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ON AN ABSTRACT RADIATION CONDITION

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INTRODUCTION

We shall present an abstract radiation condition in terms of the Mourre theory of conjugate operator method.

Let $\mathcal{H}$ be a Hilbert space and $A$ be a self-adjoint operator in $\mathcal{H}$. For $s \geq 0$ consider the Hilbert space $A^s = D((A^*)^s)$ with the graph norm, and if $s < 0$, $A^s = (A^{-s})^*$. Then, if $s \geq 0$, $A^s \subseteq \mathcal{H} \subseteq A^{-s}$ continuously and densely, and the scalar product of $\mathcal{H}$ extends to a natural duality $(\cdot, \cdot)_{s, -s} : A^s \times A^{-s} \to \mathbb{C}$ for all $s \in \mathbb{R}$. We denote by $P_\pm$ the spectral projectors of $A$ associated to the half-lines $[0, +\infty)$ and $(-\infty, 0]$, respectively.

We recall now some (Besov) spaces of operators (see [ABG]). Let $S$ be a bounded operator on $\mathcal{H}$. We say that $S \in C^K(A)$, $k$ positive integer, if the application $\mathbb{R} \ni \tau \mapsto W(\tau)[S] = e^{i\tau A}S e^{-i\tau A} \in \mathcal{B}(\mathcal{H})$ is strongly $C^K$; in this case $ad^*_S S$ can be extended as a bounded operator on $\mathcal{H}$. Consider $\theta \in (0, 1], p \in [1, \infty]$; we say that $S \in C^{\theta, p}(A)$ if $(\tau \to (W(\tau) - I)^m[S])/|\tau|^p \in L^p([0, \infty))$, where $m = 1$ if $\theta < 1$, and $m = 2$ if $\theta = 1$. (If $p = \infty$, this condition should be read as $\sup_{\tau > 0} ||(W(\tau) - I)^m[S]||/|\tau|^\theta < \infty$.) For general $\theta > 0$, we say that $S \in C^{\theta, p}(A)$ if $S \in C^l(A)$ and $ad_S S \in C^{\theta, p}(A)$, where $l$ is the largest integer $l < \theta$.

Let $L$ be a self-adjoint operator in $\mathcal{H}$. Then $L \in C^{\theta, p}(A)$ (or $C^K(A)$) if $(L - z)^{-1} \in C^{\theta, p}(A)$ (or $C^K(A)$) for some (and hence all) $z \in \mathbb{C} \setminus \sigma(L)$.

If $L$ is a self-adjoint operator of class $C^1(A)$, then the commutator $i[L, A]$ is defined as a continuous form on the domain of $L$. Then one can define the strict Mourre set $\mu^L(L)$ of $L$ with respect to $A$ as the set of $\lambda \in \mathbb{R}$ with the property that there exists $J = (\lambda - \delta, \lambda + \delta) \neq \emptyset$ and $d > 0$ such that

$$E_L(J) \langle [L, A] E_L(J) \geq d \cdot E_L(J).$$

We recall that if $L$ has a spectral gap and $L \in C^{1, 1}(A)$, then there exist $R_L(\lambda \pm i0) = \lim_{\epsilon \to 0} (L - \lambda \mp i\epsilon)^{-1}$ uniformly in $\mathcal{B}(A^s, A^{-s})$, whenever $s > 1/2$.

The following theorem was given in [BGS1] (for the proof see [BGS2]; see also [J] for some earlier results).

THEOREM 1. Let $s > 1/2$ be a real number and $L$ be a self-adjoint operator with a spectral gap and of class $C^{s+1/2, 1}(A)$. Then we have $P_\mp R_L(\lambda \pm i0)A^s \subseteq A^{s-1}$ for each $\lambda \in \mu^L(L)$.

It turns out that in some stronger hypotheses this condition characterizes $R_L(\lambda \pm i0)$. Namely, we prove the following theorem, extending some results of [B2], [M].

THEOREM 2. Let $1 \geq \theta > 1/2$ be a real number, $L \geq -M$ be a bounded from below self-adjoint operator of class $C^{1+\theta, \infty}(A)$ such that $i[L, A] \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, where $\mathcal{G}$ is the form domain of $L$, and $\lambda \in \mu^A(L)$.

Suppose $u \in A^{-s}$, $s \in (1/2, \theta]$ satisfies:

a) $(u, (L - \lambda)\varphi)_{-s, s} = 0$ for all $\varphi \in (L + M)^{-1}A^s$,

b) there exists $\alpha < \theta/2$ such that $(A)^{-\alpha}P_-(A)u \in \mathcal{H}$ (or $(A)^{-\alpha}P_+(A)u \in \mathcal{H}$).

Then $u = 0$. 

The proof follows Isozaki's proof of some type of radiation conditions which are strongly related to those presented here. (See [I1], [I2], [I3].) We only remark here that Theorem 2 provides some useful results in the study of the layered media.

One of the tools needed here is the functional calculus using almost analytic extensions of symbols. Let \( m \in \mathbb{R} \). We denote by \( S^m \) the set of symbols \( f \in C^\infty(\mathbb{R}) \) that satisfy

\[
p_k(f) = \sup_{x \in \mathbb{R}} (x)^{m-k}|f^{(k)}(x)| < \infty.
\]

Then \( S^m \), endowed with the seminorms \( p_k \) is a Fréchet space. The following result can be found in [B2], [M] (see also [DG] for the main idea).

**PROPOSITION 3.** Consider a bounded family of symbols \( \{f_\epsilon\} \subset S^m \). Then there exists a family of functions (the almost analytic extensions) \( \{\tilde{f}_\epsilon\} \subset C^\infty(\mathbb{C}) \) such that:

i. \( |Imz| \leq (Rez) \) on \( \text{supp} \tilde{f}_\epsilon \),

ii. \( |\overline{\partial}\tilde{f}_\epsilon(z)| \leq C_N(z)^{m-N-1}|Imz|^{N-1} \) for all \( N \geq 0 \) and all \( z \in \mathbb{C} \), where the constants \( C_N \) do not depend on \( z \) and \( \epsilon \).

This construction provides an useful representation for the functional calculus of a self-adjoint operator, due to Helffer–Sjöstrand ([HS]): Let \( A \) be a self-adjoint operator on \( \mathcal{H} \) and \( f \in S^{-\delta}, \delta > 0 \). Then

\[
f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A-z)^{-1} dx dy,
\]

where \( z = x + iy \) and \( \tilde{f} \) is an almost analytic extension of \( f \). If \( B \) is a bounded operator with \( ad_A n \) is a bounded form on the domain of \( A \), and \( ad_A k f^{(k)}(A) \) (respectively \( f^{(k)}(A)ad_A k \)) \( k = 1, \ldots, n-1, \) are bounded operators, then

\[
[B, f(A)] = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} ad_A^k(B)f^{(k)}(A) + R_n(A,B)
\]

where

\[
R_n(A,B) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A-z)^{-1}ad_A^n(B)(A-z)^{-n} dx dy,
\]

\[
R_n'(A,B) = \frac{(-1)^n}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A-z)^{-n}ad_A^n(B)(A-z)^{-1} dx dy.
\]

For a proof, see for instance [M].

1. COMMUTATORS

**LEMMA 1.1.** Let \( B \in C^{0,\infty}(A) \), \( 0 < \theta < 1 \), be a bounded operator and \( \alpha_1, \alpha_2 \) positive numbers such that \( 0 < \alpha_1 + \alpha_2 < \theta \). Then

\[
\|(A)^{\alpha_1}[B,(A-z)^{-1}](A)^{\alpha_2}\| \leq C(|Imz|^{-\theta-1} + |Imz|^{-1} + \langle z \rangle |Imz|^{-2} + \langle z \rangle^2 |Imz|^{-3})
\]

(1.1)
whenever $|\text{Im}z| \neq 0$.

Proof. Consider $0 < \alpha < \theta$. We consider first the operator $(A)^{\alpha}[B, (A-z)^{-1}]$. Suppose $\text{Im}z > 0$; the case $\text{Im}z < 0$ is similar.

(i) We have (weakly)

$$[B, (A-z)^{-1}] = \int_{-\infty}^{0} e^{it\lambda t}[B, e^{itA}] dt,$$

where $z = \lambda + i\mu$. Using that $B \in C^{\theta,\infty}(A)$, we get

$$\|[B, (A-z)^{-1}]\| \leq \int_{-\infty}^{0} e^{\mu t} |t|^\theta dt \leq C \mu^{-\theta-1} \int_{-\infty}^{0} e^{t} |t|^\theta dt,$$

hence

(1.2) $$\|[B, (A-z)^{-1}]\| \leq C \mu^{-\theta-1}.$$

(ii) Denote $\nu(\lambda) = \langle \lambda \rangle^\alpha$. Helffer-Sjöstrand formula gives (first as bounded operators between $A^\alpha$ and $A^{-\alpha}$)

(1.3) $$[B, (A)^{\alpha}] = \frac{1}{\pi} \int_{C} \overline{\partial} \tilde{\nu}(z)[B, (A-z)^{-1}] dxdy.$$

The norm of the integrand in (3) can be bounded by

$$\|\overline{\partial} \tilde{\nu}(z)[B, (A-z)^{-1}]\| \leq C(z)^{\alpha-1-N} |\text{Im}z|^N \theta^{-1}.$$

If one takes $N = \theta + 1$ to avoid the singularities, we get

$$\|\overline{\partial} \tilde{\nu}(z)[B, (A-z)^{-1}]\| \leq C(z)^\alpha 2^{-\theta},$$

which is integrable if $\alpha < \theta$. Hence

$$[B, (A)^{\alpha}] \in \mathcal{B}(\mathcal{H}).$$

(iii) We can write then

(1.4) $$(A)^{\alpha}[B, (A-z)^{-1}] = [B, (A)^{\alpha}(A-z)^{-1}] - [(A)^{\alpha}, B](A-z)^{-1}.$$

The norm of the second hand in the rhs of (4) is bounded by $C|\text{Im}z|^{-1}$.

(iv) We estimate now the first term in the rhs of (4). Let $g$ be a smooth function on $\mathbb{R}$, $g(t) = 1$ if $|t| \geq 1$ and $g(t) = 0$ if $|t| < 1/2$. Then

(1.5) $$[B, (A)^{\alpha}(A-z)^{-1}] = [B, g(A)(A)^{\alpha}(A-z)^{-1} + [B, (1-g(A))(A)^{\alpha}(A-z)^{-1}].$$

The second term of the rhs of (1.5) equals

$$[B, (A)^{\alpha}](A-z)^{-1} + [B, (A-z)^{-1}](1-g(A))(A)^{\alpha},$$
and has the norm less then (using (2))

\begin{equation}
C(|\text{Im}z|^{-1} + |\text{Im}z|^\theta+1).
\end{equation}

We denote \(g_z(\lambda) = g(\lambda)(\lambda)^{\alpha}(\lambda - z)^{-1}\). We shall use the following form of the Helffer–Sjöstrand form (see [BGS2], section 4):

\begin{equation}
[B, g_z(A)] = \frac{1}{\pi} \int \left( (g_z(\lambda) - \lambda g_z'(\lambda))[B, \text{Im}R_A(\lambda + i\lambda)] - \partial_\lambda (\lambda g_z(\lambda))[B, \text{Im}iR_A(\lambda + i\lambda)] \right) d\lambda
- \frac{1}{\pi} \int \left( \int_0^\lambda g_z^{(2)}(\lambda))][B, \text{Im}R_A(\lambda + i\mu)]d\mu d\lambda.
\end{equation}

The norm of the integrand in the first term of (1.7) can be estimated by (using (2) and on \text{supp}g)

\begin{equation}
C\left( \left( \frac{\langle \lambda \rangle^\alpha}{|\lambda - z|} + \frac{\langle \lambda \rangle^{\alpha+1}}{|\lambda - z|^2} \right) \langle \lambda \rangle^{-\theta-1} \leq C\langle \lambda \rangle^{-\theta-1}(|\text{Im}z|^{-1} + \langle z \rangle|\text{Im}z|^{-2})
\end{equation}

Hence the first integral in (7) can be bounded as follows

\begin{equation}
\| \int_{\mathbb{R}}(g_z(\lambda) - \lambda g_z'(\lambda) + 2(i + 1)^{-1}\partial_\lambda (\lambda g_z(\lambda))[B, R_A(\lambda + i\lambda)]d\lambda\| \leq C(|\text{Im}z|^{-1} + \langle z \rangle|\text{Im}z|^{-2}).
\end{equation}

To estimate the second integral we note first that

\begin{equation}
\| \int_0^\lambda g_z^{(2)}(\lambda))][B, R_A(\lambda + i\mu)]d\mu d\lambda\| \leq C\langle \lambda \rangle^{1-\theta}
\end{equation}
on \text{supp}g. Then

\begin{equation}
\| \int_0^\lambda g_z^{(2)}(\lambda))][B, R_A(\lambda + i\mu)]d\mu d\lambda\| \leq C\langle \lambda \rangle^{1-\theta} \left( \frac{\langle \lambda \rangle^\alpha}{|\lambda - z|^{3}} + \frac{\langle \lambda \rangle^{\alpha-1}}{|\lambda - z|^2} + \frac{\langle \lambda \rangle^{\alpha-2}}{|\lambda - z|} \right)
\leq C\langle \lambda \rangle^{-1+\alpha-\theta}(|\text{Im}z|^{-1} + \langle z \rangle|\text{Im}z|^{-2} + \langle z \rangle^2|\text{Im}z|^{-3})
\end{equation}

Summing up:

\begin{equation}
\| [B, g_z(A)] \| \leq C(|\text{Im}z|^3 + \langle z \rangle|\text{Im}z|^{-2} + \lambda^{-1+\alpha-\theta}|\text{Im}z|^{-1}).
\end{equation}

Then one gets

\begin{equation}
\| \langle A \rangle^\alpha[B, (A - z)^{-1}] \| \leq C(|\text{Im}z|^3 + \langle z \rangle|\text{Im}z|^{-2} + \lambda^{-1+\alpha-\theta}|\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).
\end{equation}

In the same way

\begin{equation}
\| [B, (A - z)^{-1}\langle A \rangle^\alpha] \| \leq C(|\text{Im}z|^3 + \langle z \rangle|\text{Im}z|^{-2} + \lambda^{-1+\alpha-\theta}|\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).
\end{equation}

The general result follows by interpolation.
LEMMA 1.2. Let \( \{\chi_t\} \in S^a \), \( a < 1 \) be a bounded family of symbols, and \( B \in \mathcal{C}^{1+\theta,\infty}(A) \) a bounded operator. Then
\[
\begin{align*}
i[B, \chi_t(A)] &= i[B, A]\chi'_t(A) + R_{1,t}, \\
i[B, \chi_t(A)] &= \chi'_t(A)i[B, A] + R_{2,t},
\end{align*}
\]
where
\[
\begin{align*}
\langle A \rangle^{\alpha_1}R_{1,t}(A)^{\alpha_2} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad ||\langle A \rangle^{\alpha_1}R_{1,t}(A)^{\alpha_2}|| \leq C \\
\langle A \rangle^{\alpha_2}R_{2,t}(A)^{\alpha_1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad ||\langle A \rangle^{\alpha_2}R_{2,t}(A)^{\alpha_1}|| \leq C,
\end{align*}
\]
whenever \( \alpha_1 + \alpha_2 + a < 1 + \theta \), \( \alpha_1 + \alpha_2 < 1 + \theta \), \( \alpha_1 < \theta \). Here \( C \) stands for constants not depending on \( t \).

Proof. We have \( i[B, \chi_t(A)] = i[B, A]\chi'_t(A) + R_{1,t} \) where
\[
R_{1,t} = \frac{1}{\pi} \int_{\mathcal{C}} \overline{\delta x_t i[D, (A - z)^{-1}](A - z)^{-1}} \, dx \, dy,
\]
with \( D = i[B, A] \in C^{\theta,\infty}(A) \), bounded.

We take \( \delta = \theta - \alpha_1 - \epsilon \) with \( \epsilon \) sufficiently small such that \( \alpha_2 - \delta < 1 \) and \( a + \alpha_2 - \delta < 1 \). (This is possible by hypothesis.) Then, by Lemma 1,
\[
\begin{align*}
||\overline{\delta x_t(A)^{\alpha_1} i[D, (A - z)^{-1}](A)^{\delta}(A - z)^{-1}(A)^{\alpha_2 - \delta}}|| &
\leq C_N \langle z \rangle^{\alpha_1 - \alpha_2 - \delta + 1 - \theta} \langle \delta x_t \rangle^{\alpha_1 - \alpha_2 - \delta + 1 - \theta} \langle \delta x_t \rangle^{\alpha_1 - \alpha_2 - \delta + 1 - \theta} ||
\end{align*}
\]
on \text{supp} \delta x_t. We take \( N = \theta + 2 \) and thus obtain that
\[
||\overline{\delta x_t(A)^{\alpha_1} i[D, (A - z)^{-1}](A - z)^{-1}(A)^{\alpha_2}}|| \leq C \langle z \rangle^{\alpha_1 - \alpha_2 - \delta}
\]
which is integrable and \( C \) does not depend on \( t \). Hence \( (A)^{\alpha_1}R_{1,t}(A)^{\alpha_2} \) extends to a bounded operator and the estimate in the statement holds. One proceed similarly to get the second assertion.

LEMMA 1.3. Let \( B \) be a bounded operator of class \( C^{\theta,\infty}(A) \), \( 0 < \theta \leq 1 \) and \( \alpha_1, \alpha_2 \) positive numbers such that \( \alpha_1 + \alpha_2 < \theta \). Then \( \langle A \rangle^{\alpha_1}B \langle A \rangle^{\alpha_2} \) extends to a bounded operator on \( \mathcal{H} \).

Proof. Recall that in the proof of Lemma 1 we proved that \( B, \langle A \rangle^{\delta} \in \mathcal{B}(\mathcal{H}) \) whenever \( \delta = \alpha_1 + \alpha_2 + \epsilon < \theta \). We denote \( \theta_i = \alpha_i / \delta \), \( i = 1, 2 \) and set \( A_\delta = \langle A \rangle^{\delta} \); this is a self-adjoint operator \( A_\delta \geq 1 \). We have then to control \( A_{\delta}^{\theta_1}i[B, h(A_\delta)] \), where \( h \in S^{\theta_1} \), \( h(s) = s^{\theta_2} \) if \( s \geq 1/2 \) and \( h(s) = 0 \) if \( s \leq 1/4 \). We have (first in form sense)
\[
A_{\delta}^{\theta_1}i[B, h(A_\delta)] = -\frac{1}{\pi} \int_{\mathcal{C}} \overline{\delta h(z)} A_{\delta}^{\theta_1}(A_\delta - z)^{-1} i[B, A_\delta](A_\delta - z)^{-1} \, dx \, dy.
\]
On the support of \( \overline{\delta h} \) the norm of the integrand can be estimated as
\[
||\overline{\delta h(z)} A_{\delta}^{\theta_1}(A_\delta - z)^{-1} i[B, A_\delta](A_\delta - z)^{-1}|| \leq C \langle z \rangle^{\theta_1 + \theta_2 - 1 - 2}.
\]
The rhs is an integrable function, since \( \theta_1 + \theta_2 < 1 \). Therefore \( A_{\delta}^{\theta_1}i[B, h(A_\delta)] \) extends to a bounded operator on \( \mathcal{H} \).
LEMMA 1.4. Let $B$ be a bounded operator of class $C^0,\infty(A)$, $0 < \theta \leq 1$ and $\alpha_1$, $\alpha_2$ positive numbers such that $\alpha_1 + \alpha_2 < \theta$, and $\{g_t\} \subset S^a$, $a \leq 0$, a bounded family of symbols. Then:

$$\|(A)^{\alpha_1}i[B,g_t(A)](A)^{\alpha_2}\| \leq C,$$

where $C$ does not depend on $t$.

Proof. (i) Consider first the case where $a < 0$. Then

$$\langle A \rangle^{\alpha_1}i[B,g_t(A)](A)^{\alpha_2} = \frac{1}{\pi} \int \bar{g}_t(z)\langle A \rangle^{\alpha_1}i[B,(A-z)^{-1}]\langle A \rangle^{\alpha_2} dx dy.$$ 

Using Lemma 1 the norm of the integrand can be majorized by $C(z)^{a-2}$.

(ii) If $a = 0$, let $\epsilon > 0$ be such that $\alpha_1 + \alpha_2 + \epsilon < \theta$ and write

$$\langle A \rangle^{\alpha_1+\epsilon}(A)^{-\epsilon}i[B,g_t(A)](A)^{\alpha_2} = \langle A \rangle^{\alpha_1+\epsilon}i[(A)^{-\epsilon}, B] \langle A \rangle^{\alpha_2} g_t(A) + \langle A \rangle^{\alpha_1+\epsilon}i[B,g_t(A)(A)^{-\epsilon}]\langle A \rangle^{\alpha_2}.$$ 

We use the proof of the previous lemma to show that the first term is a bounded operator and its norm can be bounded by a constant not depending on $t$. For the second term we use (i).

2. THE PROOF OF THEOREM 2

We can suppose, without restricting the generality, that in Theorem 2 we have $M = 1$ and $\lambda = 0$.

LEMMA 2.1. If $\Phi \in C^\infty(\mathbb{R})$ is a real function, $\Phi = 1$ on a neighborhood of 0, then

$$(2.1) \quad (u, \Phi(L)\varphi)_{-s,s} = (u, \varphi)_{-s,s}, \quad \text{for all } \varphi \in \mathcal{A}^1.$$ 

Proof. We have, for $\varphi \in \mathcal{A}^1$,

$$(2.2) \quad (u, (1 - \Phi(L))\varphi)_{-s,s} = (u, L\Psi(L)\varphi)_{-s,s},$$

where $\Psi(t) = (1 - \Phi(t))t^{-1}$. Therefore, to have (2) for $\varphi \in \mathcal{A}^1$ it suffices to prove that $\Psi(L) = (L+1)^{-1}\varphi_1$ with $\varphi_1 \in \mathcal{A}^1$. We can write $(L + 1)\Psi(L) = (1 - \Phi(L)) + \Psi(L)$. Thus, since $(1\Phi(L))\varphi \in \mathcal{A}^1$, it remains to show that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. We have

$$i[\Psi(L), A] = \frac{1}{\pi} \int \bar{\Psi}i[(L - z)^{-1}, A] dx dy$$

$$= -\frac{1}{\pi} \int \bar{\Psi}(L - z)^{-1}(L + 1)^{-1/2}(L + 1)^{-1/2}[L, A](L + 1)^{-1/2}(L + 1)^{-1/2}(L - z)^{-1} dx dy$$

The norm of the integrand can be bounded by $C(z)^{-2-2}|\text{Im}z|^2(z)|\text{Im}z|^2 = C(z)^{-3}$. We get that $i[\Psi(L), A]$ is a bounded operator and we obtain easily that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. Thus equation (1) holds for $\varphi \in \mathcal{A}^1$; the general result follows by density using the fact that $\Phi(L) \in \mathcal{B}(\mathcal{A}^1)$. \[\square\]
Remark. In fact the previous Lemma says that $\Phi(L)u = u$ for all $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi = 1$ on a neighborhood of 0; this fact can be easily seen using that $\Phi(L) \in B(A^*) \cap B(A^{-1})$ and it is symmetric with respect to the duality $(.,.)_{s,-s}$.

**Lemma 2.2.** Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$. We consider the $C_0^\infty(\mathbb{R})$ function

$$
\chi_t(y) = \int_{(y)}^\infty s^{-2\beta} \chi^2(s/t) ds,
$$

where $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Then

$$(L \phi^2(L) \chi_t(A)u, u)_{s,-s} = 0.\tag{2.3}$$

**Proof.** The Lemma follows by hypothesis as $\Phi^2(L) \chi_t(A)u \in (L+1)^{-1} A^{s}$.  

We shall set $T$ for different bounded operators with norm independent on $t$.

**Remark.** We have

$$2 \text{Re} i \langle (L \phi^2(L) \chi_t(A)u, u)_{s,-s} = 0.\tag{2.4}$$

We shall give to this relation the form and the meaning

$$i([L \Phi^2(L), \chi_t(A)]u, u)_{s,-s} = 0.$$

Set $L_1 = \Phi^2(L)L$. Then $L_1$ is a bounded operator of class $C^{\theta+1,\infty}(A)$ (Thm. 6.2.5 [ABG]).

**Lemma 2.3.** We have

$$i[L_1, \chi_t(A)] = i[L_1, A] A^{(A)}^{-2\beta-1} \chi^2(A/t) + \langle A \rangle^{s} T \langle A \rangle^{-s}.$$

**Proof.** One applies Lemma 1.2 for $R_{1,t}$, $\alpha_1 = \alpha_2 = s$, $a = 1-2\beta$. (Here $\alpha_1 + \alpha_2 + a = 2s + 1 - 2\beta < \theta + 1$ since $\beta > \theta/2$, and $s < \theta$.)

As a direct consequence we get the next Lemma.

**Lemma 2.3'.** $\sup_{t \geq 1} |i[L_1, A] A^{(A)}^{-2\beta-1} \chi^2((A)/t)u, u)_{s,-s} < \infty$.

**Lemma 2.4.** If $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi = 1$ on a small enough neighborhood of 0, then

$$\sup_{t \geq 1} |(\Phi(L)i[L, A] \Phi(L) A^{(A)}^{-2\beta-1} \chi^2((A)/t)u, u)_{s,-s} < \infty.$$

**Proof.** We know that $\Phi(L)u = u$. We have

$$A^{(A)}^{-2\beta-1} \chi^2((A)/t) \Phi(L) = \Phi(L) A^{(A)}^{-2\beta-1} \chi^2((A)/t) + [\Phi(L), A^{(A)}^{-2\beta-1} \chi^2((A)/t)].$$

Set $g_t(A) = A^{(A)}^{-2\beta-1} \chi^2((A)/t)$. Here $\{g_t\} \in S_{-2\beta}$ is a bounded family of symbols. One applies Lemma 1.2 to get

$$[\Phi(L), g_t(A)] = [\Phi(L), A] g'_t(A) + R_{1,t}.$$
(In this case $\alpha_1 = \alpha_2 = s$, $a = -\beta$.) Thus, since $2\beta + 1 > s + 1 > 2s$,

\[ A(A)^{-2\beta-1}x^2((A)/t)\tilde{\Phi}(L) = \tilde{\Phi}(L)A(A)^{-2\beta-1}x^2((A)/t) + \langle A \rangle^{-s}T(A)^{-s}. \]

Hence

\[ (i[L_1, A]A(A)^{-2\beta-1}x^2((A)/t)\tilde{\Phi}(L)u, \tilde{\Phi}(L)u)_{s,-s} = (i[L_1, A]\tilde{\Phi}(L)A(A)^{-2\beta-1}x^2((A)/t)u, \tilde{\Phi}(L)u)_{s,-s} + (i[L_1, A]\langle A \rangle^{-s}T(A)^{-s}u,u)_{s,-s}. \]

Since $i[L_1, A] \in \mathcal{B}(\mathcal{H}) \cap C^{\delta,\infty}(A)$ (Prop. 5.2.2 [ABG]), $i[L_1, A]$ is a bounded operator on $\mathcal{A}^*$, $s < \theta$ (Thm 5.3.2, Lemma 5.3.2 [ABG]). Therefore the second term in (2.6) is bounded by a constant independent on $t$. But, in form sense,

\[ \tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)\Phi(L)i[L_1, A]A\tilde{\Phi}(L) + L\tilde{\Phi}(L)i[A, 1-\Phi(L)]\tilde{\Phi}(L) + \tilde{\Phi}(L)i[A, 1-\Phi(L)]L\tilde{\Phi}(L). \]

We take $\text{supp} \tilde{\Phi}$ to be in the set where $\Phi = 1$; then

\[ \tilde{\Phi}(L)i[A, 1-\Phi(L)]\tilde{\Phi}(L) = 0 \quad \text{on} \quad \mathcal{A}^1 \times \mathcal{A}^1. \]

Therefore the bounded operator given by this form on $\mathcal{H}$ ($L \in C^1(A)$) is zero. Similarly we get that $\tilde{\Phi}(L)i[A, 1-\Phi(L)]L\tilde{\Phi}(L) = 0$. Summing up

\[ \tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L). \]

The lemma follows by (2.6), (2.5) and the previous relation.

We can denote $\tilde{\Phi}$ also by $\Phi$.

**Lemma 2.5.** $\sup_{t \geq 1} |(\Phi(L)i[L_1, A]\Phi(L)A(A)^{-\beta}x((A)/t)u, \langle A \rangle^{-\beta}x((A)/t)u) | \leq \infty$.

**Proof.** Set $B = \Phi(L)i[L_1, A]\Phi(L)$. Then $B$ is a bounded operator of class $C^{\delta,\infty}(A)$. Denote $f_t(x) = \langle x \rangle^{-\beta}x((x)/t)$, $x \in \mathcal{R}$. We take $\beta_0 < \beta$, but still $\beta_0 > s/2$, $\beta_0 < \theta/2$. We write

\[ \langle A \rangle^{s}[B, f_t(A)](A)^{s-\beta} = \langle A \rangle^{s-\beta_0}[B, f_t(A)](A)^{s-\beta} = \langle A \rangle^{s-\beta_0}[B, f_t(A)](A)^{s-\beta} = \langle A \rangle^{s-\beta_0}[B, f_t(A)](A)^{s-\beta} \leq \langle A \rangle^{s-\beta_0}[B, f_t(A)](A)^{s-\beta}. \]

But $2s - 2\beta_0 < 2s - \theta < \theta$, so the first term is a bounded operator and its norm does not depend on $t$ (Lemma 1.4). The second term is bounded since $s - 2\beta_0 < 0$ and $\langle A \rangle^{s-\beta_0}[B, f_t(A)](A)^{s-\beta_0}$ is bounded by Lemma 1.3. Now the lemma follows easily.

**Lemma 2.6.** For all $\beta > \alpha$ we have $\langle A \rangle^{-\beta}u \in \mathcal{H}$.

**Proof.** (i) Consider first $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Let $F_+$ be a smooth bounded real function, $F_+ = 1$ on $[1, \infty)$, $F_+ = 0$ on $(-\infty, 1/2]$. We shall show first that

\[ \sup_{t \geq 1} |(\Phi(L)i[L_1, A]\Phi(L)A(A)^{-\beta}F_+(A)x((A)/t)u, \langle A \rangle^{-\beta}F_+(A)x((A)/t)u) | < \infty. \]

We use again the notation $B = \Phi(L)i[L_1, A]\Phi(L)$. If $F_- = 1 - F_+$ then

\[ (BA(A)^{-\beta}x((A)/t)u, \langle A \rangle^{-\beta}x((A)/t)u) = (BA(A)^{-\beta}x((A)/t)(F_+ + F_-)(A)u, \langle A \rangle^{-\beta}(F_+ + F_-)(A)x((A)/t)u). \]
Here \((A)^{-\beta}u \in \mathcal{H}\). Moreover, by Thm 3.10 [BGS2], the fact that \(B\) is of class \(C^{s,2}\) for all \(s < \theta\) ensure that \(F_+(A)BF_-(A) \in B(A^{\beta-s}, A^{\beta-s})\). Hence \((A)^{-\beta}F_+(A)BF_-(A)(A)^{\beta} = T \in B(\mathcal{H})\), and this gives
\[
(BA(A)^{-\beta-1}\chi((A)/t)F_+(A)u, (A)^{-\beta}\chi((A)/t)F_-(A)) = (T(A)^{\beta-t}(A)^{-\beta-1}\chi((A)/t)Au, (A)^{-2\beta}\chi((A)/t)F_-(A)u).
\]
Therefore
\[
(2.8) \quad \sup_{t \geq 1}|(BA(A)^{-\beta-1}\chi((A)/t)F_+(A)u, (A)^{-\beta}\chi((A)/t)F_-(A))| < \infty.
\]
Similarly one gets
\[
(2.9) \quad \sup_{t \geq 1}|(BA(A)^{-\beta-1}\chi((A)/t)F_-(A)u, (A)^{-\beta}\chi((A)/t)F_+(A))| < \infty.
\]
Now (2.7) follows by (2.8), (2.9) and the previous lemma.

We can write \(A(A)^{-1}F_+(A) = g^2(A)F_+(A)\) with \(g \in S^0\). But \((A)^{s-\beta}[B, g(A)](A)^{s-\beta}\) is bounded by Lemma 1.4 \((2s - 2\beta < 2s - \theta < 2\theta - \theta = \theta)\). Hence
\[
\sup_{t \geq 1}|(B(A)^{-\beta}\chi((A)/t)g(A)F_+(A)u, (A)^{-\beta}\chi((A)/t)g(A)F_+(A))| < \infty.
\]
Using now the Mourre estimate we get
\[
\sup_{t \geq 1}||\Phi(L)(A)^{-\beta}\chi((A)/t)g(A)F_+(A)u|| \leq \infty.
\]
As \([\Phi(L), (A)^{-\beta}\chi((A)/t)g(A)F_+(A)](A)^s\) is a bounded operator with norm independent on \(t\) (by Lemma 1.4) it follows
\[
\sup_{t \geq 1}||(A)^{-\beta}\chi((A)/t)g(A)F_+(A)u|| \leq \infty.
\]
This provide, using Beppo-Levi Theorem,
\[
(2.10) \quad (A)^{-\beta}g(A)F_+(A)u \in \mathcal{H}.
\]
If we take \(\tilde{F}_+\) to be a smooth bounded real function on \(\mathbb{R}\), \(\tilde{F}_+ = 1\) on \([2, \infty)\) and \(\text{supp} \tilde{F}_+ \subset [1, \infty)\), we can write
\[
\tilde{F}_+(A)(A)^{-\beta} = (\tilde{F}_+/gF_+)(A)(gF_+)(A)(A)^{-\beta},
\]
and \((\tilde{F}_+/gF_+)(A)(gF_+)(A)\) is a bounded operator. Then (2.10) gives that \(\tilde{F}_+(A)(A)^{-\beta}u \in \mathcal{H}\). Thus the lemma follows in this case since \((1 - \tilde{F}_+(A))(A)^{-\beta}u \in \mathcal{H}\) by hypothesis (b) of Thm 2.

(ii) Now we can repeat the argument with \(s\) replaced by \(2\alpha\) and see that \((A)^{-\beta}u \in \mathcal{H}\) for all \(\beta < \theta/2, \beta > \alpha\).

**Lemma 2.7.** In the conditions of Thm 2, \(u \equiv 0\).

**Proof.** Denote \(u_\epsilon = (\epsilon A)^{-\beta}u\). We shall show that \(||u_\epsilon|| \leq C\), where \(C\) does not depend on \(\epsilon\). This implies that \(u \in \mathcal{H}\). Since \(u = \Phi(L)u\), \(u\) is in the domain of \(L\); and, as \(Lu = 0\), it follows that either 0 is an eigenvalue of \(L\), or \(u \equiv 0\). The first case is impossible due to the Mourre estimate.
Recall that \( L_1 = L \Phi^2(L) \). We shall denote by \( T \) different bounded operators with norm independent on \( t \) and \( \epsilon \). We begin by computing

\[
(i[L_1, A]u_{\epsilon}, u_{\epsilon}) = \lim_{t \to \infty} (i[L_1, A(A + itA)^{-1}it]u_{\epsilon}, u_{\epsilon})
\]

\[
= \lim_{t \to \infty} i[L_1A(A + itA)^{-1}it(\epsilon A)^{-\beta}u, (\epsilon A)^{-\beta}u] - \lim_{t \to \infty} i(\epsilon A)^{-\beta}u, L_1A(A - itA)^{-1}it(\epsilon A)^{-\beta}u
\]

\[
= -\lim_{t \to \infty} i((\epsilon A)^{-\beta}, L_1)[A(A + itA)^{-1}it(\epsilon A)^{-\beta}u, u]_{\beta, -\beta}
\]

\[
+ \lim_{t \to \infty} i(L_1A(A + itA)^{-1}it(\epsilon A)^{-\beta}u, \epsilon A)^{-\beta}u), (\epsilon A)^{-\beta}u_{\epsilon}, (A)^{-\beta}u_{\epsilon} \to (-\beta i[L_1, A(\epsilon A)^{-2}B_1u_{\epsilon}, u_{\epsilon}] + (T\langle A \rangle^{-1}u_{\epsilon}, \langle A \rangle^{-1}u_{\epsilon})
\]

Similarly for the second commutator. We get thus

\[
(i[L_1, A]u_{\epsilon}, u_{\epsilon}) = 2\beta(\epsilon^2A^2(\epsilon A)^{-2-\beta}B_1u_{\epsilon}, u_{\epsilon}) + (T\langle A \rangle^{-1}u_{\epsilon}, \langle A \rangle^{-1}u_{\epsilon})
\]

where \( B_1 = i[L_1, A] \). We write \( \epsilon^2A^2(\epsilon A)^{-2-\beta} = \langle \epsilon A \rangle^\beta - \langle \epsilon A \rangle^{-1}\beta \). Lemma 1.4 gives

\[
((\epsilon A)^\beta B_1u_{\epsilon}, u_{\epsilon}) = (B_1u_{\epsilon}, u_{\epsilon}) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u)
\]

and

\[
((\epsilon A)^{-2}B_1u_{\epsilon}, u_{\epsilon}) = (B_1(\epsilon A)^{-1}u_{\epsilon}, (\epsilon A)^{-1}u_{\epsilon}) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u)
\]

Hence

\[
(1 - 2\beta)(B_1u_{\epsilon}, u_{\epsilon}) = -2\beta(B_1(\epsilon A)^{-1}u_{\epsilon}, (\epsilon A)^{-1}u_{\epsilon}) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u)
\]

We use that \( u = \Phi(u) \) as in Lemma 2.4 to get

\[
(B_1(\epsilon A)^{-1}u_{\epsilon}, (\epsilon A)^{-1}u_{\epsilon}) = (\Phi(L)i[L_1, A]\Phi(L)(\epsilon A)^{-1}u_{\epsilon}, (\epsilon A)^{-1}u_{\epsilon}) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u)
\]

We have

\[
(i[L_1, (\epsilon A)^{-\beta}] = -\beta i[L_1, A]\epsilon^2A^2(\epsilon A)^{-2-\beta} + (A)^{-\beta}T\langle A \rangle^{-\beta-1}
\]

(by Lemma 1.1)
The Mourre inequality (suppose supp$\Phi$ small enough) provide

$$(1 - 2\beta)(B_{1}u_{\epsilon}, u_{\epsilon}) \leq (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

Again:

$$(B_{1}u_{\epsilon}, u_{\epsilon}) = (\Phi(L)i[L, A]\Phi(L)u_{\epsilon}, u_{\epsilon}) + (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

Then the Mourre inequality gives

$$||\Phi(L)u_{\epsilon}|| \leq C.$$

Commuting $\Phi(L)$ and $\langle \epsilon A \rangle^{-\beta}$ (by Lemma 1.4), we get

$$||\langle \epsilon A \rangle^{-\beta} u|| \leq C,$$

which gives $u \in \mathcal{H}$ and thus finishes the proof.  

REFERENCES


