数理解析研究所講究録 1208 巻 2001 年 170-182

THE FEYNMAN PATH INTEGRAL REPRESENTATION OF GREEN FUNCTIONS OF THE POSITION AND THE MOMENTUM OPERATORS

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1. Introduction

It was an interested and important problem to give the description of quantization, ie of passing from classical physical systems to the corresponding quantum ones, from the moment that quantum mechanics came into existence. In the end Heisenberg and Schrödinger succeeded in giving the description based on the notion of operators. On the other hand in 1948 Feynman proposed an essentially new description in [2] based on the notion of the so-called Feynman path integrals. His description is that the probability amplitudes can be constructed from the classical systems in a direct way with physical meanings. In 1951 Feynman himself gave the description reformulated by means of the path integrals in phase space in [3]. Now we know that his description is very useful and applied to wide areas in physics (cf. [7, 20]).

Since Feynman published his paper, much work has been done by physists and mathematicians to give the rigorous meaning to the Feynman path integrals. Some definitions of the Feynman path integrals are proposed and proved

* Research partially supported by Grant-in-Aid for Scientific No.10640176, Ministry of Education, Science, and Culture, Japanese Government.

to be well-defined under some assumptions. See [6, 14, 21] and their references. Recently the author in [9 – 12] studied the time-slicing approximate integrals, determined through broken line paths as oscillatory integrals, of the path integral in configuration space and also in phase space and then proved in a general way their convergence in L^2 space. It is noted that the approximate integrals studied in [9 – 12] are very familiar in physics (cf. [4, 5, 20, 21]).

Our aim in the present paper is to study the path integral representation of correlation functions of the position and also the momentum operators and then to give a rigorous meaning to their representation. It seems that there have been no results of this problem. Our path integral representation of correlation functions is defined by the limit of the time-slicing approximate integrals, determined through broken line paths as oscillatory integrals, similarly to the path integral in [9 – 12]. As is well known, correlation functions are some of the most important quantities in quantum mechanics and quantum field theory (cf. [16, 20]). In physics the path integral representation is well known of correlation functions of only the position operators, though it has not been rigorous. We note that in the present paper a more general representation, ie of correlation functions including the momentum operators, is given rigorously and the Feynman path integrals in phase space determined in [12] are used for obtaining our results. In addition, we note that we can give the path integral representation of the canonical commutation relations , which are the most fundamental in quantum mechanics.

The plan of the proof is as follows. The approximate integral of the Feynman path integral is determined correspondingly to each subdivision of the time interval. We consider the family of all approximate integrals. We first show the uniform boundedness of the family of approximate integrals in some weighted Sobolev spaces. This result is essential in our proof. By using this result of the boundedness we show the equi-continuity w.r.t the time variable of the family of approximate integrals in our weighted Sobolev spaces. Then, by applying the abstract Ascoli-Arzelà theorem we can prove convergence of the approximate integrals of the Feynman path integral in our weighted Sobolev spaces as the size of subdivisions tends to zero. We note that our method of proving convergence is direct compared to that in [9 – 12], where convergence in only L^2 space was proved by using the results in [8] about solutions of the corresponding Schrödinger equation. Convergence of the approximate integrals of correlation functions is proved by using the result above of convergence of the Feynman path integral in the weighted Sobolev spaces and some delicate calculus that is special to oscillatory integrals.

In the present paper we will only state main results and some remarks, which will be given in §2. See [13] for their proofs.

2. Main Results and Remarks

We consider some charged non-relativistic particles in an electromagnetic field. For the sake of simplicity we suppose charge and mass of every particle to be one and $m > 0$, respectively. We consider $x \in R^n$ and $t \in [0,T]$. Let $E(t,x) = (E_1, \cdots, E_n) \in R^n$ and $(B_{jk}(t,x))_{1\leq j<k\leq n} \in R^{n(n-1)/2}$ denote electric strength and magnetic strength tensor, respectively and $(V(t,x), A(t,x)) = (V, A_1, \cdots, A_n) \in R^{n+1}$ an electromagnetic potential, ie

$$
\begin{aligned}
&E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x}, \\
&d(\sum_{j=1}^{n} A_j dx_j) = \sum_{1\leq j<k\leq n} B_{jk} dx_j \wedge dx_k \quad \text{on } R^n, \qquad (2.1)
\end{aligned}
$$

where $\partial V/\partial x = (\partial V/\partial x_1, \cdots, \partial V/\partial x_n)$. Then the Lagrangian function $\mathcal{L}(t, x, \dot{x}$ ($\dot{x} \in R^n$) is given by

$$\mathcal{L}(t, x, \dot{x}) = \frac{m}{2}|\dot{x}|^2 + \dot{x} \cdot A - V. \tag{2.2}$$

The Hamiltonian function $\mathcal{H}(t, x, p)$ ($p \in R^n$) is defined through the Legendre transformation of $\mathcal{L}$ by

$$\mathcal{H}(t, x, p) = \frac{1}{2m}|p - A|^2 + V. \tag{2.3}$$

Let $T^*R^n = R^n_x \times R^n_p$ denote phase space, and $(R^n)^{[s,t]}$ and $(T^*R^n)^{[s,t]}$ the spaces of all paths $q : [s,t] \ni \theta \to q(\theta) \in R^n$ and $(q,p) : [s,t] \ni \theta \to (q(\theta), p(\theta)) \in T^*R^n$, respectively. The classical actions $S_c(t, s; q)$ for $q \in (R^n)^{[s,t]}$ in configuration space and $S(t, s; q, p)$ for $(q,p) \in (T^*R^n)^{[s,t]}$ in phase space are given by

$$S_c(t, s; q) = \int_s^t \mathcal{L}(\theta, q(\theta), \dot{q}(\theta))d\theta, \quad \dot{q}(\theta) = \frac{dq}{d\theta}(\theta) \tag{2.4}$$

and

$$S(t, s; q, p) = \int_s^t p(\theta) \cdot \dot{q}(\theta) - \mathcal{H}(\theta, q(\theta), p(\theta))d\theta, \tag{2.5}$$

respectively (cf. [1]).

Let $\Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T$ be a subdivision of the interval $[0, T]$. We set $|\Delta| = \max_{1 \le j \le \nu}(\tau_j - \tau_{j-1})$. Let $0 \le s \le t \le T$ and $f \in C_0^\infty(R^n)$, where $C_0^\infty(R^n)$ is the space of all infinitely differentiable functions in R^n with compact support. For Δ above we define the time-slicing approximate integrals $\mathcal{C}_\Delta(t, s)f$ and $G_\Delta(t, s)f$ of the Feynman path integrals in configuration space and in phase space, respectively as follows.

At first we define $\mathcal{C}_\Delta(t, s)f$. We set $\mathcal{C}_\Delta(s, s)f = f$. Let $0 \le s < t \le T$. We take $1 \le \mu' \le \mu \le \nu$ such that $\tau_{\mu'-1} \le s < \tau_{\mu'}$ and $\tau_{\mu-1} < t \le \tau_\mu$. For $y, x^{(j)}$ ($j = \mu', \mu' + 1, \ldots, \mu - 1$) and x in R^n let's define $q_\Delta(\theta; y, x^{(\mu')}, \ldots, x^{(\mu-1)}, x) \in (R^n)^{[s,t]}$ by the broken line path joining points

y at s, $x^{(j)}$ at τ_j $(j = \mu', \mu'+1, \ldots, \mu-1)$ and x at t in order. We define $\mathcal{C}_\Delta(t,s)f$ by

$$(\mathcal{C}_\Delta(t,s)f)(x) = \sqrt{\frac{m}{2\pi i\hbar(t-\tau_{\mu-1})}}^{\,n} \prod_{j=\mu'+1}^{\mu-1} \sqrt{\frac{m}{2\pi i\hbar(\tau_j-\tau_{j-1})}}^{\,n} \sqrt{\frac{m}{2\pi i\hbar(\tau_{\mu'}-s)}}^{\,n}$$
$$\times \mathrm{Os} - \int \cdots \int (\exp i\hbar^{-1} S_c(t,s;q_\Delta)) f(y)dydx^{(\mu')} \cdots dx^{(\mu-1)}. \tag{2.6}$$

Here $\mathrm{Os} - \int \cdots \int g(y, x^{(\mu')}, \ldots, x^{(\mu-1)})dydx^{(\mu')} \cdots dx^{(\mu-1)}$ means the oscillatory integral (cf. [15]).

We define $G_\Delta(t,s)$. For the sake of simplicity we set $s=0$. The general case can be defined in the same way that $\mathcal{C}_\Delta(t,s)$ was done. We set $G_\Delta(0,0)f = f$. For $0 < t \le T$ take a $1 \le \mu \le \nu$ such that $\tau_{\mu-1} < t \le \tau_\mu$. For $v^{(j)}$ $(j = 0, 1, \ldots, \mu-1)$ in velocity space R^n we define $v_\Delta(\theta; v^{(0)}, \ldots, v^{(\mu-1)}) \in (R^n)^{[0,t]}$ in velocity space by the piecewise constant path taking $v^{(0)}$ at $\theta = 0, v^{(j)}$ for $\tau_j < \theta \le \tau_{j+1}$ $(j = 0, 1, \ldots, \mu-2)$ and $v^{(\mu-1)}$ for $\tau_{\mu-1} < \theta \le t$. Let $q_\Delta(\theta; x^{(0)}, \ldots, x^{(\mu-1)}, x) \in (R^n)^{[0,t]}$ $(x^{(0)} = y)$ be the path in configuration space defined above. Then we determine the path $p_\Delta(\theta; x^{(0)}, \ldots, x^{(\mu-1)}, x, v^{(0)}, \ldots, v^{(\mu-1)}) \in (R^n)^{[0,t]}$ in momentum space by

$$p_\Delta(\theta) := \frac{\partial \mathcal{L}}{\partial \dot{x}}(\theta, q_\Delta(\theta), v_\Delta(\theta)) = mv_\Delta(\theta) + A(\theta, q_\Delta(\theta)). \tag{2.7}$$

We define $G_\Delta(t,0)f$ by

$$(G_\Delta(t,0)f)(x) = (2\pi\hbar)^{-n\mu}\, \mathrm{Os} - \int \cdots \int \Big(\exp i\hbar^{-1} S(t,0;q_\Delta,p_\Delta)\Big)$$
$$\times f(x^{(0)})dmv^{(0)}dx^{(0)}dmv^{(1)}dx^{(1)} \cdots dmv^{(\mu-1)}dx^{(\mu-1)}. \tag{2.8}$$

Let $L^2 = L^2(R^n)$ be the space of all square integrable functions in R^n with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $|\alpha| = \sum_{j=1}^n \alpha_j$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and $< x >= \sqrt{1+|x|^2}$. In [9 – 12] we proved the following.

Theorem A. *Let* $\partial_x^\alpha E_j(t,x)$ $(j = 1, 2, \cdots, n)$, $\partial_x^\alpha B_{jk}(t,x)$ *and* $\partial_t B_{jk}(t,x)$ $(1 \le j < k \le n)$ *be continuous in* $[0,T] \times R^n$ *for all* α. *We suppose*

$$|\partial_x^\alpha E_j(t,x)| \le C_\alpha,\ |\alpha| \ge 1,\ |\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha < x >^{-(1+\delta)},\ |\alpha| \ge 1 \tag{2.9}$$

in $[0,T] \times R^n$ *for some constants* $\delta > 0$ *and* C_α, *where* δ *is independent of* α. *Let* (V, A) *be an arbitrary potential such that* $V, \partial V/\partial x_j, \partial A_j/\partial t$ *and* $\partial A_j/\partial x_k$ $(j, k = 1, 2, \cdots, n)$ *are continuous in* $[0,T] \times R^n$. *Then we have: (1) Let* $|\Delta|$ *be small. Then both of* $\mathcal{C}_\Delta(t,s)$ *and* $G_\Delta(t,s)$ *on* C_0^∞ *are well-defined and can be extended to bounded operators on* L^2. *They are equal to one other. (2) Let* $|\Delta|$ *be small. Then there exists a constant* $K \ge 0$ *independent of* Δ *such that*

$$\|\mathcal{C}_\Delta(t,s)f\| \le e^{K(t-s)}\|f\|, \quad 0 \le s \le t \le T \tag{2.10}$$

for all $f \in L^2$. *(3) As* $|\Delta| \to 0$, $\mathcal{C}_\Delta(t,s)f$ *for* $f \in L^2$ *converges in* L^2 *uniformly in* $0 \le s \le t \le T$ *and this limit satisfies the Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t), \quad u(s) = f, \tag{2.11}$$

where

$$H(t) = \frac{1}{2m}\sum_{j=1}^n\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 + V. \tag{2.12}$$

We write $\int (\exp i\hbar^{-1}S_c(t,s;q))\, f(q(s))\mathcal{D}q$ and $\iint (\exp i\hbar^{-1}S(t,s;q,p)) \times f(q(s))\mathcal{D}p\mathcal{D}q$ for the limit of $\mathcal{C}_\Delta(t,s)f$ and $G_\Delta(t,s)f$ as $|\Delta| \to 0$, respectively.

Remark 2.1. In (2.8) we make the change of variables: $R^{n\mu} \ni (v^{(0)}, \ldots, v^{(\mu-1)}) \to (p^{(0)}, \ldots, p^{(\mu-1)}) \in R^{n\mu}$, setting $p^{(j)} = \partial\mathcal{L}(\tau_j, q_\Delta(\tau_j), v_\Delta(\tau_j))/\partial\dot{x} = mv^{(j)} + A(\tau_j, x^{(j)})$. Then $G_\Delta(t,0)f$ is written

$$(G_\Delta(t,0)f)(x) = (2\pi\hbar)^{-n\mu}\ \mathrm{Os} - \int\cdots\int \left(\exp i\hbar^{-1}S(t,0;q_\Delta,p_\Delta)\right) \times f(x^{(0)})dp^{(0)}dx^{(0)}dp^{(1)}dx^{(1)}\cdots dp^{(\mu-1)}dx^{(\mu-1)}$$

in the form of an integral on the product space of phase space.

Remark 2.2. In Theorem A only smooth electromagnetic fields are considered. We can apply Theorem A as follows to the case that electromagnetic fields have singularities. For example consider atomic Hamiltonians

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\Delta_j - \sum_{j=1}^{n}\frac{n}{|x^{(j)}|} + \sum_{1\le j<k\le n}\frac{1}{|x^{(j)} - x^{(k)}|},$$

where $x^{(j)} \in R^3$ and Δ_j denotes the Laplacian operator in $x^{(j)}$. Let χ_l ($l = 1, 2, \dots$) be real valued infinitely differentiable functions in R^3 such that $\sup_{x\in R^3} |\partial_x^\alpha \chi_l(x)| < \infty$ for $|\alpha| \ge 2$ and

$$\lim_{l\to\infty} \chi_l(x) = -\frac{1}{|x|} \quad \text{in } L^2(R^3) + L^\infty(R^3).$$

We set

$$H_l = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\Delta_j + \sum_{j=1}^{n} n\chi_l(x^{(j)}) - \sum_{1\le j<k\le n}\chi_l(x^{(j)} - x^{(k)}).$$

We know that $e^{-i\hbar^{-1}(t-s)H_l}$ converges to $e^{-i\hbar^{-1}(t-s)H}$ strongly in L^2 as $l \to \infty$. See Example 2 of §X.2 in [18] and Theorems VIII.21, VIII.25 in [17] and also see [22]. It follows from Theorem A in the present paper that $e^{-i\hbar^{-1}(t-s)H_l} f$ for $f \in L^2$ can be written in the form of our path integrals. So we see that $e^{-i\hbar^{-1}(t-s)H} f$ can be written in the form of the limit of our path integrals. The same argument can be applied to the general case of electromagnetic fields having singularities.

Let B^a ($a = 1, 2, \dots$) be the weighted Sobolev space $\{f \in L^2;\ \|f\|_{B^a} := \|f\| + \sum_{|\alpha|=a}(\|x^\alpha f\| + \|\partial_x^\alpha f\|) < \infty\}$ and B^{-a} its dual space. We write $B^0 = L^2$. As the first result in the present paper we have

Theorem 1. *Besides the assumption of Theorem A we suppose*

$$|\partial_x^\alpha A_j| \le C_\alpha,\ |\alpha| \ge 1, \quad |\partial_x^\alpha V| \le C_\alpha < x >,\ |\alpha| \ge 1 \tag{2.13}$$

in $[0,T]\times R^n$. *Let* $a=0,1,\ldots$. *Then we have: (1) Let* $|\Delta|$ *be small. Then there exists a constant* $K_a\geq 0$ *such that*

$$\|\mathcal{C}_\Delta(t,s)f\|_{B^a}\leq e^{K_a(t-s)}\|f\|_{B^a},\quad 0\leq s\leq t\leq T \tag{2.14}$$

for all $f\in B^a$. *In addition,* $\mathcal{C}_\Delta(t,s)f$ *for* $f\in B^a$ *is continuous as a* B^a*-valued function in* $0\leq s\leq t\leq T$. *(2) As* $|\Delta|\to 0$, $\mathcal{C}_\Delta(t,s)f$ *for* $f\in B^a$ *converges in* B^a *uniformly in* $0\leq s\leq t\leq T$.

Remark 2.3. Suppose that E and B_{jk} satisfy the assumption of Theorem A. We remark that then, we can find a potential (V,A) satisfying (2.13), which was proved in Lemma 6.1 of [10]. In addition, we can easily prove Theorem A from Theorem 1 where $a=0$ by using the gauge transformation as in the proof of Theorem of [10].

Remark 2.4. Let $\mathcal{E}^0_{t,s}([0,T];B^{a+2})\bigcap\mathcal{E}^1_{t,s}([0,T];B^a)$ denote the space of all B^{a+2}-valued continuous and B^a-valued continuously differentiable functions in $0\leq s\leq t\leq T$. Suppose (2.13) and consider the Schrödinger equation (2.11) for $f\in\bigcup_{a=0}^\infty B^a$. Then uniqueness of the solutions in $\bigcup_{a=-\infty}^\infty\mathcal{E}^0_{t,s}([0,T];B^{a+2})\bigcap\mathcal{E}^1_{t,s}([0,T];B^a)$ has been proved in [8]. So we write the solution of (2.11) as $U(t,s)f$ hereafter. As was noted in introduction, Theorem 1 is proved directly without the use of the results in [8]. We also note that we can prove uniqueness stated above of the solutions of (2.11) from Theorem 1 as in the proof of Theorem in [8].

Let Δ be a subdivision and $\big(q_\Delta(\theta;x^{(0)},\ldots,x^{(\nu-1)},x),p_\Delta(\theta;x^{(0)},\ldots,x^{(\nu-1)},x,$ $v^{(0)},\ldots,v^{(\nu-1)})\big)\in(T^*R^n)^{[0,T]}$ the path determined before for Δ. Let $0\leq t_1\leq$

$t_2 \leq \ldots \leq t_k \leq T$. For $z = q$ or p we write

$$\iint \left(\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta)\right)(z_\Delta)_{j_k}(t_k)\cdots(z_\Delta)_{j_1}(t_1) f(q_\Delta(0))\mathcal{D}p_\Delta\mathcal{D}q_\Delta$$
$$:= \mathrm{Os} - \int\cdots\int \left(\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta)\right)(z_\Delta)_{j_k}(t_k)\cdots(z_\Delta)_{j_1}(t_1)$$
$$\times f(x^{(0)})(2\pi\hbar)^{-n\nu} dmv^{(0)}dx^{(0)}dmv^{(1)}dx^{(1)}\cdots dmv^{(\nu-1)}dx^{(\nu-1)} \tag{2.15}$$

and

$$\int \left(\exp i\hbar^{-1} S_c(T,0;q_\Delta)\right)(q_\Delta)_{j_k}(t_k)\cdots(q_\Delta)_{j_1}(t_1) f(q_\Delta(0))\mathcal{D}q_\Delta$$
$$:= \mathrm{Os} - \int\cdots\int \left(\exp i\hbar^{-1} S_c(T,0;q_\Delta)\right)(q_\Delta)_{j_k}(t_k)\cdots(q_\Delta)_{j_1}(t_1) f(x^{(0)})$$
$$\times \prod_{j=1}^{\nu} \sqrt{\frac{m}{2\pi i\hbar(t_j - t_{j-1})}}^{\,n} dx^{(0)}dx^{(1)}\cdots dx^{(\nu-1)}, \tag{2.16}$$

where $(z_\Delta)_j$ is the j-th component of $z_\Delta \in (R^n)^{[0,T]}$.

Theorem 2. *Let $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T$ and $a = 0, 1, \ldots$. Under the assumption of Theorem 1 we have: (1) Let $|\Delta|$ is small. Then the operator (2.15) on C_0^∞ is well-defined and can be extended to a bounded operator from B^{a+k} into B^a. In more detail, we have*

$$\Big\| \iint \left(\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta)\right)(z_\Delta)_{j_k}(t_k)\cdots$$
$$\times (z_\Delta)_{j_1}(t_1) f(q_\Delta(0))\mathcal{D}p_\Delta\mathcal{D}q_\Delta \Big\|_{B^a} \leq C_l \|f\|_{B^{a+k}}, \tag{2.17}$$

where C_a is a constant independent of $\Delta, t_1, \ldots, t_{k-1}$ and t_k. (2) We assume $t_i \neq t_j$ $(i \neq j)$. Then as $|\Delta| \to 0$, (2.15) for $f \in B^{a+k}$ converges in B^a, which we write $\iint \left(\exp i\hbar^{-1} S(T,0;q,p)\right) z_{j_k}(t_k)\cdots z_{j_1}(t_1) f(q(0))\mathcal{D}p\mathcal{D}q$. This limit is equal to $U(T,t_k)\hat{z}_{j_k}U(t_k,t_{k-1})\cdots\hat{z}_{j_1}U(t_1,0)f$, where $\hat{z}_j$ denotes a multiplication operator x_j when $z = q$ and denotes $i^{-1}\hbar\partial_{x_j}$ when $z = p$. (3) Let $t \in [0,T]$

and $f \in B^{a+2}$. We take a μ for each Δ so that $\tau_{\mu-1} < t \leq \tau_\mu$. Then we have

$$\lim_{|\Delta|\to 0} \iint \left(\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta)\right)(p_\Delta)_k(t)(q_\Delta)_j(t) f(q_\Delta(0))\mathcal{D}p_\Delta \mathcal{D}q_\Delta$$
$$= U(T,t)\hat{q}_j\hat{p}_k U(t,0)f + \frac{\hbar}{i}\delta_{jk} \lim_{|\Delta|\to 0}\left(\frac{\tau_\mu - t}{\tau_\mu - \tau_{\mu-1}}\right)U(T,0)f \tag{2.18}$$

in B^l, where δ_{jk} is the Kronecker delta. It is noted that the right-hand side above is divergent if $j = k$. (4) Here we don't assume $t_i \neq t_j$ $(i \neq j)$. Let $|\Delta|$ be small. Then the operator (2.16) on C_0^∞ is well-defined and is equal to (2.15) where $z = q$. In addition, in this case, ie all $z = q$ (2.15) for $f \in B^{a+k}$ converges in B^a, as $|\Delta| \to 0$.

We write $\int (\exp i\hbar^{-1} S_c(T,0;q))\, q_{j_k}(t_k)\cdots q_{j_1}(t_1) f(q(0))\mathcal{D}q$ for the limit of (2.16) as $|\Delta| \to 0$. Let's use the notations of the Heisenberg picture of quantum mechanics, $\hat{z}_j(t) = U(t,0)^{-1}\hat{z}_j U(t,0)$, $|f,t> = U(t,0)^{-1}f$ and $< f,t| = |f,t>^*$, where g^* is the complex conjugate of g.

Corollary. *Under the assumption of Theorem 1 we have: (1) Let $0 \leq t_1 < t_2 < \ldots < t_k \leq T, g \in L^2$ and $f \in B^k$. Then we obtain the path integral representation of correlation functions*

$$< g,T|\hat{z}_{j_k}(t_k)\cdots \hat{z}_{j_1}(t_1)|f,0> \left(:= (|g,T>, \hat{z}_{j_k}(t_k)\cdots\hat{z}_{j_1}(t_1)|f,0>)\right)$$
$$= (g, \iint \left(\exp i\hbar^{-1} S(T,0;q,p)\right) z_{j_k}(t_k)\cdots z_{j_1}(t_1) f(q(0))\mathcal{D}p\mathcal{D}q). \tag{2.19}$$

We also have

$$< g,T|\hat{q}_{j_k}(t_k)\cdots \hat{q}_{j_1}(t_1)|f,0>$$
$$= (g, \int \left(\exp i\hbar^{-1} S_c(T,0;q)\right) q_{j_k}(t_k)\cdots q_{j_1}(t_1) f(q(0))\mathcal{D}q).$$

(2) Let $0 \leq t' < t \leq T$ *and* $f \in B^2$. *Then we have for* $j, k = 1, 2, \ldots, n$

$$\begin{aligned}&\lim_{t' \to t} \iint \left(\exp i\hbar^{-1} S(T, 0; q, p)\right)\left(p_j(t) q_k(t') - q_k(t) p_j(t')\right) f(q(0)) \mathcal{D}p\mathcal{D}q \\ &= \frac{\hbar}{i}\delta_{jk} \iint \left(\exp i\hbar^{-1} S(T, 0; q, p)\right) f(q(0)) \mathcal{D}p\mathcal{D}q \qquad (2.21)\end{aligned}$$

in L^2.

Proof. Since

$$U(T, t_k)\hat{z}_{j_k} U(t_k, t_{k-1}) \cdots \hat{z}_{j_1} U(t_1, 0) f = U(T, 0)\hat{z}_{j_k}(t_k) \cdots \hat{z}_{j_1}(t_1) f, \quad (2.22)$$

we can easily prove (2.19) and (2.20) from the assertions (2) and (4) of Theorem 2. It follows from the assertion (2) of Theorem 2 that the left-hand side of (2.21) is equal to

$$\lim_{t' \to t}\left(U(T, t)\hat{p}_j U(t, t')\hat{q}_k U(t', 0) f - U(T, t)\hat{q}_k U(t, t')\hat{p}_j U(t', 0) f\right).$$

Here let's use the fact that $\|U(t, s) g\|_{B^a} \leq \mathrm{e}^{K_a(t-s)}\|g\|_{B^a}$ and $U(t, s) g$ for $g \in B^a$ is continuous as a B^a-valued function in $0 \leq s \leq t \leq T$, which follows from Theorem 1. Then

$$\begin{aligned}&\|U(t, t')\hat{q}_k U(t', 0) f - \hat{q}_k U(t', 0) f\|_{B^1} \\ &\leq \mathrm{e}^{K_1(t-t')}\|\hat{q}_k \left(U(t', 0) - U(t, 0)\right) f\|_{B^1} + \|U(t, t')\hat{q}_k U(t, 0) f - \hat{q}_k U(t, 0) f\|_{B^1}\end{aligned}$$

and so $\lim_{t' \to t} U(t, t')\hat{q}_k U(t', 0) f = \hat{q}_k U(t, 0) f$ in B^1. Consequently we have

$$\lim_{t' \to t} U(T, t)\hat{p}_j U(t, t')\hat{q}_k U(t', 0) f = U(T, t)\hat{p}_j \hat{q}_k U(t, 0) f$$

in L^2. Hence we can prove (2.21). Q.E.D.

Remark 2.5. (i) The path integral representation (2.20) of correlation functions of the position operators is well known in physics, though it has not been rigorous ([16, 20]). It is noted that our result (2.19) gives a more general representation of correlation functions including the momentum operators. (ii) It follows from Theorem 2 and (2.22) that the equation (2.21) is equivalent to

$$\lim_{t'\nearrow t}\left(\hat{p}_j(t)\hat{q}_k(t')f - \hat{q}_k(t)\hat{p}_j(t')f\right) = \frac{\hbar}{i}\delta_{jk}f, \tag{2.23}$$

ie the canonical commutation relations.

Example 2.1. Let (V, A) be an electromagnetic potential such that

$$|\partial_x^\alpha V| + < x >^{1+\delta} |\partial_x^\alpha A| \le C_\alpha,\ |\alpha| \ge 2, \quad |\partial_x^\alpha \partial_t A| \le C_\alpha,\ |\alpha| \ge 1$$

in $[0, T] \times R^n$ for some constant $\delta > 0$. Then since $E_j = -\partial A_j/\partial t - \partial V/\partial x_j$ and $B_{jk} = \partial A_k/\partial x_j - \partial A_j/\partial x_k$ from (2.1), we can see that the assumption of Theorems 1 and 2 is satisfied.

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