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Kyoto University
ON THE NORM CONVERGENCE OF THE TROTTER–KATO PRODUCT FORMULA WITH ERROR BOUND

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Abstract. The norm convergence of the Trotter–Kato product formula with error bound is shown for the semigroup generated by that operator sum of two nonnegative selfadjoint operators $A$ and $B$ which is selfadjoint.

1. Introduction and Result

It is well-known ([23], [15]; [19]) that the Trotter–Kato product formula for the selfadjoint semigroup holds in strong operator topology. Namely, when $A$ and $B$ are nonnegative selfadjoint operators in a Hilbert space $\mathcal{H}$ with domains $D[A]$ and $D[B]$, then

$$
s\lim_{n\to\infty}(e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n = s\lim_{n\to\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-tC}, \quad (1.1)
$$

if $C$ is the form sum $A \dotplus B$ which is selfadjoint, or, in particular, if the operator sum $A + B$ is essentially selfadjoint on $D[A] \cap D[B]$ with $C$ its closure. The convergence is uniform on each compact $t$-interval in the closed half line $[0, \infty)$.

The aim of this note is to briefly announce our recent results on its operator-norm convergence with error bound. In [12] we have shown

**Theorem 1.1.** If $A$ and $B$ are nonnegative selfadjoint operators in $\mathcal{H}$ with domains $D[A]$ and $D[B]$ and if their operator sum $C := A + B$ is selfadjoint on $D[C] = D[A] \cap D[B]$, then the product formula in operator norm holds with error bound:

$$
\|(e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n - e^{-tC}\| = O(n^{-1/2}),
\|(e^{-tA/n}e^{-tB/n})^n - e^{-tC}\| = O(n^{-1/2}), \quad n \to \infty. \quad (1.2)
$$
The convergence is uniform on each compact $t$–interval in the open half line $(0, \infty)$, and further, if $C$ is strictly positive, uniform on the closed half line $[T, \infty)$ for every fixed $T > 0$.

One of the typical examples of such a selfadjoint operator $C = A + B$ is the Schrödinger operator

$$H = -\frac{1}{2} \Delta + P|x|^{-1} + D|x|^2 + E|x|^{2000}$$

in $L^2(\mathbb{R}^3)$, where $P$, $D$ and $E$ are nonnegative constants.

**Remark 1.1** The first result of such a norm convergence of the Trotter–Kato product formula (1.1) was proved by Rogava [20] in the abstract case under an additional condition that $B$ is $A$–bounded, with error bound $O(n^{-1/2} \log n)$. The next was by Helffer [5] for the Schrödinger operators $H = H_0 + V \equiv -\frac{1}{2} \Delta + V(x)$ with $C^\infty$ nonnegative potentials $V(x)$, roughly speaking, growing at most of order $O(|x|^2)$ for large $|x|$ with error bound $O(n^{-1})$. Each of these two results is independent of the other.

Then under some stronger or more general conditions, several further results are obtained. As for the abstract case, a better error bound $O(n^{-1} \log n)$ than Rogava’s is obtained by Ichinose–Tamura [11] (cf. [9]) when $B$ is $A^\alpha$–bounded for some $0 < \alpha < 1$, even though the $B = B(t)$ may be $t$–dependent, and by Neidhardt–Zagrebnov [16], [17] (cf. [18]) when $B$ is $A$–bounded with relative bound less than 1. As for the Schrödinger operators, a different proof to Helffer’s result was obtained by Dia–Schatzman [2]. Further, more general results were proved for continuous nonnegative potentials $V(x)$, roughly speaking, growing of order $O(|x|^\rho)$ for large $|x|$ with $\rho > 0$, together with error bounds dependent on the power $\rho$ (for instance, of order $O(n^{-2/\rho})$, if $\rho \geq 2$), by Ichinose–Takanobu [6] (cf. [7]), Doumeki–Ichinose–Tamura [3], Ichinose–Tamura [10], Decombes–Dia [1] and others, although the primary purpose of most of these papers was to prove rather a norm estimate between the Kac transfer operator and its corresponding Schrödinger semigroup. The Schrödinger operators treated in [6] and [3] may even involve bounded magnetic fields $\nabla \times A(x)$ : $H = H_0(A) + V \equiv \frac{1}{2} (-i \nabla - A(x))^2 + V(x)$. In [7] and [8] the relativistic Schrödinger operator was also dealt with.

It should be noted (see [4], [21]) that in all these cases of the Schrödinger operators the sum $H = H_0 + V$ (resp. $H = H_0(A) + V$) is selfadjoint on the domain $D[H] = D[H_0] \cap D[V]$ (resp. $D[H] = D[H_0(A)] \cap D[V]$).

Thus the present theorem not only extends Rogava’s result, but also can extend and contain all the results mentioned above, inclusive better error bounds in some cases.

**Remark 1.2.** Unless the sum $A + B$ is selfadjoint on $D[A] \cap D[B]$, the norm convergence of the Trotter–Kato product formula does not always hold, even though the sum is essentially selfadjoint there and $B$ is $A$–form–bounded with relative bound less than 1. A counterexample is due to Hiroshi Tamura [22].

The theorem also holds with the exponential function $e^{-s}$ replaced by real-valued, Borel measurable functions $f$ and $g$ on $[0, \infty)$ satisfying that

$$0 \leq f(s) \leq 1, \quad f(0) = 1, \quad f'(0) = -1,$$

(1.3)
that for every small \( \varepsilon > 0 \) there exists a positive constant \( \delta = \delta(\varepsilon) < 1 \) such that
\[
f(s) \leq 1 - \delta(\varepsilon), \quad s \geq \varepsilon,
\]
and that, for some fixed constant \( \kappa \) with \( 1 < \kappa \leq 2 \),
\[
[f]_{\kappa} := \sup_{s > 0} s^{-\kappa} |f(s) - 1 + s| < \infty,
\]
and the same for \( g \). Of course, the functions \( f(s) = e^{-s} \) and \( f(s) = (1 + k^{-1}s)^{-k} \) with \( k > 0 \) are examples of functions having these properties.

**Theorem 1.2.** If \( 3/2 \leq \kappa \leq 2 \), it holds in operator norm that
\[
\|[g(tB/2n)f(tA/n)g(tB/2n)]^{n} - e^{-tC}\| = O(n^{-1/2}),
\]
\[
\|[f(tA/n)g(tB/n)]^{n} - e^{-tC}\| = O(n^{-1/2}), \quad n \to \infty.
\]

2. Outline of Proof

To proving the theorem, it is crucial to show the following operator-norm version of Chernoff's theorem with error bounds. The case without error bounds was noted by Neidhardt–Zagrebnov [18].

**Lemma.** Let \( C \) be a nonnegative selfadjoint operator in a Hilbert space \( \mathcal{H} \) and let \( \{F(t)\}_{t \geq 0} \) be a family of selfadjoint operators with \( 0 \leq F(t) \leq 1 \). Define \( S_t = t^{-1}(1 - F(t)) \). Then in the following two assertions, for \( 0 < \alpha \leq 1 \), (a) implies (b).

(a) \[
\|(1 + S_t)^{-1} - (1 + C)^{-1}\| = O(t^\alpha), \quad t \downarrow 0.
\]

(b) For any \( \delta > 0 \) with \( 0 < \delta \leq 1 \),
\[
\|F(t/n)^n - e^{-tC}\| = \delta^{-2} t^{-1+\alpha} e^{\delta t} O(n^{-\alpha}), \quad n \to \infty,
\]
for all \( t > 0 \).

Therefore, for \( 0 < \alpha < 1 \) (resp. \( \alpha = 1 \)), the convergence in (2.2) is uniform on each compact \( t \)-interval in the open half line \((0, \infty)\) (resp. in the closed half line \([0, \infty)\)).

Moreover, if \( C \) is strictly positive, i.e. \( C \geq \eta \) for some constant \( \eta > 0 \), the error bound on the right-hand side of (2.2) can also be replaced by \( (1 + 2/\eta)^{2} t^{-1+\alpha} O(n^{-\alpha}) \), so that, for \( 0 < \alpha < 1 \) (resp. \( \alpha = 1 \)), the convergence in (2.2) is uniform on the closed half line \([T, \infty)\) for every fixed \( T > 0 \) (resp. on the whole closed half line \([0, \infty)\)).

**Sketch of Proof of Lemma.**

Put
\[
F(t/n)^n - e^{-tC} = (F(t/n)^n - e^{-tS_{t/n}}) + (e^{-tS_{t/n}} - e^{-tC}).
\]

For the first term on the right we have by the spectral theorem
\[
\|F(t/n)^n - e^{-tS_{t/n}}\| = \|F(t/n)^n - e^{-n(1-F(t/n))}\| \leq e^{-1} n^{-1},
\]
$0 \leq e^{-n(1-\lambda)} - \lambda^n \leq e^{-1}/n, \quad \text{for } 0 \leq \lambda \leq 1.$

For the second term, we use

$$(1 + S_\epsilon)^{-1}[e^{-t(\delta+S_\epsilon)} - e^{-t(\delta+C)}](1 + C)^{-1}$$

$$= \int_0^t e^{-(t-s)(\delta+S_\epsilon)}[(1 + S_\epsilon)^{-1} - (1 + C)^{-1}]e^{-s(\delta+C)}ds$$

$$= \int_0^{t/2} + \int_{t/2}^t$$

where $0 < \delta \leq 1$ and $\epsilon > 0$, to bound these two integrals on the right by $(\delta^2 t)^{-1}e^{\delta t}O(\epsilon^\alpha)$.

Taking $\epsilon = t/n$, we have

$$\|e^{-tS_{t/n}} - e^{-tC}\| \leq (\delta^2 t)^{-1}e^{\delta t}O((t/n)^\alpha) = \delta^{-2}t^{-1+\alpha}e^{\delta t}O(n^{-\alpha}).$$

**Sketch of Proof of Theorems 1.1 and 1.2.**

First note that since $C = A + B$ is itself selfadjoint and so a closed operator, by the closed graph theorem there exists a constant $a$ such that

$$\|(1+A)u\| + \|(1+B)u\| \leq a\|(1+C)u\|, \quad u \in D[C] = D[A] \cap D[B].$$

The proof of the theorem is divided into two cases, (a) the symmetric product case

$$F(t) = e^{-tB/2}e^{-tA}e^{-B/2}, \quad (2.3)$$

and (b) the non-symmetric product case

$$G(t) = e^{-tA}e^{-tB}. \quad (2.4)$$

(a) In the symmetric case we put

$$S_t = t^{-1}(1 - F(t)) = t^{-1}(1 - e^{-tB/2}e^{-tA}e^{-tB/2})$$

and use Lemma to show that

$$\|(1 + S_t)^{-1} - (1 + C)^{-1}\| = O(t^{1/2}), \quad t \downarrow 0.$$
\[ K_t = 1 + A_t + B_t/2 - \frac{1}{4}B_t^2 \geq 1, \]
\[ Q_t = \frac{t^2}{4}K_t^{-1/2}B_t/2A_tB_t/2K_t^{-1/2} - \frac{t}{2}K_t^{-1/2}(A_tB_t/2 + B_t/2A_t)K_t^{-1/2}. \]

Then we can show
\[ \|(1 + Q_t)^{-1}\| \leq 2/(3 - \sqrt{5}), \quad (2.5) \]
\[ \|(1 + S_t)^{-1}K_t^{1/2}\| = \|K_t^{-1/2}(1 + Q_t)^{-1}\| \leq 2/(3 - \sqrt{5}). \quad (2.6) \]

Then we have
\[
(1 + S_t)^{-1} - (1 + C)^{-1}
= (1 + S_t)^{-1}[A + B - (A_t + B_t/2 - \frac{t}{4}B_t/2(1 - tA_t)B_t/2]
- \frac{t}{2}(A_tB_t/2 + B_t/2A_t)](1 + C)^{-1}
= (1 + S_t)^{-1}(1 - A_t)(1 + C)^{-1} + (1 + S_t)^{-1}(1 - B_t/2)(1 + C)^{-1}
+ (1 + S_t)^{-1}[\frac{t}{4}B_t/2(1 - tA_t)B_t/2 + \frac{t}{2}(A_tB_t/2 + B_t/2A_t)](1 + C)^{-1}
\equiv R_1(t) + R_2(t) + R_3(t). \]

We can show the bounds
\[ \|R_i(t)\| \leq ct^{1/2}, \quad i = 1, 2, 3, \quad (2.8) \]
with some constant \(c > 0\). For instance, we can get the bound for \(R_1(t)\), via the expression
\[
R_1(t) \equiv [(1 + S_t)^{-1}K_t^{1/2}][K_t^{-1/2}(1 + A_t)^{1/2}]
\times [(1 + A_t)^{-1/2} - (1 + A_t)^{1/2}(1 + A)^{-1}](1 + A)(1 + C)^{-1}
\]
by (2.6) and the spectral theorem
\[ \|R_1(t)\| \leq \frac{2}{3 - \sqrt{8}}a\|(1 + A_t)^{-1/2} - (1 + A_t)^{1/2}(1 + A)^{-1}\| \leq ct^{1/2}. \]

(b) The non-symmetric case will follow from the symmetric case. We use the commutator argument to observe that
\[ \|G(t/n)^n - F(t/n)^n\| = \|(e^{-tA/n}e^{-tB/n})^n - (e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n\|
= O(1/n). \]
3. The Final Result

In a recent preprint [14], we have shown that if $\kappa = 2$, then Theorem 1.2 holds with optimal error bound $O(n^{-1})$. Further, the convergence is uniform on each compact $t$-interval in the closed half line $[0, \infty)$, and further, if $C$ is strictly positive, uniform on the whole closed half line $[0, \infty)$.

The idea of proof is simply to iterate the resolvent equation of the first identity in (2.5) with help of its adjoint form to get

$$(1 + S_t)^{-1} - (1 + C)^{-1}$$

$$= ((1 + C)^{-1} + [(1 + S_t)^{-1} - (1 + C)^{-1}](C - S_t)(1 + C)^{-1}$$

$$= (1 + C)^{-1}(C - S_t)(1 + C)^{-1} + [(C - S_t)(1 + C)^{-1}]^*(1 + S_t)^{-1}(C - S_t)(1 + C)^{-1}$$

$$\equiv R'_1(t) + R'_2(t).$$

Then by the same arguments together with (2.6) we can show the bounds

$$\|R'_i(t)\| = O(t), \quad i = 1, 2.$$ 

Therefore it turns out that the product formula (1.2) in Theorem 1.1 holds, now with ultimate error bound $O(n^{-1})$, properly extending and containing all the known previous related results.

Finally, we comment about optimality of the error bound $O(n^{-1})$. We know that if both $A$ and $B$ are bounded operators, then we have, in the symmetric product case (2.3), $\|F(t/n)^n - e^{-tC}\| = O(n^{-2})$, while, in the non-symmetric product case (2.4), $\|G(t/n)^n - e^{-tC}\| = O(n^{-1})$. But also in the symmetric product case, we can give an example of two unbounded selfadjoint operators $A$ and $B$ whose operator sum $C = A + B$ is selfadjoint on $D[A] \cap D[B]$ such that $\|F(t/n)^n - e^{-tC}\| \geq L(t)n^{-1}$, with a positive continuous function $L(t)$ of $t > 0$ independent of $n$.

Part of the present results also was briefly announced in [13].

References


