 + \theta, \alpha_1 < \theta \). Here \( C \) stands for constants not depending on \( t \).

Proof. We have \( i[B,\chi_t(A)] = i[B,A]\chi_t'(A) + R_{1,t} \) where
\[
R_{1,t} = \frac{1}{\pi} \int_{C} \partial \chi_t[D,(A-z)^{-1}](A-z)^{-1} \, dz \, dy,
\]
with \( D = i[B,A] \in C^{\theta,\infty}(A) \), bounded.

We take \( \delta = \theta - \alpha_1 - \epsilon \) with \( \epsilon \) sufficiently small such that \( \alpha_2 - \delta < 1 \) and \( \alpha + \alpha_2 - \delta < 1 \). (This is possible by hypothesis.) Then, by Lemma 1,
\[
||\partial \chi_t(A)^{\alpha_1}[D,(A-z)^{1}](A)^{\delta}(A-z)^{-1}(A)^{\alpha_2-\delta}||
\leq C_N (z)^{a-1-N} |Imz|^N (|Imz|^{-1} + |Imz|^{-\theta-1}) \langle z \rangle^\alpha |Imz|^{-1}
\]
on \text{supp}\partial \chi_t. We take \( N = \theta + 2 \) and thus obtain that
\[
||\partial \chi_t(A)^{\alpha_1}[D,(A-z)^{1}](A-z)^{-1}(A)^{\alpha_2}|| \leq C(z)^{a-3+\alpha-\delta}
\]
which is integrable and \( C \) does not depend on \( t \). Hence \( (A)^{\alpha_1}R_{1,t}(A)^{\alpha_2} \) extends to a bounded operator and the estimate in the statement holds. One proceed similarly to get the second assertion.

LEMMA 1.3. Let \( B \) be a bounded operator of class \( C^{\theta,\infty}(A), 0 < \theta \leq 1 \) and \( \alpha_1, \alpha_2 \) positive numbers such that \( \alpha_1 + \alpha_2 < \theta \). Then \( (A)^{\alpha_1}B(A)^{\alpha_2} \) extends to a bounded operator on \( \mathcal{H} \).

Proof. Recall that in the proof of Lemma 1 we proved that \( [B,(A)^{\delta}] \in \mathcal{B}(\mathcal{H}) \) whenever \( \delta = \alpha_1 + \alpha_2 + \epsilon < \theta \). We denote \( \delta_i = \alpha_i/\delta, i = 1, 2 \) and set \( A_\delta = (A)^{\delta} \); this is a self-adjoint operator \( A_\delta \geq 1 \). We have
\[
A_\delta^{\epsilon_1}i[B,h(A_\delta)] = -\frac{1}{\pi} \int_{C} \partial h(z)A_\delta^{\epsilon_1}(A_\delta - z)^{-1}i[B,A_\delta](A_\delta - z)^{-1} \, dz \, dy.
\]
On the support of \( \partial h \) the norm of the integrand can be estimated as
\[
||\partial h(z)A_\delta^{\epsilon_1}(A_\delta - z)^{-1}i[B,A_\delta](A_\delta - z)^{-1}|| \leq C(z)^{\theta_1+\theta_2-1-2}.
\]
The rhs is an integrable function, since \( \theta_1 + \theta_2 < 1 \). Therefore \( A_\delta^{\epsilon_1}i[B,h(A_\delta)] \) extends to a bounded operator on \( \mathcal{H} \).
LEMMA 1.4. Let $B$ be a bounded operator of class $C^{0,\infty}(A)$, $0 < \theta \leq 1$ and $\alpha_1$, $\alpha_2$ positive numbers such that $\alpha_1 + \alpha_2 < \theta$, and $\{g_t\} \subset S^\alpha$, $a \leq 0$, a bounded family of symbols. Then:

$$||\langle A \rangle^{\alpha_1}i[B, g_t(A)]\langle A \rangle^{\alpha_2}|| \leq C,$$

where $C$ does not depend on $t$.

Proof. (i) Consider first the case where $a < 0$. Then

$$\langle A \rangle^{\alpha_1}i[B, g_t(A)]\langle A \rangle^{\alpha_2} = \frac{1}{\pi} \int_\mathbb{C} \overline{\partial} \tilde{g}_t(\xi)\langle A \rangle^{\alpha_1}i[B, (A - \xi)^{-1}]\langle A \rangle^{\alpha_2} d\xi.$$

Using Lemma 1 the norm of the integrand can be majorized by $C|\xi|^{\alpha_2 - 2}$.

(ii) If $a = 0$, let $\epsilon > 0$ be such that $\alpha_1 + \alpha_2 + \epsilon < \theta$ and write

$$\langle A \rangle^{\alpha_1 + \epsilon}i[B, g_t(A)]\langle A \rangle^{\alpha_2} = \langle A \rangle^{\alpha_1 + \epsilon}i[B, g_t(A)]\langle A \rangle^{\alpha_2}.$$

We use the proof of the previous lemma to show that the first term is a bounded operator and its norm can be bounded by a constant not depending on $t$. For the second term we use (i).

2. THE PROOF OF THEOREM 2

We can suppose, without restricting the generality, that in Theorem 2 we have $M = 1$ and $\lambda = 0$.

LEMMA 2.1. If $\Phi \in C_0^\infty(\mathbb{R})$ is a real function, $\Phi = 1$ on a neighborhood of 0, then

$$(u, \Phi(L)\varphi)_{-s,s} = (u, \varphi)_{-s,s}, \quad \text{for all } \varphi \in A^s.$$  

Proof. We have, for $\varphi \in A^1$,

$$(u, (1 - \Phi(L))\varphi)_{-s,s} = (u, L\Phi(L)\varphi)_{-s,s},$$

where $\Phi(t) = (1 - \Phi(t))t^{-1}$. Therefore, to have (2) for $\varphi \in A^1$ it suffices to prove that $\Phi(L) = (L+1)^{-1}\varphi_1$ with $\varphi_1 \in A^1$. We can write $(L + 1)\Phi(L) = (1 - \Phi(L)) + \Phi(L)$. Thus, since $(1\Phi(L))\varphi \in A^1$, it remains to show that $\Phi(L)\varphi \in A^1$. We have

$$i[\Phi(L), A] = \frac{1}{\pi} \int_\mathbb{C} \overline{\partial} \tilde{\Phi}i[(L - z)^{-1}, A] d\xi$$

$$= -\frac{1}{\pi} \int_\mathbb{C} \overline{\partial} \tilde{\Phi}(L - z)^{-1}(L + 1)^{1/2}(L + 1)^{-1/2}[L, A](L + 1)^{-1/2}(L + 1)^{1/2}(L - z)^{-1} d\xi.$$

The norm of the integrand can be bounded by $C(z)^{-2 - 2} |\text{Im}z|^2 |\text{Im}z|^{-2} = C(z)^{-3}$. We get that $i[\Phi(L), A]$ is a bounded operator and we obtain easily that $\Phi(L)\varphi \in A^1$ if $\varphi \in A^1$. Thus equation (1) holds for $\varphi \in A^1$; the general result follows by density using the fact that $\Phi(L) \in \mathcal{B}(A^s)$. 


Remark. In fact the previous Lemma says that $\Phi(L)u = u$ for all $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi = 1$ on a neighborhood of 0; this fact can be easily seen using that $\Phi(L) \in \mathcal{B}(A^s) \cap \mathcal{B}(A^{-t})$ and it is symmetric with respect to the duality $(.,.)_{s,-s}$.

**LEMMA 2.2.** Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$. We consider the $C_0^\infty(\mathbb{R})$ function

$$\chi_t(y) = \int_{(y)}^\infty s^{-2\beta} \chi^2(s/t) ds,$$

where $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Then

$$(L\phi^2(L)\chi_t(A)u, u)_{s,-s} = 0.$$  

**Proof.** The Lemma follows by hypothesis as $\Phi^2(L)\chi_t(A)u \in (L+1)^{-1}A^s$. □

We shall set $T$ for different bounded operators with norm independent on $t$.

**Remark.** We have

$$2Re i((L\phi^2(L)\chi_t(A)u, u)_{s,-s} = 0.$$  

We shall give to this relation the form and the meaning

$$i([L\Phi^2(L), \chi_t(A)]u, u)_{s,-s} = 0.$$  

Set $L_1 = \Phi^2(L)L$. Then $L_1$ is a bounded operator of class $C^{\theta+1,\infty}(A)$ (Thm. 6.2.5 [ABG]).

**LEMMA 2.3.** We have

$$i[L_1, \chi_t(A)] = i[L_1, A]A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t) + \langle A \rangle^{-T}\langle A \rangle^{-s}.$$  

**Proof.** One applies Lemma 1.2 for $r_1, \alpha_1 = \alpha_2 = s$, $a = 1-2\beta$. (Here $\alpha_1 + \alpha_2 + a = 2s+1-2\beta < \theta+1$ since $\beta > \theta/2$, and $s < \theta$.) □

As a direct consequence we get the next Lemma.

**LEMMA 2.3’.** $\sup_{t \geq 1} |(i[L_1, A]A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{s,-s}| < \infty$.

**LEMMA 2.4.** If $\tilde{\Phi} \in C_0^\infty(\mathbb{R})$, $\tilde{\Phi} = 1$ on a small enough neighborhood of 0, then

$$\sup_{t \geq 1} |(\tilde{\Phi}(L)i[L, A]\Phi(L)A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{s,-s}| < \infty.$$  

**Proof.** We know that $\tilde{\Phi}(L)u = u$. We have

$$A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L) = \tilde{\Phi}(L)A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t) + [\tilde{\Phi}(L), A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)].$$  

Set $g_t(A) = A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)$. Here $\{g_t\} \in S_{-2\beta}$ is a bounded family of symbols. One applies Lemma 1.2 to get

$$[\tilde{\Phi}(L), g_t(A)] = [\tilde{\Phi}(L), A]g_t(A) + R_{1,t}.$$
(In this case $\alpha_1 = \alpha_2 = s$, $a = -\beta$.) Thus, since $2\beta + 1 > s + 1 > 2s$,

$$A(A)^{-2\beta - 1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L) = \tilde{\Phi}(L)A(A)^{-2\beta - 1}\chi^2(\langle A \rangle/t) + \langle A \rangle^{-s}T(A)^{-s}.$$  

Hence

$$\langle i[L_1, A]A(A)^{-2\beta - 1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L)u, \tilde{\Phi}(L)u \rangle_{s,-s}$$
$$= \langle i[L_1, A]\tilde{\Phi}(L)A(A)^{-2\beta - 1}\chi^2(\langle A \rangle/t)u, \tilde{\Phi}(L)u \rangle_{s,-s} + \langle i[L_1, A]\langle A \rangle^{-s}T(A)^{-s}u, u \rangle_{s,-s}. $$

Since $i[L_1, A] \in \mathcal{B}(\mathcal{H}) \cap C^\theta,\infty(A)$ (Prop. 5.2.2 [ABG]), $i[L_1, A]$ is a bounded operator on $A^*$, $s < \theta$ (Thm 5.3.3, Lemma 5.3.2 [ABG]). Therefore the second term in (2.6) is bounded by a constant independent on $t$. But, in form sense,

$$\tilde{\Phi}(L)[L_1, A] \tilde{\Phi}(L) = \langle \tilde{\Phi}(L)\Phi(L)[L, A]\Phi(L) + L\tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) + \tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L). $$

We take $\text{supp} \tilde{\Phi}$ to be in the set where $\Phi = 1$; then

$$\tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) = 0 \quad \text{on} \quad A^1 \times A^1.$$  

Therefore the bounded operator given by this form on $\mathcal{H}$ ($L \in C^1(A)$) is zero. Similarly we get that $\tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L) = 0$. Summing up

$$\tilde{\Phi}(L)i[L_1, A] \tilde{\Phi}(L) = \tilde{\Phi}(L)i[L, A] \tilde{\Phi}(L). $$

The lemma follows by (2.6), (2.5) and the previous relation.  

We can denote $\tilde{\Phi}$ also by $\Phi$.

**Lemma 2.5.** $\sup_{t \geq 1} |(\Phi(L)i[L, A] \Phi(L)A(A)^{-\beta}\chi(\langle A \rangle/t)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)u)| \leq \infty$.

**Proof.** Set $B = \Phi(L)i[L, A] \Phi(L)$. Then $B$ is a bounded operator of class $C^\infty(A)$. Denote $f_t(x) = \langle x \rangle^{-\beta}(\langle x \rangle/t)$, $x \in \mathbb{R}$. We take $\beta_0 < \beta$, but still $\beta_0 > s/2$, $\beta_0 < \theta/2$. We write

$$\langle A \rangle^{s-2\beta}[B, f_t(A)] (A)^{-\beta} = \langle A \rangle^{s-2\beta}[B, f_t(A)] (A)^{-\beta} $$
$$= \langle A \rangle^{s-2\beta}[B, f_t(A)(A)^{\beta_0}] (A)^{-\beta} - \langle A \rangle^{s-2\beta}[B, \langle A \rangle^{\beta_0}] (A)^{-2\beta}\chi(\langle A \rangle/t). $$

But $2s - 2\beta_0 < 2s - \theta < \theta$, so the first term is a bounded operator and its norm does not depend on $t$ (Lemma 1.4). The second term is bounded since $s - 2\beta < 0$ and $\langle A \rangle^{s-2\beta}[B, \langle A \rangle^{\beta_0}]$ is bounded by Lemma 1.3. Now the lemma follows easily.

**Lemma 2.6.** For all $\beta > \alpha$ we have $\langle A \rangle^{-\beta}u \in \mathcal{H}$.

**Proof.** (i) Consider first $\beta > \max(\alpha, s/2), \beta < \theta/2$. Let $F_+$ be a smooth bounded real function, $F_+ = 1$ on $[1, \infty)$, $F_+ = 0$ on $(-1, 1/2]$. We shall show first that

$$\sup_{t \geq 1} |(\Phi(L)i[L, A] \Phi(L)A(A)^{-\beta-1}F_+(A)\chi(\langle A \rangle/t)u, \langle A \rangle^{-\beta}F_+(A)\chi(\langle A \rangle/t)u)| < \infty. $$

We use again the notation $B = \Phi(L)i[L, A] \Phi(L)$. If $F_- = 1 - F_+$ then

$$(BA(A)^{-\beta-1}\chi(\langle A \rangle/t)(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)u)$$
$$= (BA(A)^{-\beta-1}\chi(\langle A \rangle/t)(F_+ + F_-)(A)u, \langle A \rangle^{-\beta}(F_+ + F_-)(A)\chi(\langle A \rangle/t)u).$$
Here \((A)^{-\beta} u \in \mathcal{H}\). Moreover, by Thm 3.10 [BGS2], the fact that \(B\) is of class \(C^{s,2}\) for all \(s < \theta\) ensure that \(F_+(A)BF_-(A) \in \mathcal{B}(A^{2-s}, A^{2-s})\). Hence \((A)^{-\beta} F_+(A)BF_-(A)(A)^{\beta} = T \in \mathcal{B}(\mathcal{H})\), and this gives

\[(BA(A)^{-\beta} \chi((A)/t)F_+(A)u, (A)^{-\beta} \chi((A)/t)F_-(A))\]
\[= (T(A)^{-\beta} F_+(A)(A)^{-\beta} \chi((A)/t)F_-(A)u)\]

Therefore

\[\sup_{t \geq 1} |(BA(A)^{-\beta} \chi((A)/t)F_+(A)u, (A)^{-\beta} \chi((A)/t)F_-(A))| < \infty. \tag{2.8} \]

Similarly one gets

\[\sup_{t \geq 1} |(BA(A)^{-\beta} \chi((A)/t)F_-(A)u, (A)^{-\beta} \chi((A)/t)F_+(A))| < \infty. \tag{2.9} \]

Now (2.7) follows by (2.8), (2.9) and the previous lemma.

We can write \(A(A)^{-1}F_+(A) = g^2(A)F_+(A)\) with \(g \in S^0\). But \(A^{2-s} [B, g(A)](A)^{-\beta}\) is bounded by Lemma 1.4 \((2s - 2\beta < 2s - \theta < 2\beta - \theta = \theta)\). Hence

\[\sup_{t \geq 1} ||(BA(A)^{-\beta} \chi((A)/t)g(A)F_+(A)u, (A)^{-\beta} \chi((A)/t)g(A)F_+(A))|| < \infty. \]

Using now the Mourre estimate we get

\[\sup_{t \geq 1} ||\Phi(L)(A)^{-\beta} g(A)F_+(A)u|| \leq \infty. \tag{2.10} \]

As \([\Phi(L)(A)^{-\beta} g(A)F_+(A)](A)^{s}\) is a bounded operator with norm independent on \(t\) (by Lemma 1.4) it follows

\[\sup_{t \geq 1} ||(A)^{-\beta} \chi((A)/t)g(A)F_+(A)u|| \leq \infty. \]

This provide, using Beppo-Levi Theorem,

If we take \(\tilde{F}_+\) to be a smooth bounded real function on \(\mathbb{R}\), \(\tilde{F}_+ = 1\) on \([2, \infty)\) and \(\text{supp}\tilde{F}_+ \subset [1, \infty)\), we can write

\[\tilde{F}_+(A)(A)^{-\beta} = (\tilde{F}_+/gF_+(A))(gF_+(A))(A)^{-\beta},\]

and \((\tilde{F}_+/gF_+(A))(gF_+(A))\) is a bounded operator. Then (2.10) gives that \(\tilde{F}_+(A)(A)^{-\beta} u \in \mathcal{H}\). Thus the lemma follows in this case since \((1 - \tilde{F}_+(A)(A)^{-\beta} u \in \mathcal{H}\) by hypothesis (b) of Thm 2.

(ii) Now we can repeat the argument with \(s\) replaced by \(2\alpha\) and see that \((A)^{-\beta} u \in \mathcal{H}\) for all \(\beta < \theta/2, \beta > \alpha\). □

**Lemma 2.7.** In the conditions of Thm. 2, \(u \equiv 0\).

**Proof.** Denote \(u_\epsilon = (\epsilon A)^{-\beta} u\). We shall show that \(||u_\epsilon|| \leq C\), where \(C\) does not depend on \(\epsilon\). This implies that \(u \in \mathcal{H}\). Since \(u = \Phi(L)u, u\) is in the domain of \(L\); and, as \(Lu = 0\), it follows that either 0 is a eigenvalue of \(L\), or \(u \equiv 0\). The first case is impossible due to the Mourre estimate.
Recall that $L_1 = L\Phi^2(L)$. We shall denote by $T$ different bounded operators with norm independent on $t$ and $\epsilon$. We begin by computing

\[
(i[L_1, A]u_\epsilon, u_\epsilon) = \lim_{t \to \infty} (i[L_1, A(A + itA)^{-1}it]u_\epsilon, u_\epsilon)
\]

\[
= \lim_{t \to \infty} i[(L_1A + itA)^{-1}(\epsilon A)^{-\beta}u, (\epsilon A)^{-\beta}u)
\]

\[
- \lim_{t \to \infty} i((\epsilon A)^{-\beta}u, L_1(A - itA)^{-1}it(\epsilon A)^{-\beta}u)
\]

\[
- \lim_{t \to \infty} i(\langle\epsilon A\rangle^{-\beta}, L_1)A(A + itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u, u_\beta, -\beta
\]

\[
- \lim_{t \to \infty} i(u, L_1A(A + itA)^{-1}it(\epsilon A)^{-2\beta}u)_{-\beta, +\beta}
\]

\[
+ \lim_{t \to \infty} i((L_1A + itA)^{-1}it(\epsilon A)^{-2\beta}u, u)_{\beta, -\beta}
\]

\[
+ \lim_{t \to \infty} i(u, (\epsilon A)^{-\beta}, L_1)A(A - itA)^{-1}it(\epsilon A)u)_{-\beta, +\beta}
\]

We have

\[
i[L_1, (\epsilon A)^{-\beta}] = -\beta i[L_1, A]\epsilon^2\langle\epsilon A\rangle^{-\beta - 2} + \langle A\rangle^{-\beta}T\langle A\rangle^{-\beta - 1}
\]

(by Lemma 2.1, with $1 + 2\beta < 1 + \theta$, $\beta < \theta$, $a = 1$) and also:

\[
i[L_1, (\epsilon A)^{-\beta}] = -\beta \epsilon^2 A\epsilon\langle\epsilon A\rangle^{-\beta - 1}i[L_1, A] + \langle A\rangle^{-\beta - 1}T\langle A\rangle^{-\beta}.
\]

It follows then

\[
- i((\epsilon A)^{-\beta}, L_1)itA(A + itA)^{-1}(\epsilon A)^{-\beta}u, u)_{\beta, -\beta}
\]

\[
= -\beta i[L_1, A]\epsilon^2 A^2(\epsilon A)^{-2\beta - 2}u, u)_{\beta, -\beta} + (T\langle A\rangle^{-\beta - 1}itA(A + itA)^{-1}u_e, \langle A\rangle^{-\beta}u_e \to
\]

\[
\to (-\beta i[L_1, A]\epsilon^2 A^2(\epsilon A)^{-2\beta - 2}u, u)_{\beta, -\beta} + (T\langle A\rangle^{-\beta - 1}u_e, \langle A\rangle^{-\beta}u_e)
\]

Similarly for the second commutator. We get thus

\[
(i[L_1, A]u_e, u_e) = 2\beta \epsilon^2 A^2(\epsilon A)^{-2\beta}B_1u, u_e) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u),
\]

where $B_1 = i[L_1, A]$. We write $\epsilon^2 A^2(\epsilon A)^{-2\beta} = (\epsilon A)^{\beta} - (\epsilon A)^{-1\beta}$. Lemma 1.4 gives

\[
((\epsilon A)^{\beta}B_1u, u_e) = (B_1u, u_e) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u)
\]

and

\[
((\epsilon A)^{-2}B_1u, u_e) = (B_1(\epsilon A)^{-1}u_e, (\epsilon A)^{-1}u_e) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).
\]

Hence

\[
(1 - 2\beta)(B_1u, u_e) = -2\beta(B_1(\epsilon A)^{-1}u_e, (\epsilon A)^{-1}u_e) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).
\]

We use that $u = \Phi(L)u$ as in Lemma 2.4 to get

\[
(B_1(\epsilon A)^{-1}u_e, (\epsilon A)^{-1}u_e) = (\Phi(L)i[L, A]\Phi(L)(\epsilon A)^{-1}u_e, (\epsilon A)^{-1}u_e) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).
\]
The Mourre inequality (suppose $\text{supp}\Phi$ small enough) provide
\[(1 - 2\beta)(B_1 u_\varepsilon, u_\varepsilon) \leq (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).
\]
Again:
\[(B_1 u_\varepsilon, u_\varepsilon) = (\Phi(L)i[L, A]\Phi(L)u_\varepsilon, u_\varepsilon) + (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).
\]
Then the Mourre inequality gives
\[||\Phi(L)u_\varepsilon|| \leq C.
\]
Commuting $\Phi(L)$ and $(\epsilon A)^{-\beta}$ (by Lemma 1.4), we get
\[||(<\epsilon A)^{-\beta} u|| \leq C,
\]
which gives $u \in \mathcal{H}$ and thus finishes the proof. □

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