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short-crested waveの線形安定性

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The stability of resonant short-crested waves is investigated. We found the similar kind of *bubble* of instability near the turning point as Ioualalen *et al.* (1996), the instability which is related to harmonic resonance. Their results are, however, different from ours for the growth rate and the instability region. Furthermore we discovered another *bubble* of instability that is very close to the turning point.

Harmonic resonances of short-crested water waves are thus associated with two *bubbles* of instability; the first one is located on a branch with the turning point and the second one is located on another branch that is continuous in the vicinity of the bifurcation point.

1 Introduction

In a linear description, short-crested wave fields are defined as a superposition of two two-dimensional progressive wave trains of equal wavelengths and intersecting at an angle γ. The description of the geometry of the propagation can be found in Hsu *et al.* (1979) who defined an angle θ so that $\theta = (\pi - \gamma)/2$. The three-dimensional fields admit two two-dimensional limits: the progressive Stokes wave for $\theta = 90°$ for which the two waves propagate in the same direction and the standing wave for $\theta = 0°$ for which

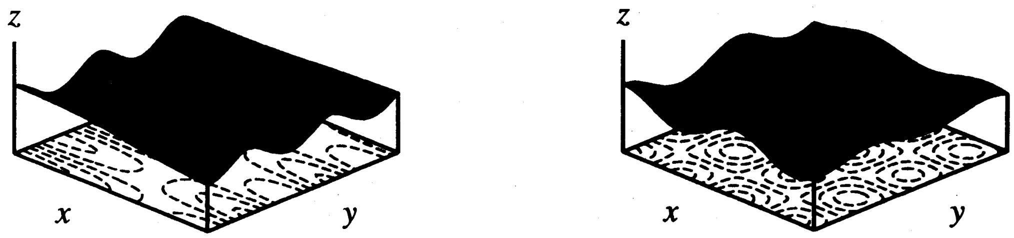

図 1: Short-crested wave fields for angles $\theta = 10°$ (left) and $\theta = 45°$ (right) on deep water.

the two waves propagate in opposite directions. Figure 1 shows two typical wave patterns on deep water: the left in the figure represents a short-crested wave field for $\theta = 10°$ that is close to standing waves, exhibiting long crests in the x-direction compared to the other horizontal direction y; the right in the figure represents a wave at angle $\theta = 45°$ exhibiting equal wave lengths in both horizontal directions.

The properties of short-crested waves have been discussed in Roberts (1983) for deep water and in Marchant & Roberts (1987) on water of finite depth. In particular, the authors showed how short-crested wave fields may be unsteady through harmonic resonance phenomena. Roberts & Peregrine (1983) calculated low order analytical solutions for $\theta \to 90°$ and found that harmonic resonances correspond to multiple-like solutions. Okamura (1996) calculated both weakly nonlinear and fully nonlinear short-crested waves in deep water for $\theta \approx 0°$ and found that harmonic resonances correspond to multiple-like solutions. Marchant & Roberts (1987) showed that harmonic resonances occur for short-crested waves in finite depth when harmonic (m, n), i.e. $\sin(m\alpha(x - ct)) \cos(n\beta y) \cosh[\kappa_{mn}(z + d)]$, is a solution of the homogeneous differential equation derived from the nonlinear surface conditions, where $\kappa_{mn} = (m^2\alpha^2 + n^2\beta^2)^{1/2}$, $\alpha = \sin\theta$ and $\beta = \cos\theta$.

Such cases occur at critical depths for which

$$\kappa_{mn} \tanh(\kappa_{mn} d) = m^2 \tanh d. \tag{1}$$

The critical angles for which a harmonic resonance occurs are given in Marchant & Roberts (1987) and in Ioualalen *et al.* (1996) for different depths. We have chosen the harmonic resonance (2,6) occurring at depth $d = 1$. In the linear description, the critical angle for which this harmonic resonance occurs is $\theta_c = 65.8354°$ [*e.g.*, Ioualalen *et al.* (1996)].

Ioualalen *et al.* (1993, 1996) computed the stability problem associated with the harmonic resonance phenomenon and found that harmonic resonances are associated with sporadic and weak superharmonic instabilities that have a *bubble*-like shape in the wave steepness parameter space. This was the first attempt to characterize the stability of resonant short-crested waves. However, the stability problem studied by the authors was associated with non-bifurcated solutions and not fully nonlinear solutions.

The aim of the present study is to extend their work to fully nonlinear solutions exhibiting a multiple-like solution behaviour.

2 Mathematical formulation of the problem

We consider surface gravity waves on an inviscid, incompressible fluid of finite depth where the flow is assumed irrotational. The governing equations are given in a dimensionless form with respect to the reference length $1/k$ and the reference time $(gk)^{-1/2}$, where g is the gravitational acceleration and k the wavenumber of the incident wave train.

Let us define a frame of reference (x^*, y^*, z^*, t^*, ϕ^*) so that $x^* = x - ct$, $y^* = y$, $z^* = z$, $t^* = t$ and $\phi^* = \phi - cx^*$ where c represents the propagation velocity of the short-crested wave train and is equal to ω/α, ω being the frequency of the wave and $\alpha = \sin\theta$ is the x-direction wave number, the y-direction wave number being $\beta = \cos\theta$. If we omit the asterisks for sake of simplicity, the governing equations are:

$$\Delta\phi = 0\,, \quad \text{for } -d < z < \eta(x,y,t), \tag{2}$$

$$\phi_z = 0\,, \quad \text{on } z = -d, \tag{3}$$

$$\phi_t + \eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2 - c^2) = 0\,, \quad \text{on } z = \eta(x,y,t), \tag{4}$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0\,, \quad \text{on } z = \eta(x,y,t), \tag{5}$$

where d is the depth of the fluid, $\phi(x,y,z,t)$ is the velocity potential and $z = \eta(x,y,t)$ is the equation of the free surface. In this new frame of reference propagating at a speed c, the system of equations (2)–(5) admits doubly periodic solutions of permanent form $\bar{\eta}(x,y)$ and $\bar{\phi}(x,y,z)$.

We define the following functions to construct a stability problem:

$$\eta(x,y,t) = \bar{\eta}(x,y) + \eta'(x,y,t)\,, \tag{6}$$

$$\phi(x,y,z,t) = \bar{\phi}(x,y,z) + \phi'(x,y,z,t)\,, \tag{7}$$

where we assume that the surface elevation and the velocity potential are superposition of a steady unperturbed wave $(\bar{\eta},\bar{\phi})$ and infinitesimal perturbations (η',ϕ'). After substituting expressions (6) and (7) into equations (2)–(5) and linearizing we obtain the zeroth order system of equations for which permanent short-crested waves are solutions and the first order perturbation equations representing the stability problem. Both systems of equations will be resolved in the frame of reference moving with the wave.

We look for non-trivial solutions of the following superharmonic form

$$\eta' = e^{-i\sigma t}\sum_{J=-\infty}^{\infty}\sum_{K=-\infty}^{\infty} a_{JK}e^{i(J\alpha x + K\beta y)}\,, \tag{8}$$

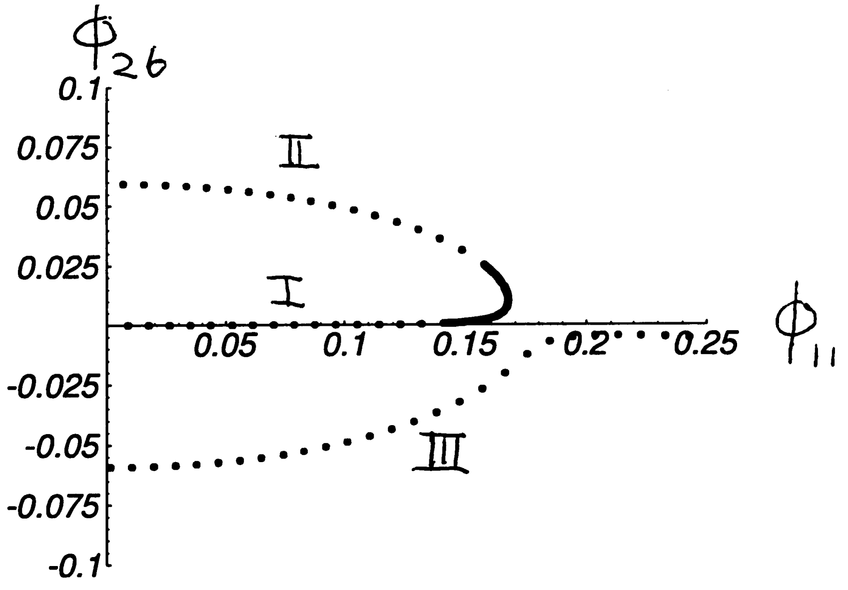

図 2: Coefficient ϕ_{26} versus coefficient ϕ_{11} for depth $d = 1$ and angle $\theta = 66°$. The different branches of the solutions are numbered I–III.

$$\phi' = e^{-i\sigma t} \sum_{J=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} b_{JK} e^{i(J\alpha x + K\beta y)} \frac{\cosh[\kappa_{JK}(z+d)]}{\cosh(\kappa_{JK} d)}, \tag{9}$$

where $\kappa_{JK} = [(J\alpha)^2 + (K\beta)^2]^{1/2}$.

2.1 Computation of the multiple-like solutions of permanent form

The velocity potential $\bar{\phi}$ is expressed as follows:

$$\bar{\phi} = -cx + \sum_{k=0}^{N} \sum_{j=2-(k \bmod 2)}^{N} \phi_{jk} \sin(j\alpha x) \cos(k\beta y) \frac{\cosh[\kappa_{jk}(z+d)]}{\cosh(\kappa_{jk} d)}, \tag{10}$$

where N is the maximum order of expansion and is chosen to be odd. All calculations are carried out using $N = 19$ in this paper.

In this work we are interested in the harmonic coefficient ϕ_{26} whose mode (2,6) is responsible for a harmonic resonance at angle $\theta_c = 65.8354°$ for depth $d = 1$ in the linear approximation. Figure 2 exhibits the multiple-like structure of the coefficient ϕ_{26} as a function of the coefficient ϕ_{11} of the fundamental mode for the wave parameters $d = 1$ and $\theta = 66°$ near the critical angle θ_c. The turning point is $(\phi_{11}, \phi_{26}) = (0.167091, 0.0106346)$.

We have obtained all solutions of the resonant short-crested waves: branch I, branch II including the turning point and branch III. Ioualalen *et al.* (1996 obtained branch I and the part of branch III on the right side of the turning point and failed to find branch II. Then they used the Shanks transform to match artificially branches I and III in the vicinity of the turning point that we obtain here. They computed the stability of the solution corresponding to a non-bifurcated solution. Because their solutions are much different from ours near the critical point, we are now interested in studying the stability of the multiple-like solutions that represent the fully nonlinear wave field along all the three branches and around the turning point to characterize definitely the stability behaviour of harmonic resonances.

2.2 Resolution of the stability problem

The stability analysis consists in determining the set of eigenvalues σ and the coefficients a_{JK} and b_{JK} of their associated eigenvectors. Since the system of perturbation equations is real-valued, the eigenvalues σ appear in complex conjugate pairs. Thus an instability corresponds to $\Im(\sigma) \neq 0$. For $h = 0$, the unperturbed wave is given by $\bar{\eta} = 0$ and $\bar{\phi} = -c_0 x$ with $c_0 = \omega_0/\alpha = (\tanh d)^{1/2}/\alpha$. Then the eigenvalues are $\sigma^s_{JK} = -(J\alpha)c_0 + s[\kappa_{JK} \tanh(\kappa_{JK} d)]^{1/2}$ with $s = \pm 1$, $\mathrm{sign}[s\Im(-i\sigma)]$ being the signature of the perturbation [*e.g.*, MacKay & Saffman (1986)]. The real-valued set of eigenvalues $\{\sigma^s_{JK}\}$ causes the wave to be neutrally stable for $h = 0$. Instabilities arise as the wave steepness h increases. We apply the necessary condition for instability in terms of collision of eigenvalues of opposite signatures s or at zero-frequency. An instability can arise if for some wave steepness h, two modes have the same frequency, that is, $\sigma^{s}_{J_1K_1} = \sigma^{-s}_{J_2K_2}$. This condition takes the following form for $s = 1$ ($s = -1$ corresponds to an opposite direction of propagation):

$$[\kappa_{J_1K_1} \tanh(\kappa_{J_1K_1} d)]^{1/2} + [\kappa_{J_2K_2} \tanh(\kappa_{J_2K_2} d)]^{1/2} = (J_1 - J_2)(\tanh d)^{1/2}.$$

The perturbation equations lead to a generalized eigenvalue problem of the form: $\boldsymbol{A u} = i\sigma \boldsymbol{B u}$, where σ is the set of eigenvalues to be computed with the corresponding eigenvectors $\boldsymbol{u} = (a_{jk}, b_{jk})^T$.

3 Resonant interactions: superharmonic instabilities associated with the harmonic resonance (2,6)

In this section, the superharmonic instability of short-crested waves subject to a harmonic resonance (2,6) is computed for depth $d = 1$ and angle $\theta = 66°$.

Following Ioualalen *et al.* (1996), a superharmonic instability associated with a harmonic resonance $(\pm m, n)$ may arise only if the two eigenvalues $\sigma^{s}_{mn} = \sigma^{-s}_{-mn}$ of opposite signature are equal at a given wave steepness h. The collision of the two eigenmodes $(\pm m, n)$ is then interpreted as Ioualalen *et al.* (1993)'s class Ia (m, n) instability and it can be interpreted as a resonance between the two eigenmodes $(\pm m, n)$ and the $2m$ modes $(1, \pm 1)$ of the basic short-crested wave, that is,

$$\Omega_1 = -\Omega_2 + m\Omega_{01} + m\Omega_{02}, \tag{11}$$

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + m\boldsymbol{k}_{01} + m\boldsymbol{k}_{02}, \tag{12}$$

where $\Omega_i = [|\boldsymbol{k}_i| \tanh(\kappa_{mn} d)]^{1/2}$, $\Omega_{0i} = (\tanh d)^{1/2}$ for $i = 1, 2$ and $\boldsymbol{k}_1 = (\alpha m, \beta n)$, $\boldsymbol{k}_2 = (-\alpha m, \beta n)$, $\boldsymbol{k}_{01} = (\alpha, \beta)$, and $\boldsymbol{k}_{02} = (\alpha, -\beta)$.

In Figure 3 are plotted the frequencies of the eigenvalues $\sigma_{\pm 26}$ along all the branches of the short-crested wave solutions shown in Figure 2.

The stability of branch III shows that frequencies of the modes $(\pm 2, 6)$ coalesce between $\phi_{11} \approx 0.160$ and $\phi_{11} \approx 0.214$. The two modes are neutrally stable with a non-zero frequency for infinitesimal wave steepness h, i.e. $\phi_{11} \to 0$, and then give rise to a *bubble* of instability in that range of ϕ_{11} with zero-frequency. The *bubble* of instability is physically associated with a resonant interaction; the coalescence of the two eigenmodes with zero-frequency simply means that the harmonics $(\pm 2, 6)$ propagate at the same phase speed as the basic wave, bearing in mind that the stability problem has been computed in the frame of reference moving with the basic wave. This phase-locking of the resonant modes with the basic wave is what we have expected.

More interestingly, the superharmonic instability is not anymore sporadic as mentioned by Ioualalen *et al.* (1996). The authors found that the instability occurs in a range of h of order h^4 while ours occurs in the range of order h^3. The difference comes out of the short-crested wave solutions. Our short-crested wave, branch III, is computed numerically through a fully nonlinear method while theirs is obtained by matching artificially the solutions by using the Shanks transform. Thus they obtained a solution that is not strictly similar to our branch III.

The stability of branches I and II shows that no instability appears on branch I while a *bubble* of instability occurs for $0.1571 < \phi_{11} < 0.1656$ on branch II, the region which is very close to the turning point ($\phi_{11} = 0.167091$). The superharmonic instability is more sporadic and of lower intensity (larger time scale) than that of branch III. Note that the insta-

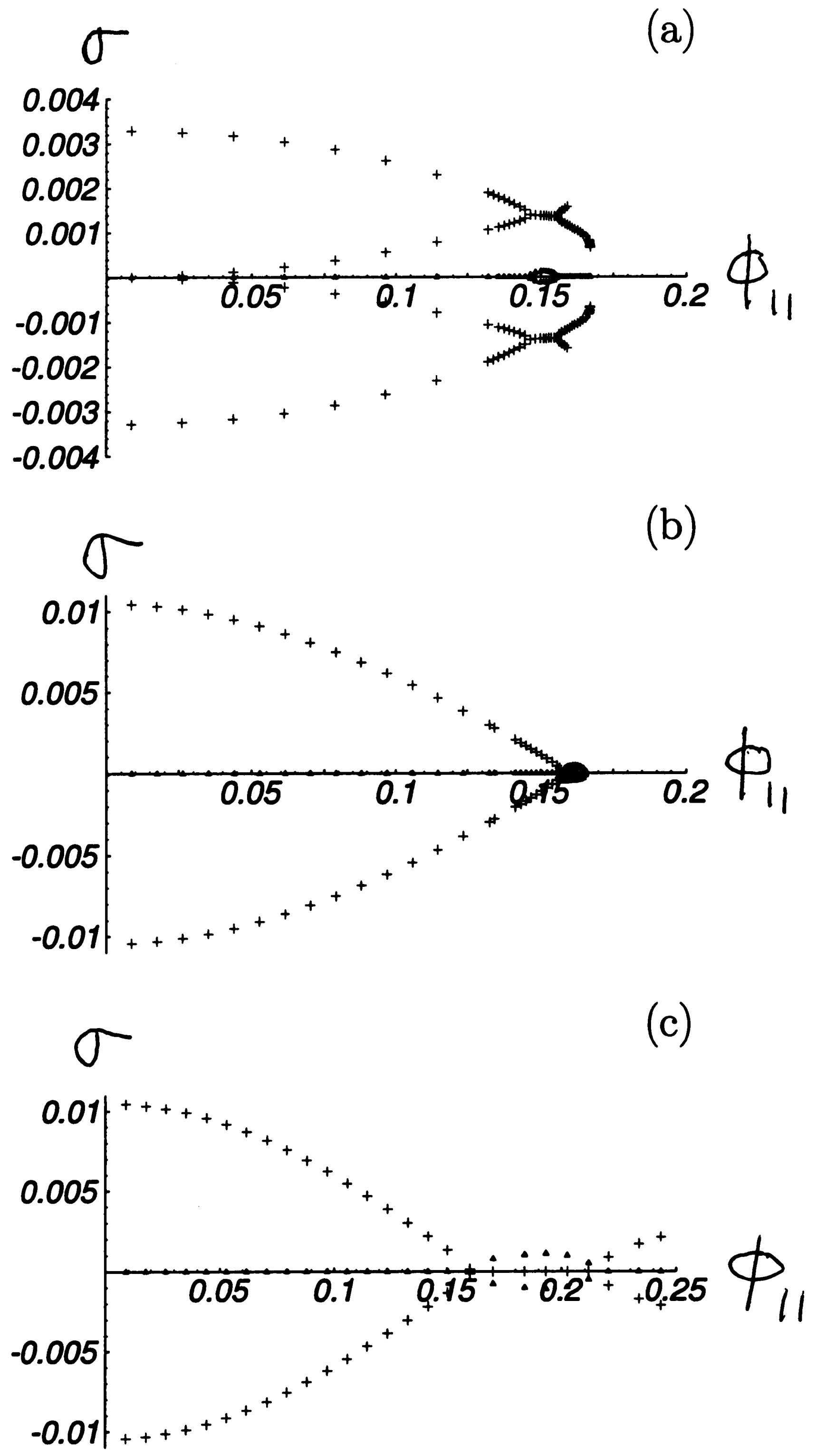

図 3: Frequency $[-\Re(\sigma_{\pm 26})]$ (+) and growth rate $[-\Im(\sigma_{\pm 26})]$ (▲) as a function of coefficient ϕ_{11} for depth $d = 1$ and angle $\theta = 66°$: (a) branch I; (b) branch II; (c) branch III. See Figure 2 for definition of the branches.

bility between $\phi_{11} = 0.146$ and $\phi_{11} = 0.155$ on branch I is simply due to the presence of a four-mode class I Benjamin-Feir instability, which is not related to harmonic resonance.

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